Lesson 37. Using conformal mappings: Fluid flow

We associate a complex function $v(z) = v_1(z) + iv_2(z)$, where $z = x + iy$, with a two-dimensional velocity field $\vec{v}(x,y) = (v_1(x,y), v_2(x,y))$.

If the flow is incompressible ($\operatorname{div}\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0$) and irrotational ($\operatorname{curl}\vec{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$) then $\overline{v(z)}$ is analytic.

A **streamline** is a path such that $\vec{v}$ is its tangent at each point.

An analytic function $F(z) = \Phi(z) + i\Psi(z)$ is the **complex potential** for the flow if $\overline{v(z)} = F'(z)$.

The function Φ is the **velocity potential**, its level curves are **equipotential lines**. The function Ψ is the **stream function**, its level curves are the **streamlines**.

We have $v(z) = \overline{F'(z)} = \dfrac{\partial \Phi}{\partial x} - i\dfrac{\partial \Psi}{\partial x} = \dfrac{\partial \Phi}{\partial x} + i\dfrac{\partial \Phi}{\partial y}$
so $v_1 = \frac{\partial \Phi}{\partial x}$, $v_2 = \frac{\partial \Phi}{\partial y}$, and $\vec{v} = \nabla \Phi$.
In particular, Φ, Ψ, v_1, v_2 are harmonic.

Let $(x(t), y(t))$ be a streamline. Then,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Psi(x(t), y(t)) = \nabla \Psi \cdot (x', y').$$

But $\nabla \Psi$ is normal to the curve $\Psi = \text{const}$, thus (x', y') is parallel to $\vec{v} = \nabla \Phi$. In particular, **boundaries of flows must be streamlines**.

Example. Flow through an opening. Domain D is the complex plane without infinite rays $(-\infty, -1]$ and $[1, \infty)$. We want to verify that $F(z) = \cosh^{-1} z$ gives a complex potential for this flow.

To see this write $w = \cosh^{-1} z$ or $z = \cosh w$ and consider the level curves $\Psi = \operatorname{Im} w = \text{const}$. This means $z = \cosh(u + ic)$ where c is a constant.

$$z = \frac{1}{2}(e^{u+ic} + e^{-u-ic}) =$$

$$\frac{1}{2}\Big(e^{u}(\cos c + i \sin c) + e^{-u}(\cos c - i \sin c)\Big) =$$

$$\frac{1}{2}\Big((e^{u} + e^{-u}) \cos c + i(e^{u} - e^{-u}) \sin c\Big).$$

Thus $x = \frac{1}{2}(e^u + e^{-u})\cos c$ and $y = \frac{1}{2}(e^u - e^{-u})\sin c$,

$$\frac{x^2}{\cos^2 c} - \frac{y^2}{\sin^2 c} = 1.$$

These are hyperbolas with foci 1 and -1 (the foci for a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\pm\sqrt{a^2 + b^2}$).

Note that $z = \cosh w$ maps the domain $D' = \{0 < \operatorname{Im} w < \pi\}$ conformally one-to-one onto our domain D. Our flow is transported from the flow in D' with the complex potential w and streamlines $\operatorname{Im} w = \text{const}$.

Example. Flow around a corner. Complex potential is $F(z) = z^2 = (x^2 - y^2) + 2ixy$, equipotential lines are $\Phi = x^2 - y^2 = \text{const}$, streamlines are $\Psi = 2xy = \text{const}$, $\overline{F'(z)} = 2\bar{z} = 2x - 2iy$, $\vec{v}(z) = (2x, -2y)$.

Example. Flow around a cylinder. Complex potential is $F(z) = z + \frac{1}{z} = re^{i\theta} + \frac{1}{r}e^{-i\theta} = (1 + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$. Streamlines are $\Psi = (r - \frac{1}{r})\sin\theta = \text{const.}$ $\overline{F'(z)} = 1 - \frac{1}{\bar{z}^2}$, $\vec{v}(z) = (1 - \frac{1}{r^2}\cos 2\theta, -\frac{1}{r^2}\sin 2\theta)$.

Basic Theorem. If the domain of a flow is simply connected and the flow is irrotational and incompressible, then there exists an analytic complex potential $F(z)$ for the flow.

This follows from the fact that an analytic function $\overline{v(z)}$, where $v(z) = v_1(z) + iv_2(z)$, has an analytic antiderivative $F(z)$ in D.