Lesson 14. Cauchy-Riemann equations

If $f(z) = u(z) + iv(z)$ is analytic in a domain D, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{in } D.$$

Proof. In $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ we write $z = x + iy$, $\Delta z = \Delta x + i\Delta y$, and approach z in two different directions, through real and imaginary values ($\Delta z = \Delta x$ and $\Delta z = i\Delta y$, respectively. We get $f'(z) =$

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},$$

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Thus $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$. The real and imaginary parts of this equation should be equal:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are exactly the Cauchy-Riemann equations.

Theorem. If u and v have continuous partial derivatives and satisfy the Cauchy-Riemann equations in a domain D, then $f = u + iv$ is analytic in D.

Differentiating the Cauchy-Riemann equations with respect x and y, we get

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 u}{\partial y^2}$. Thus $\nabla^2 u = 0$.

Similarly, $\nabla^2 v = 0$, i.e., u and v are **harmonic**.

If u is harmonic and v is chosen so that $u+iv$ is analytic, then v is called a **harmonic conjugate** of u.

Example. Let $u = e^x \cos y + xy$. Then $\nabla^2 u = 0$.
To find v, note that $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = e^x \cos y + y$.
Thus $v = e^x \sin y + \frac{1}{2}y^2 + h(x)$.
Now $\frac{\partial v}{\partial x} = e^x \sin y + h'(x) = -\frac{\partial u}{\partial y} = e^x \sin y - x$.
Thus $h'(x) = -x$, $h(x) = -\frac{1}{2}x^2 + C$, and we get

$$v = e^x \sin y + \frac{1}{2}(y^2 - x^2) + C.$$

Example. Exponential function $e^z = e^x(\cos y + i \sin y)$ is analytic. (More about it next time.)

Cauchy-Riemann equations in polar coordinates.

$x = r\cos\theta,\ y = r\sin\theta.$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

$$\begin{aligned}\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} &= -\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta \\ &= r\left(\frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta\right) = r\frac{\partial u}{\partial r}.\end{aligned}$$

So $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$. Similarly, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

This implies that the Cauchy-Riemann equations are invariant under rotation and dilation.

Consequences of the Cauchy-Riemann equations.

1. If f is analytic and pure real (or pure imaginary) then $f \equiv$ const. More generally, if $\arg f \equiv$ const then $f \equiv$ const.

2. If f is analytic and $|f(z)| \equiv$ const then $f \equiv$ const.

Proof. If $f(z) \equiv 0$ there is nothing to prove. Suppose that $u^2 + v^2 \equiv$ const > 0. Then $2uu_x + 2vv_x = 0$ and $2uu_y + 2vv_y = 0$, thus $uu_x - vu_y = 0$ and $uu_y + vu_x = 0$. This implies $u^2u_x + v^2u_x = 0$, hence $u_x = 0$. Similarly, $u_y = 0$. Thus $u \equiv$ const, hence $f \equiv$ const.