Lesson 2. Divergence and curl

Let $\mathbf{v} = (v_1, v_2, v_3)$ be a differentiable vector field. Then $\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$.

Example. For $\mathbf{v} = (e^{x \sin y}, xy^2, z^2)$,

$$\operatorname{div} \mathbf{v} = (\sin y) e^{x \sin y} + 2xy + 2z.$$

The **Laplacian** $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ can be written as

$$\Delta f = \nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \nabla \cdot \nabla f.$$

Fluid flow. Let $\mathbf{v} = (v_1, v_2, v_3)$ be the velocity vector of a fluid in a region R where there are no sources and sinks. Let ρ be the mass density of the fluid, so that the mass flow is $\mathbf{u} = \rho\mathbf{v}$.

Let $W = [x, x+\delta x] \times [y, y+\delta y] \times [z, z+\delta z]$ be a small box with the sides δx, δy, δz and the volume $\delta V = \delta x\, \delta y\, \delta z$. Let us calculate the flux (mass per unit time) leaving W through the walls. Through the x-walls, it is approximately

$$u_1(x+\delta x, y, z)\,\delta y\,\delta z - u_1(x, y, z)\,\delta y\,\delta z \approx \frac{\partial u_1}{\partial x}\,\delta V.$$

Similarly, the flux through the y-walls is approximately $\dfrac{\partial u_2}{\partial y}\,\delta V$, and the flux through the z-walls is approximately $\dfrac{\partial u_3}{\partial z}\,\delta V$.

Thus the total mass of fluid leaving W through the walls is approximately $\operatorname{div}\mathbf{u}\,\delta V$.

On the other hand, the loss of mass per unit time is approximately $-\frac{\partial\rho}{\partial t}\,\delta V$. Thus $\operatorname{div}\mathbf{u}\,\delta V \approx -\frac{\partial\rho}{\partial t}\,\delta V$. In the limit $\delta V \to 0$, we get the continuity equation

$$\operatorname{div}\mathbf{u} = \operatorname{div}(\rho\mathbf{v}) = -\frac{\partial\rho}{\partial t}.$$

For the steady flow this becomes $\operatorname{div}(\rho\mathbf{v}) = 0$.
If the fluid is **incompressible**, $\rho = \text{const}$, thus $\operatorname{div}\mathbf{v} = 0$.

Curl. For a vector field $\mathbf{v} = (v_1, v_2, v_3)$, define

$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z},\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x},\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right).$$

Example. For $\mathbf{v} = (e^{x \sin y}, xy^2, z^2)$,

$$\operatorname{curl}\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x \sin y} & xy^2 & z^2 \end{vmatrix} = (0, 0, y^2 - x \cos y\, e^{x \sin y}).$$

Curl and rotation. The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude is twice the angular speed. A vector field $\mathbf{v}$ is **irrotational** if $\text{curl}\,\mathbf{v} = 0$.

Relations between grad, curl and div. For any function f and any vector field $\mathbf{v}$,

$$\text{curl}\,(\text{grad}\,f) = 0,$$

$$\text{div}\,(\text{curl}\,\mathbf{v}) = 0,$$

$$\text{div}\,(\text{grad}\,f) = \nabla^2 f \quad \text{is the Laplacian of } f.$$

Conversely, if $\text{curl}\,\mathbf{v} = 0$ then $\mathbf{v} = \text{grad}\,f$ for some function f, thus irrotational vector fields are **conservative**. If $\text{div}\,\mathbf{v} = 0$ then $\mathbf{v} = \text{curl}\,\mathbf{u}$ for some vector field $\mathbf{u}$.

Two-dimensional flow may be interpreted as a three-dimensional flow with $\mathbf{v} = \langle v_1, v_2, 0 \rangle$ where $v_1 = v_1(x, y)$ and $v_2 = v_2(x, y)$ do not depend on z. Then

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \quad \text{and} \quad \operatorname{curl} \mathbf{v} = \left\langle 0, 0, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right\rangle.$$

In particular, $\operatorname{curl} \mathbf{v}$ for a two-dimensional vector field $\mathbf{v}$ has only one non-zero component, and is a scalar field.

Example: Gravity field. If a mass M is placed at the origin and a mass m at $\mathbf{r} = (x, y, z)$ then the gravity force acting on the second mass is $\mathbf{v} = -mMG\dfrac{\mathbf{r}}{r^3}$ where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and G is the gravitational constant. Since $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\mathbf{r}}{r^3}$, we have $\operatorname{curl}\mathbf{v} = \mathbf{0}$. Also, $\operatorname{div}\mathbf{v} = mMG\nabla^2\left(\dfrac{1}{r}\right) =$

$$= -mMG\left(\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right) =$$

$$mMG\left(\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) + \left(\frac{3y^2}{r^5} - \frac{1}{r^3}\right) + \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\right) = 0,$$

the potential $\dfrac{1}{r}$ satisfies **Laplace equation** $\nabla^2\left(\dfrac{1}{r}\right) = 0$.