Lesson 33. Linear fractional transformations (Möbius transformations)

Let a, b, c, d be complex numbers, $ad - bc \neq 0$. Define $w = T(z) = \dfrac{az+b}{cz+d}$. If $c \neq 0$ then $T(z) = \dfrac{a}{c} - \dfrac{1}{c}\,\dfrac{ad-bc}{cz+d}$, thus $T'(z) = \dfrac{ad-bc}{(cz+d)^2} \neq 0$, so $T(z)$ is a conformal mapping when $z \neq -\frac{d}{c}$. If $c = 0$ then $ad \neq 0$ and $T(z) = (az+b)/d \quad \Rightarrow \quad T'(z) = a/d \neq 0$.

Note: Multiplication of a, b, c, d by a non-zero complex number does not change $T(z)$.

It is natural to think of $T(z)$ as a transformation of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto itself, setting $T(-\frac{d}{c}) = \infty$ and $T(\infty) = \frac{a}{c}$ if $c \neq 0$, or $T(\infty) = \infty$ if $c = 0$. Then $T(z) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a conformal mapping.

Composition of two linear fractional transformations is again a linear fractional transformation:

$$\frac{\alpha\,\frac{az+b}{cz+d}+\beta}{\gamma\,\frac{az+b}{cz+d}+\delta}=\frac{(\alpha a+\beta c)z+(\alpha b+\beta d)}{(\gamma a+\delta c)z+(\gamma b+\delta d)}.$$

Note that

$$\begin{pmatrix}\alpha a+\beta c & \alpha b+\beta d\\ \gamma a+\delta c & \gamma b+\delta d\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

Inverse of a linear fractional transformation is again a linear fractional transformation: $w=T(z)=\dfrac{az+b}{cz+d}\Rightarrow$ $w(cz+d)=az+b\Rightarrow z(cw-a)=b-dw\Rightarrow z=\dfrac{-dw+b}{cw-a}$.

Thus $T^{-1}(w)=\dfrac{-dw+b}{cw-a}$.

The identity mapping is $T(z)=z$.

There are four building blocks for the linear fractional transformations.

1. Translation $T(z) = z + b$.
2. Rotation $T(z) = e^{i\alpha} z$ (α real)
3. Dilation $T(z) = cz$ ($c \neq 0$ real)
4. Inversion $T(z) = \frac{1}{z}$.

Every linear fractional transformation is a composition of these mappings. For example, when $c \neq 0$,

$$T(z) = \frac{az + b}{cz + d} = T_5(T_4(T_3(T_2(T_1(z))))),$$

where $\quad T_1(z) = cz, \quad T_2(z) = z + d, \quad T_3(z) = \dfrac{1}{z},$

$$T_4(z) = -\frac{ad - bc}{c} z, \quad T_5(z) = z + \frac{a}{c}$$

Note that translations, rotations and dilations take lines to lines and circles to circles.

We want to show that inversion takes lines and circles either to lines or to circles.

Inversion takes a line through the origin to a line through the origin, and a line not passing through the origin to a circle passing through the origin. If $\operatorname{Re} z = 1$ then $\dfrac{1}{z} = \dfrac{1 - iy}{y^2 + 1}$, thus $\dfrac{1}{z} - \dfrac{1}{2} = \dfrac{1 - 2iy - y^2}{2(y^2 + 1)} = \dfrac{(1 - iy)^2}{2(y^2 + 1)}$ and $\left|\frac{1}{z} - \frac{1}{2}\right| = \frac{1}{2}$, so $\frac{1}{z}$ is on the circle of radius $\frac{1}{2}$, center $\frac{1}{2}$.

The result for any other line follows from combining inversion with rotation and dilation, since $\dfrac{1}{az} = \dfrac{1}{a}\dfrac{1}{z}$.

Let now z belong to the circle $|z-1| = r$ (any circle with center $\neq 0$ may be mapped to this one by rotation and dilation) and $w = \frac{1}{z}$. Then $x^2 + y^2 - 2x = r^2 - 1$, thus $2\mathrm{Re}\, w + (r^2-1)w\bar{w} = 1$. If $r^2 = 1$ then $\mathrm{Re}\, w = \frac{1}{2}$, thus w belongs to a line. Otherwise, as $2\mathrm{Re}\, w = w + \bar{w}$, we get $\left|w + \dfrac{1}{r^2-1}\right|^2 = \dfrac{1}{r^2-1} + \dfrac{1}{(r^2-1)^2} = \dfrac{r^2}{(r^2-1)^2}$, which is a circle centered at $-\frac{1}{r^2-1}$, radius $r/|(r^2-1)|$.

The case of a circle centered at 0 is easy.

Theorem. Any linear fractional transformation takes circles and lines to either circles or lines.

Note that a line in $\mathbb{C}$ becomes a circle passing through ∞ on the Riemann sphere $\overline{\mathbb{C}}$.

Fixed points. z is a fixed point of $T(z) = \dfrac{az+b}{cz+d}$ if $T(z) = z$. Then, $az + b = cz^2 + dz$, thus fixed points are solutions of a quadratic equation $cz^2 - (a-d)z - b = 0$:

$$z = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c} \text{ if } c \neq 0,$$

$$z = \frac{b}{d-a}, \infty \text{ if } c = 0,\ d \neq a,$$

$$z = \infty \text{ if } c = 0,\ d = a,\ b \neq 0,$$

all z are fixed ($T(z) = z$ is identity) if $c = b = 0$, $d = a$.

Theorem A linear fractional transformation other than identity has at most two fixed points.

Important observation. A linear fractional transformation is uniquely determined by the images of any three distinct points. That is, for any given distinct points z_1, z_2, z_3 and distinct points w_1, w_2, w_3 there is a unique linear fractional transformation $w = T(z)$ such that $w_1 = T(z_1)$, $w_2 = T(z_2)$, $w_3 = T(z_3)$.

Example. $T(z) = \dfrac{z_2 - z_3}{z_2 - z_1}\,\dfrac{z - z_1}{z - z_3} = (z, z_1, z_2, z_3)$ (**cross ratio**) maps $z_1 \to 0$, $z_2 \to 1$, $z_3 \to \infty$.

In general, $T(z)$ preserves the cross ratio:

$$\frac{w_2 - w_3}{w_2 - w_1}\,\frac{w - w_1}{w - w_3} = \frac{z_2 - z_3}{z_2 - z_1}\,\frac{z - z_1}{z - z_3}$$

A linear fractional transformation $T(z)$ preserving ∞ is linear: $T(z) = az + b$, $a \neq 0$.

A linear fractional transformation $T(z) = \dfrac{az+b}{cz+d}$ with real a, b, c, d and $ad - bc > 0$ maps the upper half plane $\mathbf{H}$ onto itself: it maps the real line onto itself, and
$T(i) = \dfrac{(b+ai)(d-ci)}{c^2+d^2} = \dfrac{(ac+bd)+i(ad-bc)}{c^2+d^2} \in \mathbf{H}$.

Let z_0 be a point inside the unit disk $\mathbf{U}$. Then
$T(z) = e^{i\alpha}\dfrac{z-z_0}{1-\bar{z}_0 z}$ maps $\mathbf{U}$ onto itself, and $T(z_0) = 0$.

To see this, let $|z| = 1$. Then
$|z - z_0| = |\bar{z} - \bar{z}_0| = |z|\,|\bar{z} - \bar{z}_0| = |z\bar{z} - \bar{z}_0 z| = |1 - \bar{z}_0 z|$,
thus $T(z)$ maps the unit circle onto itself. Since $z_0 \in \mathbf{U}$ and $T(z_0) = 0$, it maps $\mathbf{U}$ onto itself.