Lesson 15. Exponential function

$$\exp(z) = e^x(\cos y + i \sin y),\ u = e^x \cos y,\ v = e^x \sin y,$$

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Thus $\exp(z)$ is analytic in $\mathbb{C}$. A function analytic in $\mathbb{C}$ is called **entire**, thus $\exp(z)$ is an entire function.

$$\exp(z)' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \exp(z). \quad \exp(z_1 + z_2) =$$

$$e^{x_1+x_2}(\cos(y_1 + y_2) + i \sin(y_1 + y_2)) = \exp(z_1)\exp(z_2).$$

$$\exp(iy) = \cos y + i \sin y.$$

In particular, $\exp(\frac{\pi}{2}i) = i$, $\exp(\pi i) = -1$, $\exp(2\pi i) = 1$.

We will write $e^z = \exp(z)$. This agrees with the calculus definition of e^x when z is real, and satisfies $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for any two complex numbers z_1 and z_2.

Notice also that $|e^z| = e^x$, $\dfrac{1}{e^z} = e^{-z}$, and $e^{z+2\pi i} = e^z$. Thus $e^z \neq 0$, and e^z is $2\pi i$-periodic.

We may write $z = re^{i\theta}$ as the polar form of a complex number $z = r(\cos\theta + i\sin\theta)$. Thus any number $z \neq 0$ equals e^w where $w = \ln r + i\theta$.

If $e^{z_1} = e^{z_2}$ then $e^{z_2 - z_1} = 1$, thus $z_2 = z_1 + 2k\pi i$ for $k = 0, \pm 1, \pm 2, \ldots$.
Equation $e^z = c$ has no solutions for $c = 0$ and infinitely many solutions for any $c \neq 0$.

The exponential function e^z is **real**, i.e., $e^{\bar{z}} = \overline{e^z}$.

Mapping properties of $w = e^z$.

Every vertical line $x = c$ maps to a circle $|w| = e^c$, with each point of the circle covered infinitely many times. In particular, the imaginary axis maps to the unit circle.

Every horizontal line $y = \theta$ maps to a ray

$$\arg w = \theta,\ 0 < |w| < \infty.$$

In particular, the real line maps to the positive real axis (0 excluded).

The left open half plane maps to the **punctured** open unit disk $0 < |w| < 1$, with each of its points covered infinitely many times. The right open half-plane maps to the complement of the closed unit disk.

The strip $0 < y < \pi$ maps to the open upper half plane.

The strip $-\pi < y < 0$ maps to the open lower half plane.

The strip $0 \leq y \leq 2\pi$ maps to the **punctured** complex plane $\mathbb{C} \setminus \{0\}$, with the positive real axis covered twice.

The strip $-\pi < y \leq \pi$ is the **fundamental region** for e^z: each value of e^z is attained once in that region.