Lesson 23. Power Series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

where z_0 is **center** and a_j are **coefficients**.

Example. Geometric series $\sum_{n=0}^{\infty} z^n$ converges absolutely for $|z| < 1$, diverges for $|z| \geq 1$.

Example. The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely for all z (by ratio test).

Example. The series $\sum_{n=0}^{\infty} n! z^n$ diverges for all z except $z = 0$ (by ratio test).

These three examples illustrate the three cases that can occur for a power series.

Theorem. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $z = z_1 \neq z_0$ then it converges absolutely for all z in the disk $|z-z_0| < |z_1 - z_0|$. If it diverges for $z = z_2$ then it diverges for all z such that $|z - z_0| > |z_2 - z_0|$. Thus a power series converges either only at z_0, or for all z, or it converges in some disk $|z - z_0| < R$ and diverges for $|z - z_0| > R$.

Remark. This theorem does not say anything about convergence of a power series on the circle $|z - z_0| = R$. The series may converge (absolutely or not) at some points of that circle, and diverge at some other points.

Proof of the Theorem. If the series $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges then $a_n(z_1 - z_0)^n \to 0$ as $n \to \infty$. Thus $a_n(z_1 - z_0)^n$ is bounded. Let $|a_n(z_1 - z_0)^n| < M$, for some $M > 0$. Then, if $|z - z_0| < |z_1 - z_0|$, we have

$$|a_n(z - z_0)^n| = |a_n(z_1 - z_0)^n| \left|\frac{z - z_0}{z_1 - z_0}\right|^n \le M \left|\frac{z - z_0}{z_1 - z_0}\right|^n.$$

Let $\rho = \left|\frac{z - z_0}{z_1 - z_0}\right| < 1$. Then $|a_n(z - z_0)^n| \le M\rho^n$, thus the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely by the comparison test, since the geometric series $M\sum_{n=0}^{\infty} \rho^n$ converges.

If, on the other hand, $\sum_{n=0}^{\infty} a_n(z_2 - z_0)^n$ diverges, then $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ cannot converge for any z such that $|z - z_0| > |z_2 - z_0|$ by the first part of the Theorem which has been proven already.

The **radius of convergence** R is defined as follows:
$R = 0$ if $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges only for $z = z_0$,
$R = \infty$ if $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for all z,
Otherwise, we take R to be that value for which the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges if $|z - z_0| < R$ and diverges if $|z - z_0| > R$.

Cauchy-Hadamard formula. If the limit $L = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists then L is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. This follows from ratio test.

Example. For the series $\frac{2^n}{n} z^n$, $L = \lim_{n\to\infty} \left|\frac{\frac{2^n}{n}}{\frac{2^{n+1}}{n+1}}\right| = \frac{1}{2}$.

Using ratio test, $\lim_{n\to\infty} \left|\frac{\frac{2^{n+1}z^{n+1}}{n+1}}{\frac{2^n z^n}{n}}\right| = 2|z| < 1 \Rightarrow |z| < \frac{1}{2}$.

Warning. Using Cauchy-Hadamard you may have trouble with a series like $\sum_{n=0}^{\infty} 2^n z^{2n}$ where limit does not exist, while ratio test easily gives you $2|z^2| < 1 \Rightarrow |z| < \frac{1}{\sqrt{2}}$.

Example. Applying ratio test to $\sum_{n=1}^{\infty} \frac{n!z^n}{n^n}$,

$$\left|\frac{\frac{(n+1)!z^{n+1}}{(n+1)^{n+1}}}{\frac{n!z^n}{n^n}}\right| = \frac{(n+1)n^n}{(n+1)^{n+1}}|z| = \left(\frac{n}{n+1}\right)^n |z| = \frac{|z|}{\left(1+\frac{1}{n}\right)^n}$$

converges to $\frac{|z|}{e}$ as $n \to \infty$. Thus $R = e$.

Example. Applying ratio test to $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}(z-2)^n$,

$$\left|\frac{\frac{\ln(n+1)}{(n+1)^2}(z-2)^{n+1}}{\frac{\ln n}{n^2}(z-2)^n}\right| = \frac{\ln(n+1)}{\ln n}\left(\frac{n}{n+1}\right)^2 |z-2|$$

converges to $|z-2|$. Thus $R = 1$.

Example. Applying ratio test to $\sum_{n=0}^{\infty} \frac{z^{n!}}{n!}$ (note that Cauchy-Hadamard does not work!)

$$\left|\frac{\frac{z^{(n+1)!}}{(n+1)!}}{\frac{z^{n!}}{n!}}\right| = \left|\frac{(z^{n!})^{n+1}}{z^{n!}}\right| \frac{1}{n+1} = \frac{\left|(z^{n!})^{n}\right|}{n+1}$$

converges to 0 if $|z| \leq 1$, to ∞ if $|z| > 1$. Thus $R = 1$.