Lesson 10. Applications of Divergence Theorem

Consider incompressible fluid of constant density ρ in a region T with the boundary S. We take $\rho \equiv 1$. Let $\mathbf{v}$ be the velocity field of the fluid. If $\mathbf{n}$ is the outward unit normal to S then $\iint_S \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A$ is the mass of fluid leaving T through S per unit time. Since the fluid is incompressible, this should be equal total source of the fluid inside T. By Divergence Theorem, this is $\iiint_T \operatorname{div} \mathbf{v} \, \mathrm{d}V$, thus divergence represents **source intensity** of the fluid flow. In particular, divergence does not depend on the choice of Cartesian coordinates. If there are no sources or sinks, $\operatorname{div} \mathbf{v} \equiv 0$ for the incompressible fluid flow.

Let now $P = (x, y, z)$ be a point in T. Consider a small ball B_ϵ centered at P, with the boundary S_ϵ. Then

$$\iiint_{B_\epsilon} \operatorname{div} \mathbf{v} \, \mathrm{d}V = \iint_{S_\epsilon} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A.$$

Dividing both sides by the volume $V(B_\epsilon) = \frac{4}{3}\pi\epsilon^3$ of B_ϵ, in the limit $\epsilon \to 0$ we get

$$\operatorname{div} \mathbf{v}(P) = \lim_{\epsilon \to 0} \frac{1}{V(B_\epsilon)} \iint_{S_\epsilon} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A.$$

Thus the limit in the right-hand side equals source intensity of fluid at P.

Heat Equation. The rate of heat flow is proportional to the temperature gradient: $\mathbf{v} = -K\nabla U$ where K is thermal conductivity (which we assume constant) and U is temperature. From the Divergence Theorem, the rate of heat leaving a region T through its surface S is

$$-\frac{\partial H}{\partial t} = \iint_S \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}A = \iiint_T \operatorname{div}\mathbf{v}\,\mathrm{d}V =$$
$$-K\iiint_T \operatorname{div}\nabla U\,\mathrm{d}V = -K\iiint_T \nabla^2 U\,\mathrm{d}V.$$

The total heat content in T is $H = \iiint_T \sigma\rho U\,\mathrm{d}V$ where σ is specific heat and ρ is the mass density. Thus

$$\frac{\partial H}{\partial t} = \iiint_T \sigma\rho\frac{\partial U}{\partial t}\,\mathrm{d}V = K\iiint_T \nabla^2 U\,\mathrm{d}V.$$

Since this holds for any subregion of T, we have

$$\frac{\partial U}{\partial t} = c^2 \nabla^2 U \quad \text{where } c^2 = \frac{K}{\sigma \rho}.$$

In particular, the equilibrium temperature is **harmonic**:

$$\nabla^2 U = 0.$$

Then, Divergence Theorem becomes

$$0 = \iiint_T \nabla^2 U \, \mathrm{d}V = \iiint_T \operatorname{div} \nabla U \, \mathrm{d}V = \iint_S \nabla U \cdot \mathbf{n} \, \mathrm{d}A.$$

Since $\nabla U \cdot \mathbf{n} = \frac{\partial U}{\partial \mathbf{n}}$ is the normal derivative of U, we have

$$\iint_S \frac{\partial U}{\partial \mathbf{n}} \, \mathrm{d}A = 0.$$

Green's identities. From the Product Rule,

$$\text{div}(f\mathbf{v}) = f\text{div}\,\mathbf{v} + \nabla f \cdot \mathbf{v}.$$

When $\mathbf{v} = \nabla g$ is the gradient of a function g, we have
$(*)$ $\text{div}(f\nabla g) = f\,\text{div}\,\nabla g + \nabla f \cdot \nabla g = f\nabla^2 g + \nabla f \cdot \nabla g.$
Exchanging f and g,
$(**)$ $\text{div}(g\nabla f) = g\nabla^2 f + \nabla g \cdot \nabla f.$
Subtracting $(**)$ from $(*)$, we get

$$\text{div}(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f.$$

Applying Divergence Theorem and equality $\nabla g \cdot \mathbf{n} = \dfrac{\partial g}{\partial \mathbf{n}}$,

$$\iiint_T (f\nabla^2 g - g\nabla^2 f)\,\mathrm{d}V = \iint_S \left(f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}\right)\,\mathrm{d}A.$$

If g is harmonic and $f \equiv 1$, we get again

$$0 = \iiint_T \nabla^2 g \, \mathrm{d}V = \iint_S \frac{\partial g}{\partial \mathbf{n}} \, \mathrm{d}A.$$

If g is harmonic and $f = g$, from Divergence Theorem and $(*)$ we get,

$$\iint_S g \frac{\partial g}{\partial \mathbf{n}} \, \mathrm{d}A = \iiint_T \mathrm{div}(g \nabla g) \, \mathrm{d}V = \iiint_T |\nabla g|^2 \, \mathrm{d}V.$$

The right-hand side is called the **energy integral** or **Dirichlet integral**.

This implies uniqueness of a solution of the Laplace equation $\nabla^2 g = 0$ in T with the Dirichlet boundary condition $g\big|_S = g_0$.
For the Neumann boundary condition $\frac{\partial g}{\partial \mathbf{n}}\Big|_S = g_1$, solution is unique up to adding an arbitrary constant.

Example. Let $\mathbf{r} = (x, y, z)$, and let S be a smooth surface with a continuous unit normal field $\mathbf{n}$. Let α be the angle between $\mathbf{r}$ and $\mathbf{n}$. We assume $0 \le \alpha \le \pi$. Then $\mathbf{r} \cdot \mathbf{n} = |\mathbf{r}| \cos \alpha$, so $\iint_S \mathbf{r} \cdot \mathbf{n} \, \mathrm{d}A = \iint_S |\mathbf{r}| \cos \alpha \, \mathrm{d}A$.
If S is the boundary of a solid region T and $\mathbf{n}$ is the outer normal, then

$$\iiint_T \operatorname{div} \mathbf{r} \, \mathrm{d}V = \iint_S \mathbf{r} \cdot \mathbf{n} \, \mathrm{d}A.$$

But $\operatorname{div} \mathbf{r} = 3$, thus

$$3 \, \mathrm{Volume}(T) = \iint_S |\mathbf{r}| \cos \alpha \, \mathrm{d}A.$$

Note that $|\mathbf{r}|$ is the variable ρ in spherical coordinates. Compare this with the formula

$$2 \, \mathrm{Area}(R) = \oint_C r^2 \, \mathrm{d}\theta = \oint_C r \cos \alpha \, \mathrm{d}s$$

where R is a region in the plane, $C = \partial R$ a curve, and s is the normal parameter on C.