# Lesson 19. Cauchy's theorem

A **contour** is a path in $\mathbb{C}$ (piecewise smooth, oriented).

A **closed** contour is a contour with its ending point equal its starting point.

A **simple** contour is a path that does not intersect itself.

A **simple closed** contour is a closed contour that does not intersect itself at any point other than its starting and ending points.

A **domain** is a connected open subset of $\mathbb{C}$.

A domain $D$ is **simply connected** (has no holes) if its complement $\mathbb{C} \setminus D$ is connected. Equivalently, $D$ is simply connected if, for any simple closed contour $C$ in $D$, the domain bounded by $C$ is contained in $D$.

**Cauchy's theorem.** If $f(z)$ is **analytic** in a **simply connected** domain $D$ then $\int f(z)\,\mathrm{d}z$ is **independent of path** in $D$, i.e., $\int_C f(z)\,\mathrm{d}z = 0$ for any closed contour $C$ contained in $D$.

To see this for a simple closed contour $C$ bounding a domain $D_0$ contained in $D$, recall that, if $f = u + iv$,

$$\int_C f(z)\,\mathrm{d}z = \int_C (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\int_C (u\,\mathrm{d}y + v\,\mathrm{d}x).$$

Applying Green's theorem (assuming that $C$ is oriented as the boundary of $D_0$) we get

$$\int_C u\,\mathrm{d}x - v\,\mathrm{d}y = -\iint_{D_0}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

$$\int_C u\,\mathrm{d}y + v\,\mathrm{d}x = \iint_{D_0}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Both integrals in the right-hand side are 0 by Cauchy-Riemann equations, thus $\int_C f(z)\,\mathrm{d}z = 0$.

If a closed contour contains finitely many self-intersection points, we can partition it into simple closed contours, thus Cauchy's theorem still holds.

**Example.** Although we cannot compute $\oint_C e^{z^4}\,\mathrm{d}z$ by calculus methods, we know that it is 0 for any closed contour in $\mathbb{C}$.

If $f$ has an antiderivative $F$ in $D$ then $\int f(z)\,\mathrm{d}z = F(z(b)) - F(z(a))$ is independent of path in $D$.
Conversely, Cauchy's theorem implies that any function $f(z)$ analytic in a simply connected domain $D$ has an antiderivative $F(z) = \int_{z_0}^{z} f(s)\,\mathrm{d}s$, which is analytic in $D$.
Here $\int_{z_0}^{z}$ means an integral over any path in $D$ from a fixed point $z_0 \in D$ to an arbitrary point $z \in D$.

**Example.** Let $f(z) = \dfrac{1}{z^2+1}$. Then $f(z)$ is analytic everywhere except $z = \pm i$. Thus $\int f(z)\,\mathrm{d}z$ is independent of path in any domain $D$ that does not contain a loop around one of these points. From calculus we know that $F(x) = \arctan x$ is antiderivative of $f(x)$ for real $x$. In the complex domain, $\arctan z = \dfrac{i}{2} \ln \dfrac{i+z}{i-z}$. For the real $z = x$, this becomes $\dfrac{i}{2} \ln \dfrac{i+x}{i-x}$. One can check that this is equal to $\arctan x$ (in particular, this is a real number) if the principal value $\operatorname{Ln} z$ of $\ln z$ is taken (note that $\dfrac{i+x}{i-x} = \dfrac{1-x^2-2ix}{x^2+1}$ does not cross the negative $x$-axis). Thus the antiderivative of $f(z)$ in $D$ is $F(z) + c$ where $F(z)$ is a single valued branch of the multi-valued function $\arctan z$ in $D$.

Suppose that $f(z)$ is analytic in a domain $D$ which is not simply connected. Let $C_1$ and $C_2$ be two non-intersecting simple closed contours in $D$, oriented counterclockwise, such that $C_2$ is inside $C_1$ and the region $D_0$ between $C_1$ and $C_2$ is contained in $D$. Then $C_1 - C_2$ is the boundary of $D_0$. Applying Green's theorem,

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z.$$

**Example.** Let $C$ be a closed contour going once counterclockwise around each of the two points $i$ and $-i$. Let $C_1$ and $C_2$ be small circles centered at $i$ and $-i$, respectively, and oriented counterclockwise. Then,

$$\int_C \frac{\mathrm{d}z}{z^2+1} = \int_C \frac{\mathrm{d}z}{(z+i)(z-i)} = \frac{i}{2}\int_C \frac{\mathrm{d}z}{z+i} - \frac{i}{2}\int_C \frac{\mathrm{d}z}{z-i}$$

$$= \frac{i}{2}\int_{C_2} \frac{\mathrm{d}z}{z+i} - \frac{i}{2}\int_{C_1} \frac{\mathrm{d}z}{z-i} = \pi - \pi = 0.$$

**Example.** Evaluate $\int_C \frac{(z+4)(z-5)}{z(z-1)(z+2)(z-3)}\,\mathrm{d}z$ where $C$ is the circle of radius 4 centered at 0.
**Hint.** Do not use partial fractions!

This function is analytic **outside** $C$. Thus the integral is the same as over a circle of an arbitrary large radius $R$ centered at 0. But for large $R$, the length of the circle grows as $R$, and the absolute value of the function decreases as $1/R^2$. Thus the integral is bounded by a constant divided by $R$. Hence the answer is 0.

**Theorem.** If $P(z)$ and $Q(z)$ are two polynomials such that all zeros of $Q$ are inside the disk $D$ of radius $r$, and $\deg Q \geq \deg P + 2$ then $\int_C \frac{P(z)}{Q(z)}\,\mathrm{d}z = 0$ for any closed contour $C$ outside $D$.