Lesson 27. Zeros and singularities of analytic functions

Let $f(z)$ be analytic in a **punctured disk** $0 < |z - z_0| < R$, with a singular point at $z = z_0$. Then we say that $f(z)$ has an **isolated singularity** at z_0.

Example. e^z/z and $e^{1/z}$ have isolated singularities at 0, while $\dfrac{1}{\sin \frac{1}{z}}$ has a non-isolated singularity at 0.

Let $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ be the Laurent series converging to $f(z)$ when $0 < |z - z_0| < R$.

Its **principal part** is $\displaystyle\sum_{n=-\infty}^{-1} a_n(z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}$.

We distinguish 3 possibilities:

1. The principal part of the Laurent series of f has a finite number k of terms, i.e.,

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \ldots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$$

Then f has a **pole of order** k at z_0. A pole of order $k = 1$ is called a **simple pole** (e.g., $\frac{e^z}{z-z_0}$).

2. The principal part has infinitely many terms (e.g., $e^{1/(z-z_0)}$). Then f has an **essential singularity** at z_0. By definition, a non-isolated singularity is essential.

3. There are no non-zero terms in the principal part (e.g., $\frac{\sin z}{z}$). Then f has a **removable singularity** at z_0, and can be extended as an analytic function in the disk $|z - z_0| < R$ by setting $f(z_0) = a_0$.

At a pole z_0, we have $|f(z)| \to \infty$ as $z \to z_0$.

At an essential singularity the function has a chaotic behavior. The great Picard Theorem says that in every neighborhood of an essential singularity f takes on all values, with at most one exception (e.g., $e^{1/z}$ takes all values except 0 in any neighborhood of 0).

We can also speak of **singularity of f at infinity**.
By definition, this is the singularity of $f(1/z)$ at $z = 0$.
Adding a point ∞ to the complex plane, we get the **Riemann sphere**.

Example. A polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ of degree n has a pole of order n at infinity, since $p(\frac{1}{z}) = \frac{1}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \ldots + a_0$ has a pole of order n at 0.

Example. $f(z) = e^z$ has an essential singularity at ∞. $f(z) = e^{1/z}$ has a removable singularity at ∞ since $g(z) = f(1/z)$ can be extended to e^z by $g(0) = 1$.

If f has an essential singularity at z_0, and $g \not\equiv 0$ is either analytic at z_0 or has a pole of order k, then fg and $f+g$ have essential singularities at z_0.

If $f \not\equiv 0$ is analytic at z_0 and $f(z_0) = 0$ then
$f(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \ldots =$
$(z - z_0)^k(a_k + a_{k+1}z + \ldots) = (z - z_0)^k H(z)$
where H is analytic at z_0 and $a_k = H(z_0) \neq 0$.
Then f has a **zero of order** k at z_0.
A zero of order $k = 1$ is a **simple zero**.

Note that f has a zero of order k at z_0 if, and only if, $1/f$ has a pole of order k at z_0.

If $f(z)$ has a zero of order k at z_0, and $g(z)$ has a zero of order l, then $f(z)g(z)$ has a zero of order $k+l$ at z_0. If $l > k$ then $f(z) + g(z)$ has a zero of order k at z_0.

Warning. If $l = k$ then $f(z) + g(z)$ may have a zero of any order $n \geq k$ at z_0.

If $f(z)$ has a pole of order k at z_0, and $g(z)$ has a pole of order l, then $f(z)g(z)$ has a pole of order $k+l$ at z_0. If $l < k$ then $f(z) + g(z)$ has a pole of order k at z_0.

Warning. If $l = k$ then $f(z) + g(z)$ may be analytic at z_0 or have a pole of any order $n \leq k$.

Isolation of zero principle. If f is analytic in a domain D, then the zeros of f are isolated in D.

To see this, let z_0 be a zero of f. Then $H(z)$ above is analytic at z_0, $|H(z)|$ is continuous, and $|H(z_0)| > 0$. Thus $|H(z)| > 0$ in some neighborhood of z_0.

This implies that two analytic functions in D that are equal at some sequence $\{z_k\}$ converging to a point $z_0 \in D$ must be equal everywhere in D.

This principle has far reaching applications, e.g.,
High school trig formulas that hold for real numbers remain valid for complex numbers.

Example. $\sin\frac{1}{z}$ has essential (non-isolated) singularity at 0 and removable singularity (a simple zero) at ∞.

Example. $z^2\sin\frac{1}{z}$ has essential singularity at 0 and a simple pole at ∞.

Example. $\dfrac{1}{z} - \dfrac{1}{\sin z} = \dfrac{\sin z - z}{z\sin z} = \dfrac{-z^3/3! + z^5/5! - \dots}{z^2 - z^4/3! + \dots}$
has removable singularity (a simple zero) at 0, and simple poles at $z = n\pi$, $n = \pm 1, \pm 2, \dots$

Example. $f(z) = \dfrac{z^2+4}{(z^2-9)(z^2+16)}$ has simple poles at $3, -3, 4i, -4i$ and removable singularity (a zero of order 2) at ∞:

$$f\left(\frac{1}{z}\right) = \frac{\frac{1}{z^2}+4}{\left(\frac{1}{z^2}-9\right)\left(\frac{1}{z^2}+16\right)} = \frac{z^2+4z^4}{(1-9z^2)(1+16z^2)}.$$