BRIEF COMMUNICATIONS

# FORMAL RELATIONS BETWEEN ANALYTIC FUNCTIONS

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In connection with the theorem on approximation of formal solutions of an analytic equation by convergent solutions, Artin [1] formulated the following two questions.

(1) Suppose that between the convergent series $t_1(x), \ldots, t_N(x)$ $(x = (x_1, \ldots, x_n))$ there exists a relation that can be expressed by a formal series. Does it hence follow that these series are analytically dependent?*

(2) Let $x = (x_1, \ldots, x_n)$, $t = (t_1, \ldots, t_k)$, $y = (y_1, \ldots, y_N)$, $z = (z_1, \ldots, z_M)$ and let $f(x, t, y, z) = 0$ be a system of analytic equations $(f = (f_1, \ldots, f_m))$. We shall assume that a solution $(\bar{y}(x, t), \bar{z}(t))$ of this system is given in formal power series. Will this solution be a limit of convergent solutions $(y(x, t), z(t))$ in Krull's topology?

In this note we construct a counterexample of (1) for the case N = 4 and a counterexample of (2) for the case k = 2. (In the case k = 1 the question (2) has a positive answer). Thus, in the general case these assertions are not true. We can formulate, however, the following assertion.

THEOREM. Let us assume that under conditions (1) the dimension of the ring factor of formal power series of N variables with respect to the ideal of the relations between functions $t_i(x)$ is equal to the rank at the common point of the mapping $t(x) : \mathbf{C}_x^n \to \mathbf{C}_t^N$. Then all the formal relations between $t_i(x)$ will be generated by the analytic relations between these functions.

COROLLARY. Assertion (1) holds in the case N = 3.

Counterexample (1). Let $x = (x_1, x_2)$. We shall find four analytically independent functions $t_1(x), \ldots, t_4(x)$, that satisfy the formal relation $\bar{z}(t_1, \ldots, t_4)$.

Let us write $t_1(x) = x_1$, $t_2(x) = x_1x_2$, $t_3(x) = x_1x_2e^{x_2}$. It is well known (see, for example, [2], p. 115) that there are no nontrivial formal relations between these functions. For $k \geq 1$ we shall write

$$\Phi_k(t_1, t_2, t_3) = t_1^{k-1}t_3 - \sum_{i=1}^{k} \frac{1}{(i-1)!} t_1^{k-i} t_2^i.$$

Let us note that $\Phi_k(t_1(x), t_2(x), t_3(x)) = \sum_{j=0}^{\infty} \frac{1}{(k+j)!} x_1^k x_2^{k+j+1}$.

Let us write

$$t_4(x) = \sum_{k=1}^{\infty} k!\,\Phi_k(t_1(x), t_2(x), t_3(x)) = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{k!}{(k+j)!} x_1^k x_2^{k+j+1}.$$

It is evident that $t_4(x)$ is an analytic function. On the other hand

$$\bar{P}(t_1, t_2, t_3) = \sum_{k=1}^{\infty} k!\Phi_k(t_1, t_2, t_3) = \sum_{k=1}^{\infty} k! \left( t_1^{k-1}t_3 - \sum_{i=1}^{k} \frac{1}{(i-1)!} t_1^{k-i} t_2^i \right)$$

*Here Artin refers to Abiancard.

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is a divergent power series of $t_1$, $t_2$, and $t_3$, whereas the series $\bar{z}(t) = t_4 - \bar{P}(t_1, t_2, t_3)$ represents a formal relation between $t_i(x)$. Let us show that there are no nontrivial analytic relations between $t_i(x)$. In fact, let $z(t)$ be such a relation. It is possible to assume that the function z is irreducible in a ring of analytic functions, since if $z = z_1 z_2$, then $z_1(t(x)) z_2(t(x)) \equiv 0$ in $C^2$, and hence either $z_1(t(x)) \equiv 0$, or $z_2(t(x)) \equiv 0$. Since $\bar{z} = t_4 - \bar{P}(t_1, t_2, t_3)$ it follows from the preparatory theorem for formal series that $\bar{z}(t) = \bar{z}(t)g(t) + \bar{h}(t_1, t_2, t_3)$. But since z and $\bar{z}$ are relations between $t_1(x), \ldots, t_4(x)$, the series $\bar{h}$ must be a relation between $t_1(x)$, $t_2(x)$, and $t_3(x)$; but since there are no such relations, we have $\bar{h} \equiv 0$ and z is divisible by $\bar{z}$. Since the function z is irreducible in a ring of analytic functions, it will be irreducible also in formal series. Hence $\bar{g}$ is an invertible series and $z \sim \bar{z}$. In particular, the function z is regular in $t_4$ of order 1, and according to the preparatory theorem for analytic functions we have $z \sim t_4 - P(t_1, t_2, t_3)$, where P is an analytic function. From the uniqueness of such a representation in formal series it follows that $P = \bar{P}$, which cannot be the case, since P is a convergent series and $\bar{P}$ a divergent series.

Counterexample (2). Let us note that in the previous example the relation between the functions $t_1(x) = x_1$, $t_2(x) = x_1 x_2$, $t_3(x)$, $t_4(x)$ was written in the form $z_1(t_1, t_2)\, t_3 + \bar{z}_2(t_1, t_2) + t_4$. Let $t(x_1, x_2, t_2) = t_2 - x_1 x_2$. Let us consider in $C^3_{x_1, x_2, t_2}$ the set $X = \{t = 0\}$. Since $\bar{z}_1(x_1, t_2) t_3(x_1, x_2) + \bar{z}_2(x_1, t_2) + t_4(x_1, x_2) \equiv 0$ on $X_i$, there exists a formal series $\bar{y}(x_1, x_2, t_2)$, such that

$$\bar{z}_1(x_1, t_2)\, t_3(x_1, x_2) + \bar{z}_2(x_1, t_2) + t_4(x_1, x_2) + \bar{y}(x_1, x_2, t_2)\, t(x_1, x_2, t_2) \equiv 0 \text{ в } C^3.$$

This row is an analytic equation whose formal solution is $(\bar{z}_1, \bar{z}_2, \bar{y})$. If there exists a convergent solution $(z_1, z_2, y)$ of this equation, with $z_1$ and $z_2$ being independent of $x_2$, then the function $z_1(t_1, t_2) \cdot t_3 + z_2(t_1, t_2) + t_4$ will be a nontrivial analytic relation between $t_i(x)$; but we have already shown that there are no such relations.

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## LITERATURE CITED


1. M. Artin, Algebraic Spaces (mimeographed), Yale University (1969).
2. M. Érwe, Functions of Several Complex Variables [Russian translation], Mir, Moscow (1965).