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Polar Curves and Intersection Matrices of Singularities

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This paper presents formulas connecting an intersection matrix of an isolated singularity $f\colon \mathbb{C}^n \to \mathbb{C}$ with an intersection matrix of $f+z^2$ or $f|_{z=0}$ and the simplest invariants of a polar curve of f related to z, z being a linear function.

1. Polar curves

Here we give some basic notations and well-known statements concerning polar curves (see [1]).

Let f be a germ of an analytic function with an isolated singularity at $0 \in \mathbb{C}^n$, $df(0) = 0$. Let $z\colon \mathbb{C}^n \to \mathbb{C}$ be a linear function.

Definition. A polar curve of f related to z is the set $\Gamma_z(f)$ of critical points of mapping $(z,f)\colon \mathbb{C}^n \to \mathbb{C}^2$.

One can easily verify that $\Gamma_z(f)$ is really a curve, i.e. $\dim_{\mathbb{C}} \Gamma_z(f) = 1$.

Let $\Gamma_z(f) = \bigcup \Gamma_i$ be a decomposition of $\Gamma_z(f)$ into irreductible components.

Let μ_i be the number of critical points (with multiplicities) of $f - \varepsilon z$, $\varepsilon \neq 0$, belonging to Γ_i. The Milnor number $\mu(f)$ of f is obviously equal to $\sum_i \mu_i$.

If $\Gamma_i \not\subset \{z=0\}$ then $f|_{\Gamma_i} \not\equiv 0$. Let $a_i z^{\alpha_i}$ be the first term of the Puiseux expansion of $f|_{\Gamma_i}$ (more precisely that of a curve Σ_i, which is the image of Γ_i under the mapping $(z,f)\colon \mathbb{C}^n \to \mathbb{C}^2$). One can easily show that $\alpha_i > 1$.

If $\Gamma_i \subset \{z=0\}$ ($f|_{z=0}$ having a non-isolated singularity) then $f|_{\Gamma_i} \equiv 0$. Set $\alpha_i = 1$ in this case. Let ν_i be the number of critical points (with multiplicities) of $f|_{z=\varepsilon}$, $\varepsilon \neq 0$, belonging to Γ_i. If $f|_{z=0}$ has an isolated singularity then its Milnor number $\mu(f|_{z=0})$ is equal to $\sum_i \nu_i$.

We have a following simple assertion.

Proposition 1. *Jf* $\alpha_i > 1$ *then* $\mu_i = \nu_i(\alpha_i - 1)$.

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Corollary. *If $a_i \neq -1$ for $\alpha_i = 2$ then the Milnor number of $f + z^2$ is equal to* $\sum_{i:\alpha_i < 2} \mu_i + \sum_{i:\alpha_i \geqq 2} \nu_i$.

In particular, $\mu(f+z^2) \leqq \mu(f|_{z=0})$ and equality is achieved iff $\alpha_i \geqq 2$ for all i.

Proposition 2. *If z is generic and $f \in \mathfrak{m}^k$ then $\alpha_i \geqq k$ for all i.*

2. Results

Proposition 3. *Let $F_\varepsilon = f + (z-\varepsilon)^2$ be a small deformation of $f + z^2$. Critical values of F_ε at its critical points belonging to Γ_i and tending to 0 as $\varepsilon \to 0$ are as follows:*

$$F_\varepsilon = a_i \varepsilon^{\alpha_i} + o(\varepsilon^{\alpha_i}) \qquad \text{if } \alpha_i > 2,$$
$$F_\varepsilon = \varepsilon^2 a_i/(a_i+1) + o(\varepsilon^2) \qquad \text{if } \alpha_i = 2$$

(*we suppose that $a_i \neq -1$ for $\alpha_i = 2$*),

$$F_\varepsilon = \varepsilon^2 + o(\varepsilon^2) \qquad \text{if } \alpha_i < 2.$$

The proof consists in a direct calculation of the asymptotics of solutions of the equation

$$(F_\varepsilon)_z|_{\Gamma_i} = a_i \alpha_i z^{\alpha_i - 1} + 2(z-\varepsilon) + O(z^{\alpha_i - 1}) = 0$$

as $\varepsilon \to 0$ and substitution of these asymptotics into $F_\varepsilon|_{\Gamma_i}$.

Let A be the set of all the values α_i. According to Proposition 3 we can choose for a small $\varepsilon \neq 0$ positive numbers r'_α and r''_α ($\alpha \in A$, $\alpha \geqq 2$) so that for each $\alpha \in A$, $\alpha > 2$ (resp. $\alpha \leqq 2$) all the critical values of F_ε at its critical points belonging to Γ_i with $\alpha_i = \alpha$ contain in a ring $\{u: r'_\alpha < |u| < r''_\alpha\}$ (resp. $\{u: r'_2 < |u| < r''_2\}$) and $r''_\alpha < r'_\beta$ for $\alpha > \beta$.

Let $\sigma(r)$ be a continuous monotonous non-increasing function defined for $r \geqq 0$, $\sigma(r) = \alpha - 1$ for $r'_\alpha \leqq r \leqq r''_\alpha$.

Set $V_m = \{u: \arg u + 2\pi\sigma(|u|) \geqq \pi(2m-1)\}$ where $-\pi \leqq \arg u \leqq \pi$, $m = 0, 1, 2, \ldots$.

Proposition 4. *Let the following condition hold:*

$$(*) \quad (-a_i)^{q_i} \notin \mathbb{R}_+ \qquad \text{for } \alpha_i = p_i/q_i, \ (p_i, q_i) = 1.$$

Then for a small $\varepsilon \in \mathbb{R}_+$ critical values of F_ε don't contain neither in $\mathbb{R}_-$ nor in boundaries of all V_m.

The proof is deduced easily from Proposition 3.

Let us choose a system of paths between critical values of F_ε and its non-critical value 0, defining a distinguished basis of vanishing cycles for $f + z^2$ (i.e. the paths are not self-intersecting and different paths intersect only at 0, see [2]). If F_ε is not a Morse function, we replace it by a close Morse function. Suppose that our system of paths satisfies the following condition:

(V) *all the paths intersect $\mathbb{R}_-$ only at* 0; *all the paths drawn from critical values belonging to V_m*, contain in V_m.

The order of such a system of paths is defined by decreasing of arg u, u being a point of intersection of path with a small circle centered at 0.

Proposition 5. *Let a system of paths consist of straight segments between critical values of F_ε and 0. Then it satisfies the condition (V).*

Theorem 1. *Let $\{e_j\}$ be a distinguished basis of vanishing cycles for $f+z^2$ defined by a system of paths satisfying the condition (V).*

a) *There exists a distinguished basis $\{e_j^m\}$ $(m=1,2,\ldots)$ for f with the lexicographic order of (m,j) and with a following intersection matrix:*

$$(e_j^m, e_{j'}^m) = (e_j, e_{j'}),$$

$$(e_j^m, e_j^{m'}) = (m'-m)^{n-1} \quad \text{for } |m'-m|=1,$$

$$(e_j^m, e_{j'}^{m'}) = (-1)^n (e_j, e_{j'}) \quad \text{for } |m'-m|=1,\ (m'-m)(j'-j)<0,$$

$$(e_j^m, e_{j'}^{m'}) = 0 \quad \text{for } |m'-m|>1 \text{ or } (m'-m)(j'-j)>0.$$

Here a pair (m,j) is admissible iff a cycle e_j vanishes along a path containing in V_m.

b) $e_j^m = h_* e_j^{m-1}$ *for $m>0$ where h_* is a monodromy operator of f.*

c) *For a system of paths from the proposition 5 the condition on the admissible pairs (m,j) can be reformulated as follows: the first μ_i pairs from each set of pairs (m,j) with a cycle e_j vanishing in a point belonging to Γ_i are admissible.*

The proof of the theorem will be given in n° 4.

Let now $f|_{z=0}$ have an isolated singularity. Let $G_\varepsilon = f|_{z=\varepsilon}$ be a small deformation of $f|_{z=0}$. Critical values of G_ε at its critical points belonging to Γ_i are $a_i\varepsilon^{\alpha_i} + O(\varepsilon^{\alpha_i})$ as $\varepsilon \to 0$. Consequently we can choose positive numbers r'_α and r''_α for all $\alpha \in A$ so that all the critical values of G_ε at its critical points belonging to Γ_i with $\alpha_i = \alpha$ contain in a ring $\{u: r'_\alpha < |u| < r''_\alpha\}$ and $r''_\alpha < r'_\beta$ for $\alpha > \beta$. Let us define a function $\sigma(r)$, sets V_m and a condition (V) on a system of paths between critical values of G_ε and its non-critical value 0 as it was defined for F_ε with the only difference that all $\alpha \in A$ are considered, not only $\alpha \geqq 2$.

Let $\{h'_j\}$ be a distinguished basis for $f|_{z=0}$ defined by a system of paths satisfying the condition (V). Critical values of $G_\varepsilon + z^2$ being the same as those of G_ε, such a system defines also a distinguished basis $\{h_j\}$ for $f|_{z=0} + z^2$.

Proposition 6 (see [2]). *Intersection matrices of the bases $\{h'_j\}$ and $\{h_j\}$ are connected by the following relation:*

$$(h_j, h_{j'}) = (-1)^{n-1} (h'_j, h'_{j'}) \quad \text{for } j'>j.$$

Theorem 2. *Let $f|_{z=0}$ have an isolated singularity. Then $f+z^2$ and e_j in Theorem 1 can be replaced by $f|_{z=0}+z^2$ and h_j respectively.*

Proof. We'll deduce this theorem from Theorem 1 in the case of $\min(\alpha \in A) \geqq 2$ (if this is not so then $\mu(f|_{z=0}) > \mu(f+z^2)$ and the proof is some more complicated as we have to look after extra cycles in a basis for $f|_{z=0}$, see also Proposition 2).

Let f^t be a function obtained from f by replacing z by $tz+(1-t)\varepsilon$, $t \in [0,1]$. Set $F_\varepsilon^t = f^t + (z-\varepsilon)^2$. One can easily show that critical values of F_ε^t at its critical points belonging to Γ_i with $\alpha_i > 2$ have the same asymptotics as those in Proposition 3, independently of t. Critical values of F_ε^t at its critical points belonging to Γ_i with $\alpha_i = 2$ have the asymptotics $F_\varepsilon^t = \varepsilon^2 a_i/(a_i t^2+1)+O(\varepsilon^2)$. Consequently we can choose numbers r'_α and r''_α so that critical values of F_ε^t at its critical points belonging to Γ_i contain in a ring $\{u: r'_{\alpha_i} < |u| < r''_{\alpha_i}\}$ for all t. It easily follows from the definition of sets V_m that a system of paths for $F_\varepsilon^0 = G_\varepsilon + (z-\varepsilon)^2$ satisfying the condition (V) can be deformated into a system of paths for $F_\varepsilon^1 = F_\varepsilon$ satisfying the same condition as t changes form 0 to 1. Corresponding bases $\{h_j\}$ and $\{e_j\}$ have then equal intersection matrices, q.e.d.

Theorem 3. *Let $f|_{z=0}$ have an isolated singularity. Let $\hat{f}$ be a singularity obtained from f by replacing z by z^2. Let $\{h_j\}$ be a distinguished basis of vanishing cycles for $f|_{z=0}+z^2$ defined by a system of paths satisfying the condition (V).*

a) *There exists a distinguished basis of vanishing cycles $\{\hat{e}_j^m\}$ $(m=0, \pm 1, \pm 2, \ldots)$ for $\hat{f}$ with an intersection matrix*

$$(\hat{e}_j^m, \hat{e}_{j'}^m) = (h_j, h_{j'}),$$

$$(\hat{e}_j^m, \hat{e}_j^{m'}) = (|m'|-|m|)^{n-1} \quad \text{for } |m'-m|=1,$$

$$(\hat{e}_j^m, \hat{e}_{j'}^{m'}) = (-1)^n (h_j, h_{j'}) \quad \text{for } |m'-m|=1,\ (|m'|-|m|)(j'-j)<0,$$

$$(\hat{e}_j^m, \hat{e}_{j'}^{m'}) = 0 \quad \text{for } |m'-m|>1 \text{ or } (|m'|-|m|)(j'-j)>0.$$

The order of pairs (m,j) is defined as follows: $(m,j)<(m',j)$ if $|m|<|m'|$ or $|m|=|m'|$, $j<j'$ or $m'=-m>0$, $j=j'$.

A pair (m,j) is admissible iff h_j vanishes along a path containing in $V_{|m|}$.

b) *The involution $z \mapsto -z$ action in the homology of a nonsingular level of $\hat{f}$ is defined by $\hat{e}_{m,j} \mapsto -\hat{e}_{-m,j}$ i.e. the basis $\{\hat{e}_j^m\}$ defines a distinguished basis for a pair $(f, f|_{z=0})$ where $\hat{e}_j^0$ are short cycles and $\hat{e}_j^m + \hat{e}_j^{-m}$ for $m>0$ – long cycles (see [5]).*

The proof of this theorem is based on the same ideas as that of Theorem 1 and omitted here.

3. Distinguished bases for f

Proposition 7. *Critical values of $f_\varepsilon = f - 2\varepsilon z$ at its critical points belonging to Γ_i and tending to 0 as $\varepsilon \to 0$ are as follows:*

$$f_\varepsilon = -a_i(\alpha_i - 1)\left(\frac{2\varepsilon}{a_i \alpha_i}\right)^{\alpha_i/(\alpha_i-1)} + o\left(\varepsilon^{\alpha_i/(\alpha_i-1)}\right) \quad \text{for } \alpha_i > 1,$$

$$f_\varepsilon = 0 \quad \text{for } \alpha_i = 1.$$

The proof is trivial.

According to Proposition 7 we can choose for a small ε positive numbers R'_α and R''_α for $\alpha \in A$, $\alpha > 2$ and a number R'_2 so that for $\alpha \in A$, $\alpha > 2$ (resp. $\alpha \leqq 2$) all the critical values of f_ε at its critical points belonging to Γ_i with $\alpha_i = \alpha$ contain in a ring $\{u: R''_\alpha < |u| < R'_\alpha\}$ (resp. in a circle $\{u: |u| < R'_2\}$) and $R''_\alpha > R'_\beta$ for $\alpha > \beta$. Let us choose a non-critical value $u^* \in \mathbb{R}_-$ of f_ε, $u^* < -\max_\alpha R'_\alpha$.

Let $\xi(r)$ be a continuous monotonous non-increasing function defined for $0 \leqq r \leqq -u^*$ satisfying the following conditions:

$$\xi(r)=1 \qquad \text{for } r<R'_2,$$

$$\xi(r)=1/(\alpha-1) \qquad \text{for } R''_\alpha \leqq r \leqq R'_\alpha,\ \alpha>2,$$

$$\xi(r)>0 \qquad \text{for } r<-u^*, \quad \xi(-u^*)=0.$$

Set $D=\{u\colon |u| \leqq -u^*, |\arg(-u)| \leqq \pi\xi(|u|)\}$ where $-\pi \leqq \arg(-u) < \pi$.

Set $W_m=\{u \in D, \arg(-u)+\pi(2m-1)\xi(|u|) \leqq 2\pi\}$ for $m=1, 2, \ldots$. In particular, $W_1=D$.

Let us define a homeomorphism $\tau_\varphi\colon \mathbb{C} \to \mathbb{C}$ by

$$\tau_\varphi(u)=u e^{2\pi i \varphi(\xi(|u|)+1)}.$$

From now on let f satisfy the condition (*) from Proposition 4 and ε be a small positive number.

Proposition 8. *No critical values of $f_{e^{2\pi i\varphi}\varepsilon}$ contain in boundaries of all W_m.*

The proof is easily obtained from Proposition 7 and the condition (*).

Proposition 9. *The sets $\tau_{m-1}(W_m)$ haven't common inner points for different m. Each critical value of f_ε contains in $\tau_{m-1}(W_m)$ for some m.*

Proof. One can easily show that the interior of $\tau_{m-1}(W_m)$ coincides with the interior of $\tau_{m-1}(D) \setminus (\bigcup_{0 \leqq l \leqq m-1} \tau_l(D))$. As each critical value of f_ε contains in $\tau_{m-1}(D)$ for some m but not in boundaries of $\tau_l(D)$, the proposition follows.

Let us choose a system of paths between critical values of f_ε containing in D and its non-critical value u^* so that all the paths are not self-intersecting, different paths intersect only at u^* and the following condition holds:

(*W*) *all the paths drawn from points belonging to W_m contain in W_m.*

The order for the system of paths is defined by decreasing of $\arg(u-u^*)$, u being a point of intersection of a path with a small circle centered at u^*, $-\pi < \arg(u-u^*) < \pi$.

Let $\{e_j^1\}$ be a system of vanishing cycles in the homology of a nonsingular level $f_\varepsilon = u^*$ of f_ε defined by the chosen system of paths.

Let $\Theta_\varphi\colon D \to \mathbb{C}$ be a continuous family of homeomorphic inclusions satisfying the following conditions:

$\Theta_0\colon D \to D$ is an identity,

$\Theta_\varphi = \tau_\varphi$ at boundaries of W_m,

Θ_φ maps critical values of f_ε into critical values of $f_{e^{2\pi i\varphi}\varepsilon}$.

For critical values of f_ε containing in $\tau_{m-1}(W_m)=\Theta_{m-1}(W_m)$ we define a system of paths between these values and u^* as a result of applying Θ_{m-1} to the paths between critical values of f_ε containing in W_m and u^*.

Let e_j^m be a cycle defined by a path obtained by applying of Θ_{m-1} to a path defining e_j^1.

Proposition 10. *The system $\{e_j^m\}$ of vanishing cycles with the lexicographic order forms a distinguished basis for f.*

Proof. It's easily deduced from Proposition 9 that the paths defining the system $\{e_j^m\}$ are not self-intersecting, different paths intersect only at u^* and all the critical values of f_ε are connected with u^* by the paths. One can easily show also that the lexicographic order of pairs (m,j) agrees with the definition of the order of paths defining a distinguished basis (see [2]).

4. Proof of Theorem 1

Suppose $n \equiv 3 \pmod 4$. The assertion for any n is deduced then by adding a sum of squares of new variables to all functions and recalculating intersection matrices by the formulas from [2].

Lemma 1. *Let $\{e_j^m\}$ be a distinguished basis for f defined by system of paths satisfying the condition (W).*

Then

a) $e_j^m = h_* e_j^{m-1}$ *for* $m>0$ *where* h_* *is a monodromy operator of* f;

b) $(e_j^m, e_{j'}^m) = (e_j^1, e_{j'}^1)$,

$(e_j^m, e_j^{m'}) = 1 \quad$ *for* $|m'-m|=1$,

$(e_j^m, e_{j'}^{m'}) = -(e_j', e_{j'}') \quad$ *for* $|m'-m|=1,\ (m'-m)(j'-j)<0$,

$(e_j^m, e_{j'}^{m'}) = 0 \quad$ *for* $|m'-m|>1$ *or* $(m'-m)(j'-j)>0$.

Proof. The assertion a) follows from the construction of cycles e_j^m as $\Theta_\varphi(u^*) = \tau_\varphi(u^*) = e^{2\pi i\varphi}u^*$. The assertion b) is deduced from the assertion a) as follows. Since the monodromy preserves intersections then $(e_j^m, e_{j'}^m) = (e_j^1, e_{j'}^1)$. Let now decompose h_* into product of operators $T_j^m = (e \mapsto e + (e, e_j^m) e_j^m)$ (reflections in the vanishing cycles e_j^m). Since $h_* e_j^m = e_j^{m+1}$ then the result of action on e_j^m of all $T_{j'}^{m'}$ for $(m',j') > (m+1, j)$ is equal to e_j^m, the result of action of T_j^{m+1} on e_j^m is equal to $e_j^m + e_j^{m+1}$, the result of action of $T_{j'}^{m+1}$ for $j'<j$ on $e_j^m + e_j^{m+1}$ is equal to $e_j^m + e_j^{m+1}$. This means that $(e_j^m, e_{j'}^{m'}) = 0$ for $m'>m+1$ and for $m'=m+1$, $j'>j$, $(e_j^m, e_j^{m+1}) = 1$ and $(e_j^m, e_{j'}^{m+1}) = -(e_j^{m+1}, e_{j'}^{m+1}) = -(e_j^1, e_{j'}^1)$ for $j'<j$, q.e.d.

Let us consider a deformation $f_{\delta,\varepsilon} = f + \delta z^2 - 2\varepsilon z$ of f. Evidently $f_{0,\varepsilon} = f_\varepsilon$, $f_{1,\varepsilon} = F_\varepsilon - \varepsilon^2$.

Lemma 2. *As δ changes from 0 to 1, the critical values of $f_{0,\varepsilon} = f_\varepsilon$ contained in D and only they pass to the critical values of $f_{1,\varepsilon} = F_\varepsilon - \varepsilon^2$ tending to 0 as $\varepsilon \to 0$. The critical values contained in W_m and only they pass to the critical values contained in $V_m - \varepsilon^2$. A system of paths between critical values of $f_{0,\varepsilon} = f_\varepsilon$ contained in D and its non-critical value u^* satisfying the condition (W) can be deformated into a system of paths between critical values of $f_{1,\varepsilon} = F_\varepsilon - \varepsilon^2$ and its non-critical value obtained by $(-\varepsilon^2)$-shift from a system of paths satisfying the condition (V). Systems of vanishing cycles $\{e_j'\}$ and $\{e_j\}$ defined by the corresponding systems of paths have equal intersection matrices.*

The proof of this lemma is elementary and based on the study of critical values of the function $az^\alpha + \delta z^2 - 2\varepsilon z$ ($a \in \mathbb{C}$, $\delta \in \mathbb{R}_+$, $\varepsilon \in \mathbb{R}_+$, $\alpha \in \mathbb{Q}$). However it's rather complicated and omitted here.

The assertion of Theorem 1 is obtained by combining the assertions of Lemmas 1 and 2.

Table 1

Notation	Formula	Numbers M_j
$J_{k,i}$	$x^3+x^2z^k+z^{3k+i}$	$3k+i-1,\ 3k-1$
E_{6k}	x^3+z^{3k+1}	$3k,\ 3k$
E_{6k+1}	x^3+xz^{2k+1}	$3k+1,\ 3k$
E_{6k+2}	x^3+z^{3k+2}	$3k+1,\ 3k+1$
$X_{k,p}$	$x^4-x^2z^{2k}+z^{4k+p}$	$4k-1,\ 4k-1,\ 4k+p-1$
$Y^k_{r,s}$	$x^2(x-z^k)^2+x^2z^{2k+s}+(x-z^k)^2z^{2k+r}$	$4k+r-1,\ 4k+s-1,\ 4k-1$
$Z^k_{i,p}$	$x^4-x^3z^k+x^2z^{2k+i}-z^{4k+3i+p}$	$4k-1,\ 4k+3i+p-1,\ 4k+3i-1$
$Z^k_{12k+6i-1}$	$x^4-x^3z^k+z^{4k+3i+1}$	$4k+3i,\ 4k-1,\ 4k+3i$
Z^k_{12k+6i}	$x^4-x^3z^k+xz^{3k+2i+1}$	$4k+3i+1.\ 4k-1,\ 4k+3i$
$Z^k_{12k+6i+1}$	$x^4-x^3z^k+z^{4k+3i+2}$	$4k+3i+1,\ 4k-1,\ 4k+3i+1$
W_{12k}	x^4+z^{4k+1}	$4k,\ 4k,\ 4k$
W_{12k+1}	x^4-xz^{3k+1}	$4k+1,\ 4k,\ 4k$
$W_{k,i}$	$x^4-x^2z^{2k+1}+z^{4k+2+i}$	$4k+1,\ 4k+1,\ 4k+i$
$W^{\#}_{k,2q-1}$	$(x^2-z^{2k+1})^2+xz^{3k+q+1}$	$4k+q+1,\ 4k+q,\ 4k+1$
$W^{\#}_{k,2q}$	$(x^2-z^{2k+1})^2+x^2z^{2k+q+1}$	$4k+q+1,\ 4k+q+1,\ 4k+1$
W_{12k+5}	x^4-xz^{3k+2}	$4k+2,\ 4k+1,\ 4k+2$
W_{12k+6}	x^4+z^{4k+3}	$4k+2,\ 4k+2,\ 4k+2$
$Q_{k,i}$	$x^3+(z-x)y^2+x^2z^k+z^{3k+i}$	$2,\ 2,\ 3k-1,\ 3k+i-1$
Q_{6k+4}	$x^3+(z-x)y^2+z^{3k+1}$	$2,\ 2,\ 3k,\ 3k$
Q_{6k+5}	$x^3+(z-x)y^2-xz^{2k+1}$	$2,\ 2,\ 3k+1,\ 3k$
Q_{6k+6}	$x^3+(z-x)y^2+z^{3k+2}$	$2,\ 2,\ 3k+1,\ 3k+1$
S_{12k-1}	$x^2y+(z-y)y^2+z^{4k}$	$2,\ 4k-1,\ 4k-1,\ 4k-1$
S_{12k}	$x^2y+(z-y)y^2+xz^{3k}$	$2,\ 4k,\ 4k-1,\ 4k-1$
$S_{k,i}$	$x^2y+(z-y)y^2+x^2z^{2k}+z^{4k+i+1}$	$2,\ 4k,\ 4k,\ 4k+i$
$S_{k,2q-1}$	$x^2y+(z-y)y^2-yz^{2k+1}+xz^{3k+q}$	$2,\ 4k+q,\ 4k+q-1,\ 4k$
$S_{k,2q}$	$x^2y+(z-y)y^2-yz^{2k+1}+x^2z^{2k+q}$	$2,\ 4k+q,\ 4k+q,\ 4k$
S_{12k+4}	$x^2y+(z-y)y^2+xz^{3k}$	$2,\ 4k+1,\ 4k,\ 4k+1$
S_{12k+5}	$x^2y+(z-y)y^2+z^{4k+2}$	$2,\ 4k+1,\ 4k+1,\ 4k+1$
$T_{p,q,r}$	$xy(z-x-y)+x^p+y^q+(z-x-y)^r$	$p-1,\ q-1,\ r-1,\ 2$
U_{12k}	$x^3+xy^2+z^{3k+1}$	$3k,\ 3k,\ 3k,\ 3k$
$U_{k,2q-1}$	$x^3+xy^2-xz^{2k+1}+y^2z^{k+q}$	$3k+q,\ 3k+q,\ 3k,\ 3k+1$
$U_{k,2q}$	$x^3+xy^2-xz^{2k+1}+yz^{2k+q+1}$	$3k+q+1,\ 3k+q,\ 3k,\ 3k+1$
U_{12k+4}	$x^3+xy^2+z^{3k+2}$	$3k+1,\ 3k+1,\ 3k+1,\ 3k+1$

5. Examples

The problem of calculation of intersection matrix for a singularity f is reduced by the Theorem 2 to the problem of calculation of intersection matrix for $f|_{z=0}$ and of the exponents α_j for $\Gamma_z(f)$. Corresponding calculations were carried out for all the singularities from the Arnol'd list [3] excluding the series $\mathbf{V}$, in particular for all the bimodular singularities. The results of these calculations are presented in Table 1.

In this table after the notation of a singularity in terms of [3] a formula of a representative of this singularity used for the calculations and numbers $M_1, \ldots, M_{\mu'}$ $(\mu'=\mu(f|_{z=0}))$ are presented.

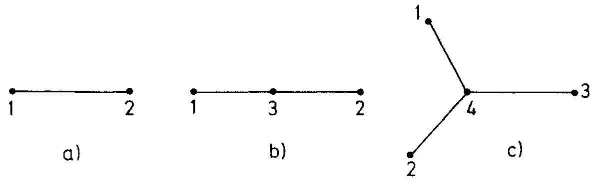

Fig. 1. Dynkin diagram of a distinguished basis for $f|_{z=0}$ satisfying the condition (V) for f belonging to the series: a) $\mathbb{J}$, $\mathbb{E}$; b) $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$; c) $\mathbb{Q}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{U}$

An intersection matrix of a distinguished basis for f can be obtained form an intersection matrix of a distinguished basis for $f|_{z=0}$ satisfying the condition (V) (corresponding Dynkin diagrams are presented in Fig. 1) by the formulas from Theorem 1, and the basis for f is formed by cycles e_j^m, $1 \leqq m \leqq M_j$.

Remark. Some of the singularities are presented in a form not satisfying the condition $(*)$ but convenient for calculations of intersection matrices of $f|_{z=0}$ (all the critical points of $f|_{z=\varepsilon}$, $\varepsilon > 0$, are real and we can use the Gusein-Zade method [4]). The condition $(*)$ holds then after multiplication of f by a non-real number.

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