

FROM 12 TO 6: SHARPENING THE THREE-CHARGE BOUND OF GABRIELOV–NOVIKOV–SHAPIRO

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ABSTRACT. In [4] we proved that, for every $\alpha > 0$, the potential of three point charges has at most 12 nondegenerate equilibrium points. We also observed that the same method would give the sharper bound 6 if a certain auxiliary polynomial system $Q = R = 0$ had at least four solutions, counted with multiplicity, in each open quadrant of the (f, g) -plane. Here we prove this four-solution statement. The main new ingredient is a separation argument at the unique saddle point of a separated-variable first integral. Consequently, the upper bound for three charges improves from 12 to 6.

The article was created with the help of Claude Fable 5 (Anthropic).

1. INTRODUCTION

In § 113 of his treatise [6], J. C. Maxwell discussed the maximal number of isolated equilibrium points generated by point charges in $\mathbb{R}^3$ and asserted the bound $(k - 1)^2$ for k charges. The argument supporting this assertion is incomplete; see the appendix to [4].¹ We refer to the problem of bounding the number of equilibrium points of a finite configuration of point charges as *Maxwell’s problem*.

A general finiteness bound for a natural generalization of Maxwell’s problem was established in Theorem 1.5(a) of [4] and subsequently improved, for example, in [9]. Very recently Arathoon, Ball, and Kvalheim [1], building on ideas from [3], subsequently constructed five positive charges in $\mathbb{R}^3$ whose Coulomb potential has at least 24 nondegenerate critical points. Thus Maxwell’s proposed bound is false in general.

For three positive charges, Theorem 1.5(b) of [4] gives the upper bound 12, whereas Maxwell’s proposed bound is 4. Our goal is to improve 12 to 6. For three equal charges, the bound 4 is known [7].

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¹For historical discussion of contemporary criticism of inconsistencies and incomplete arguments in Maxwell’s treatise, including comments by J. J. Thomson, J. C. McConnel, H. Poincaré, and O. Heaviside, see [8, Chapter 6].

Our set-up is as follows. We may assume that the three charges are noncollinear since the collinear situation is trivial. Consider positive charges $\zeta_1, \zeta_2, \zeta_3$ situated at non-collinear points $p_1, p_2, p_3 \in \mathbb{R}^n$, respectively. Their potential is

$$V_\alpha(x) = \sum_{i=1}^3 \zeta_i \rho_i^{-\alpha}, \quad \rho_i = r_i^2 = |x - p_i|^2, \quad \alpha > 0.$$

Every critical point of V_α lies in the plane containing the charges, see Lemma 3.1. As recalled in [4], Morse-theoretic considerations give the parity relation

$$\#\{\text{equilibria}\} = 2m_0 + 2, \quad (1.1)$$

where m_0 is the number of planar local minima. In particular, the number of nondegenerate equilibria is even. The main result of this note is as follows.

Theorem 1.1. *For every $n \geq 2$ and every $\alpha > 0$, a potential generated by three point charges in $\mathbb{R}^n$ has at most 6 nondegenerate equilibrium points.*

The proof retains the two Rolle–Khovanskii reductions and the mixed-volume count from [4]. The only new step is a four-contact lemma for a compact component of the auxiliary contact curve. We first recall the reduction and isolate the statement that must be proved.

2. ACKNOWLEDGMENT

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3. THE GABRIELOV–NOVIKOV–SHAPIRO COUNT

Lemma 3.1. *V_α has no critical points outside the plane containing the charges.*

Indeed, for a point x outside of this plane, the vectors $x - p_i$, proportional to $\nabla \rho_i^{-\alpha}(x)$, are linearly independent. Thus x cannot be a critical point of V_α . $\square$

Therefore, we can and will assume $n = 2$. By rotation and rescaling, we can further assume that $p_1 = (0, 0)$, $p_2 = (1, 0)$ and $p_3 = (a, b)$, with $b \neq 0$, and that $\zeta_3 = 1$.

Following [4], introduce

$$f = \left(\frac{\rho_1}{\rho_2}\right)^{\alpha+1}, \quad g = \left(\frac{\rho_1}{\rho_3}\right)^{\alpha+1}.$$

Critical points of V_α correspond bijectively to intersections, in the open positive quadrant $\mathbb{R}_{>0}^2$, of two curves γ_1 and γ_2 :

$$\begin{aligned}\gamma_1 &= \left\{ f^{1/(\alpha+1)} \frac{\xi_2}{\xi_1} = 1 \right\}, \\ \gamma_2 &= \left\{ g^{-1/(\alpha+1)} \xi_2 = f^{-1/(\alpha+1)} \xi_3 \right\},\end{aligned}$$

where

$$\begin{aligned}\xi_1(f, g) &= (ag + \zeta_2 f)^2 + b^2 g^2, \\ \xi_2(g) &= ((a-1)g - \zeta_1)^2 + b^2 g^2, \\ \xi_3(f) &= ((a-1)\zeta_2 f + a\zeta_1)^2 + b^2 (\zeta_2 f + \zeta_1)^2.\end{aligned}$$

The form ξ_1 is positive definite, while ξ_2 and ξ_3 are strictly positive quadratic polynomials in one variable.

The curves γ_1 and γ_2 are integral curves of closed logarithmic one-forms

$$\eta_1 = d \log \Xi, \quad \eta_2 = d \log \Psi, \quad (3.1)$$

where Ξ and Ψ are their respective defining functions. Both γ_1 and γ_2 are separating solutions in the sense of Khovanskii, so the Rolle–Khovanskii theorem applies twice; see [4, 5].

Let

$$N = \#(\gamma_1 \cap \gamma_2 \cap \mathbb{R}_{>0}^2). \quad (3.2)$$

All intersections considered below are isolated, see Remark 3.2 below, and are counted with their local intersection multiplicities.

The first Rolle–Khovanskii step gives

$$N \leq N_1 + N_2, \quad (3.3)$$

where $N_1 = 2$ is the number of unbounded components of γ_2 in $\mathbb{R}_{>0}^2$ and

$$N_2 = \#(\gamma_2 \cap \Gamma \cap \mathbb{R}_{>0}^2). \quad (3.4)$$

Here $\Gamma = \{Q = 0\}$ is the contact curve determined by

$$Q \, df \wedge dg = fg \, \xi_1 \xi_2 \xi_3 \, \eta_1 \wedge \eta_2. \quad (3.5)$$

Its Newton polygon is

$$\text{NP}(Q) = \{(p, q) : 2 \leq p + q \leq 6, \, 0 \leq p, q \leq 4\}. \quad (3.6)$$

The second Rolle–Khovanskii step gives

$$N_2 \leq N_3 + N_4, \quad (3.7)$$

where $N_3 = 0$ and

$$N_4 = \#(\Gamma \cap \{R = 0\} \cap \mathbb{R}_{>0}^2). \quad (3.8)$$

The equality $N_3 = 0$ follows because Γ has no unbounded components and meets the coordinate axes only at the origin, which is an isolated point of Γ . The polynomial R is defined by

$$R \, df \wedge dg = fg \, \xi_2 \xi_3 \, dQ \wedge \eta_2. \quad (3.9)$$

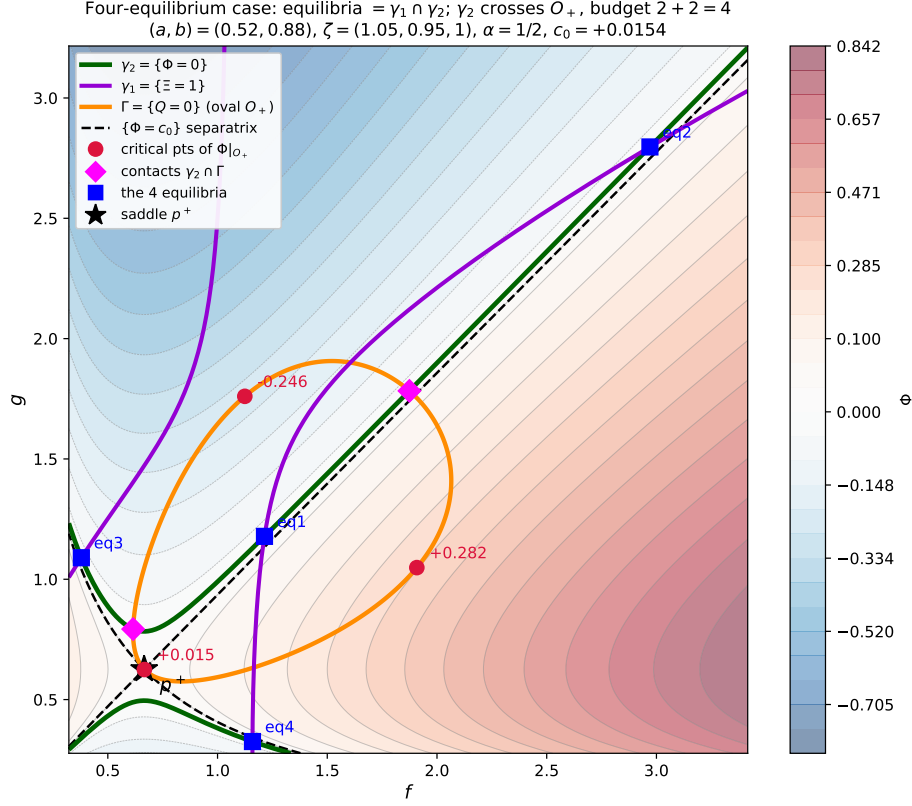


FIGURE 1. Khovanskii reduction in a generic case of four critical points, by Claude (Anthropic). A positive-quadrant configuration with four equilibria.

The orange curve is the oval $O \subset \Gamma$; the green and purple curves are γ_2 and γ_1 , respectively; and the dashed curve is the separatrix level $\Phi = c_0$. Red points mark critical points of $\Phi|_O$, magenta diamonds mark intersections $\gamma_2 \cap \Gamma$, blue squares mark the four equilibria $\gamma_1 \cap \gamma_2$, and the star marks the saddle p . Note that p is too close to another critical point to distinguish them in the picture.

Remark 3.2. One can immediately see that the critical points of the potential have to be isolated. Indeed, otherwise γ_1 and γ_2 would have a common component, which would then necessarily be a component of Γ . But the components of γ_2 are unbounded, and Γ has no unbounded components, see [4], a contradiction.

After replacing R by a reduction $\tilde{R} = R - qQ$ for some suitable polynomial $q = q(f, g)$, the Bernstein–Kushnirenko theorem gives

$$\#\{Q = \tilde{R} = 0 \text{ in } (\mathbb{C}^*)^2\} \leq 2 \operatorname{Vol}(\operatorname{NP}(Q), \operatorname{NP}(\tilde{R})) = 28, \quad (3.10)$$

where solutions are counted with intersection multiplicity; compare [2, 4]. Two nonreal conjugate common zeros of ξ_1, ξ_2, ξ_3 contribute multiplicity at least 6 each. The proof in [4] also finds at least two real solutions of $Q = R = 0$ in each of the three open quadrants different from $\mathbb{R}_{>0}^2$. Hence

$$N_4 \leq 28 - 12 - 6 = 10, \quad N \leq 2 + 0 + 10 = 12. \quad (3.11)$$

Proposition 3.3 below shows that each open quadrant contains at least four real solutions, as conjectured in [4], implying Theorem 1.1.

Proposition 3.3 (Four-solution proposition). *In each open quadrant*

$$\mathcal{Q}_\sigma = I \times J \subset \mathbb{R}^2, \quad I, J \in \{(0, \infty), (-\infty, 0)\},$$

the system $Q = R = 0$ has at least four solutions counted with multiplicity.

Deduction of Theorem 1.1 from Proposition 3.3. The two complex points contribute total multiplicity at least 12 by [4]. The three open quadrants outside $\mathbb{R}_{>0}^2$ contribute at least another 12. These solutions lie in $(\mathbb{C}^*)^2$ and are pairwise distinct. Therefore (3.10) gives

$$N_4 \leq 28 - 12 - 12 = 4, \quad (3.12)$$

Together with (3.3) and (3.7), this yields

$$N \leq 2 + 0 + 4 = 6. \quad (3.13)$$

□

The remainder of the paper proves Proposition 3.3.

4. THE SEPARATED SADDLE OF η_2

From now on, we fix an open quadrant $\mathcal{Q}_\sigma$, the argument being the same in all four cases.

The closed one-form η_2 separates in the variables f and g :

$$\eta_2 = P(f) df + S(g) dg, \quad (4.1)$$

where

$$P(f) = -\frac{1}{(\alpha+1)f} + \frac{\xi_3'(f)}{\xi_3(f)}, \quad S(g) = \frac{1}{(\alpha+1)g} - \frac{\xi_2'(g)}{\xi_2(g)}. \quad (4.2)$$

Write

$$\xi_3(f) = Af^2 + Bf + C, \quad \xi_2(g) = \hat{A}g^2 + \hat{B}g + \hat{C},$$

where $A, C, \hat{A}, \hat{C} > 0$ as ξ_2 and ξ_3 are strictly positive quadratic polynomials in one variable. Clearing denominators in (4.2) gives

$$P(f) = \frac{\pi(f)}{(\alpha+1)f\xi_3(f)}, \quad S(g) = -\frac{\sigma(g)}{(\alpha+1)g\xi_2(g)}, \quad (4.3)$$

with

$$\pi(f) = (\alpha + 1)f\xi_3'(f) - \xi_3(f) = (2\alpha + 1)Af^2 + \alpha Bf - C, \quad (4.4)$$

$$\sigma(g) = (\alpha + 1)g\xi_2'(g) - \xi_2(g) = (2\alpha + 1)\hat{A}g^2 + \alpha\hat{B}g - \hat{C}. \quad (4.5)$$

The product of the two roots of π is $-C/((2\alpha + 1)A) < 0$. Thus π has one simple positive root and one simple negative root. The same argument applies to σ .

Lemma 4.1. *The form η_2 has a single-valued real-analytic primitive Φ in $\mathcal{Q}_\sigma$,*

$$\Phi(f, g) = u(f) + s(g). \quad (4.6)$$

This primitive has exactly one critical point $p = (f_0, g_0)$ which is a nondegenerate saddle. Moreover, u is a one-well function: it is strictly decreasing and then strictly increasing, and tends to $+\infty$ at both ends of I . Similarly, s is a one-hill function and tends to $-\infty$ at both ends of J .

Proof. The functions $f\xi_3(f)$ and $g\xi_2(g)$ do not vanish on I and J . Hence P and S admit single-valued real-analytic primitives on these intervals.

Let $f_- < 0 < f_+$ be the roots of π . On $(0, \infty)$, the denominator of P is positive, so $P < 0$ on $(0, f_+)$ and $P > 0$ on (f_+, ∞) . On $(-\infty, 0)$, the sign of the factor f reverses the sign of the denominator; because π is positive on $(-\infty, f_-)$ and negative on $(f_-, 0)$, one again has $P < 0$ before the root and $P > 0$ after it. Consequently, u decreases and then increases on either interval. Furthermore,

$$\begin{aligned} P(f) &\sim -\frac{1}{(\alpha + 1)f} && \text{as } f \rightarrow 0, \\ P(f) &\sim \frac{2\alpha + 1}{(\alpha + 1)f} && \text{as } |f| \rightarrow \infty. \end{aligned}$$

After integration, these asymptotics show that $u(f) \rightarrow +\infty$ at both ends of I .

The same sign analysis, applied to the second expression in (4.3), shows that s first increases and then decreases; the analogous asymptotics give $s(g) \rightarrow -\infty$ at both ends of J . Hence the unique zeros f_0 of P and g_0 of S give the unique critical point p of Φ in $\mathcal{Q}_\sigma$.

At that point,

$$u''(f_0) = \frac{\pi'(f_0)}{(\alpha + 1)f_0\xi_3(f_0)} > 0, \quad s''(g_0) < 0. \quad (4.7)$$

Indeed, $\pi'(f_0)$ and f_0 have the same sign, and the corresponding statement for σ gives the second inequality. Therefore

$$\text{Hess}_p \Phi = \begin{pmatrix} u''(f_0) & 0 \\ 0 & s''(g_0) \end{pmatrix} \quad (4.8)$$

has signature $(+, -)$. □

Set $c_0 = \Phi(p) = u(f_0) + s(g_0)$.

Corollary 4.2. *The germ of the curve $\{\Phi = c_0\}$ at p is a union of germs of two smooth transversal curves.*

Indeed, by Morse Lemma, $\Phi = xy$ in a suitable holomorphic coordinates (x, y) near p .

Lemma 4.3 (Separation). *The set $\{\Phi > c_0\} \cap (I \times J)$ has exactly two connected components,*

$$\begin{aligned}\mathcal{L} &= \{f < f_0\} \cap \{\Phi > c_0\}, \\ \mathcal{R} &= \{f > f_0\} \cap \{\Phi > c_0\}.\end{aligned}$$

In particular, no connected subset of $\{\Phi > c_0\}$ meets both sides of the line $\{f = f_0\}$. Symmetrically, $\{\Phi < c_0\}$ has exactly two components, separated by the line $\{g = g_0\}$.

Proof. On the line $f = f_0$,

$$\Phi(f_0, g) = u(f_0) + s(g) \leq u(f_0) + s(g_0) = c_0, \quad (4.9)$$

with equality only at $g = g_0$. Thus $\{\Phi > c_0\}$ does not meet $\{f = f_0\}$, and the sets in (??) are disjoint open sets whose union is $\{\Phi > c_0\}$. Both are nonempty because u tends to $+\infty$ on either side of f_0 .

To prove that $\mathcal{L}$ is connected, note that the ray $\mathcal{L}^{sc} = \{(f, g_0) | f < f_0\} \subset \mathcal{L}$, as u is a one-well function. Also, for a fixed $f_1 < f_0$ the intersection $\{f = f_1\} \cap \mathcal{L}$ is an interval, as s is a one-hill function, continuously depending on f_1 . Thus, $\mathcal{L}^{sc}$ is a continuous retract of $\mathcal{L}$, and the claim follows. $\square$

5. THE OVAL AND ITS FOUR CONTACTS

The following identity is proved in [4]:

$$Q = -\frac{1+2\alpha}{(\alpha+1)^2} \xi_1 \xi_2 \xi_3 - fgQ_1, \quad (5.1)$$

where $\text{NP}(fgQ_1)$ lies strictly inside $\text{NP}(\xi_1 \xi_2 \xi_3)$. It follows that $\Gamma = \{Q = 0\}$ has no unbounded components, that $Q < 0$ on both coordinate axes away from the origin, and that the origin is an isolated point of Γ . Since $\eta_2(p) = 0$, equation (3.5) also gives $Q(p) = 0$.

Let O be the boundary of a bounded connected component of $\mathcal{Q}_\sigma \setminus \Gamma$ such that $p \in O$. Clearly, O is homeomorphic to a circle and is contained in the open quadrant.

Proof of Proposition 3.3. We consider three cases.

Case 1. O is smooth and transversal to both components of $\{\Phi = c_0\}$ at p . Then O remains either in $\{\Phi \geq c_0\}$ or in $\{\Phi \leq c_0\}$ near p . Assume the first, the second case being similar, so O intersects both $\mathcal{L}$ and $\mathcal{R}$ near p .

The curve $O' = O \setminus \{p\}$ is connected, and intersects both $\bar{\mathcal{L}} \setminus \{p\}$ and $\bar{\mathcal{R}} \setminus \{p\}$, which are disjoint. Thus, O' intersects $(I \times J) \setminus (\bar{\mathcal{L}} \cup \bar{\mathcal{R}}) \subset \{\Phi < c_0\}$.

It implies that the restriction of $\Phi - c_0$ to O has at least three zeros: one of multiplicity two at p , and two more on O' . Thus, the restriction of Φ to O

has at least four different local extrema: one at p , and, by Rolle, three more between consecutive zeros of $\Phi - c_0$. At each one of these extrema, $d\Phi \wedge dQ$ vanishes, so R vanishes as well. Thus, the total number of the zeros of R on O is at least 4, and Proposition 3.3 follows in this case.

Case 2. Q is smooth at p but not transversal to one of the components of $\{\Phi = c_0\}$ at p . In this case the restriction of $\Phi - c_0$ to O has zero of multiplicity three at p or higher. Also, Φ has at least two additional extrema on O , in the first case (as the restriction of Φ to O is monotone at p), and at least one additional extrema on O in the second case.

Thus, the restriction of $d\Phi \wedge dQ$ to O has a zero of multiplicity two at p , and two additional zeros in the first case, and a zero of multiplicity three at p , and an additional zero in the second case. In both cases, the total multiplicity of the zeros of R on O is at least 4, and Proposition 3.3 follows in this case as well.

Case 3. O is not smooth at p . In this case, $Q \in \mathfrak{m}^2$, and $d\Phi \wedge dQ \in \mathfrak{m}^2 dx \wedge dy$, as both dQ and $d\Phi$ vanish at p (here $\mathfrak{m}$ denotes the maximal ideal at the local ring at p). Thus, the local multiplicity of $R = Q = 0$ at p is at least 4, and Proposition 3.3 follows. $\square$

6. CONCLUDING REMARKS

Why the method stops at six. All inequalities in the argument except one are now sharp. The Bernstein–Kushnirenko count 28 is attained in the computations; the two complex points have multiplicity exactly $6 + 6$; generically, each quadrant contains exactly four real solutions; and $N_1 = 2$, $N_3 = 0$ are sharp. The remaining possible improvement lies in the combination of the first two Rolle steps:

$$N \leq 2 + \#(\gamma_2 \cap \Gamma \cap \mathbb{R}_{>0}^2).$$

The oval O in the positive quadrant carries four contact points. The level curve γ_2 meets O in 0, 2, or 4 points, depending on the position of its level relative to the four critical values of $\Phi|_O$. Four intersections occur exactly when the level lies between the two middle critical values.

In this formulation, Maxwell’s bound for three charges would follow from proving that, whenever $N_2 = 4$, the first Rolle step overcounts by exactly two; equivalently, at most one of the two noncompact components of γ_2 meets O , see Figure 1.

Recent results. 1. Edelsbrunner, Fillmore, and Oliveira [3] found a counterexample to Conjecture 1.8 of [4], which asserted that, for every $\alpha > 0$, the number of nondegenerate equilibrium points of a finite configuration of positive charges does not exceed the number of its effective Voronoi cells. One of the main results of [4] proves this assertion for every fixed configuration and all sufficiently large α . Thus it remains possible that some configuration of three charges has exactly six equilibrium points for special positive values of α .

2. In view of a recent counterexample [1], the question of finding new conjectures/results for the maximal number of points of equilibrium for configurations of point charges has resurfaced anew.

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