ON A THEOREM OF HORMANDER

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In his paper at the Tokiiskoi conference of 1969, L. Hormander proved that for an arbitrary differential polynomial P(D) in R^n there exists a fundamental solution having a singularity in some cone $K^*(P)$ (defined in paragraph 1). If P is a homogeneous polynomial and grad $P(x) \neq 0$ for $P(x) = 0$, $x \neq 0$, then the cone $K^*(P)$ is simply the inverse cone to the cone defined by the roots of the polynomial P. For the general case Hormander indicates that the geometrical meaning of $K^*(P)$ is not only vaguely understood, but it is not even known whether or not $K^*(P)$ differs from the entire space (i.e., one wishes to prove an inclusion theorem). In this paper we prove that $K^*(P)$ is contained in some cone $X^*(P)$, $\dim X^*(P) < n$, which constitutes a semi-algebraic set in R^n. The cone $X^*(P)$ is constructed by means of a regular Whitney stratification dependent on the asymptotic behavior of the roots of the polynomial P at infinity.

1. DEFINITION OF K*(P)

Let P_n^k be the linear space of all polynomials of order not greater than k in R^n, $P \in P_n^k$, $P \not\equiv 0$. Let $P^{(\alpha)}$ be the derivative of P of order $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\tilde{P}(x) = \left(\sum_\alpha |P^{(\alpha)}(x)|^2\right)^{1/2}$. Let L(P) be the set of all polynomials $Q \in P_n^k$, for which a sequence η_i exists such that $\lim_{i\to\infty} |\eta_i| = \infty$ and $\lim_{i\to\infty} P(x+\eta_i)/\tilde{P}(\eta_i) = Q(x)$. (Using a theorem of Tarskii and Zaidenberg (see [1]) it is easily shown that it suffices to examine the limits along semi-algebraic curves.*) Now let $Q \in L(P)$. We denote by $\Lambda'(Q)$ the orthogonal complement to the space $\Lambda(Q) = \{y \in R^n \mid Q(x+y) \equiv Q(x)\}$ of variables of which Q is independent. Then $K^*(P) = \overline{\bigcup_{Q \in L(P)} \Lambda'(Q)}$.

2. THE REGULAR STRATIFICATION OF WHITNEY

An algebraic variety in C^n is a set of the form $X \setminus Y$, where X and Y are algebraic sets.

Definition 2.1 (Whitney [2]). Let M be an algebraic variety in C^n, and M' a non-singular sub-variety in $\overline{M}$. The pair (M, M') is said to be *a*-regular if the following condition is met:

(a). Let x_0 be any point in M' and $T_{x_0}(M')$ be the tangent plane to M' at the point x_0. Let $x_i \to x_0$ be a sequence of simple points of the set M and $\lim_{i\to\infty} T_{x_i}(M) = T$. Then $T \supset T_{x_0}(M')$.

THEOREM 2.2 (Whitney [2]). Let X be an algebraic variety in C^n, X' an algebraic sub-variety in $\overline{X}$. Then there exists an algebraic sub-variety $S \subset X'$, $\dim S < \dim X'$, such that $(X, X' \setminus S)$ is an *a*-regular pair.

Definition 2.3. Let X be an algebraic variety in C^n. A decomposition of X into non-self-intersecting subsets X_i ("striations") is called a stratification of the set X, if

1) Each X_i is a connected algebraic variety;

2) If $\overline{X}_i \cap X_j \neq \phi$ then $\overline{X}_i \supset X_j$.

COROLLARY 2.4 (Whitney [2]). Let $X \subset C^n$ be an algebraic set. Then there exists a stratification $\{X_i\}$ of the set X (an *a*-regular stratification in the sense of Whitney) such that if $X_i \supset X_j$, then the pair (X_i, X_j) is *a*-regular.

*A "semi-algebraic curve" in R^n is a continuous mapping λ: $(0, 1] \to R^n$, whose graph is a semi-algebraic set.

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3. THE CASE OF A HOMOGENEOUS POLYNOMIAL

Assume that the space C^n is embedded in CP^n, and that the infinite hyperplane for $CP^{n-1} \subset CP^n$ is identified with the set of complex lines in C^n passing through 'he origin.

Let $y \in R^m$ (respectively C^n); we will set $R_y = \{x \in R^n | x = \alpha y, \alpha \in R\}$ (respectively $C_y = \{z \in C^n | z = \alpha y, \alpha \in C\}$).

If X is a set in C^n, then we denote by p(X) the set $\bar{X} \cap CP^{n-1}$, where $\bar{X}$ is the closure in CP^n. If X' is a set in CP^{n-1}, then we denote by k(X') the set $\{z \in C^n | z \neq 0, C_z \in X'\}$.

The set of simple points of an algebraic variety X is denoted by X_{sp}, while the set of singular points is denoted by X_{sg}.

Let P be a homogeneous polynomial in R^n, $X \subset C^n$ the cone of its complex roots. Let $X' = p(X)$, $\{X_j'\}$ an a-regular stratification of the set X', $X_j = k(X'_j)$. Then $\{X_j\}$ is an a-regular stratification of the set $X \setminus \{0\}$. Let $Y_{0,j} = X_j \cap R^n$, $Y_{l,j} = (Y_{l-1,j})_{sg}$, $X_{l,j} = Y_{l,j} \setminus Y_{l+1,j}$. We will set $NX_{l,j} = \{z \in R^n | \exists x \in X_{l,j} : z \perp T_x(X_{l,j})\}$. It is easily proved that $\overline{NX}_{l,j}$ is a semi-algebraic set in R^n, $\dim NX_{i,j} < n$. Let us set $X^*(P) = (\bigcup_{l,j} \overline{NX}_{l,j})$.

THEOREM 3.1. $X^*(P) \supset K^*(P)$.

The following lemma, which is necessary for our work, will be stated here without proof.

LEMMA 3.2. Let $\lambda : s \to Q(s)$, $s \in [0, 1]$ be a semi-algebraic curve in P_n^k, $Q(0) \not\equiv 0$. Let Y_s be the set of complex roots of the polynomial Q(s) and $y \in (Y_0)_{sp}$. Then there exist sequences $s_i \to 0$, $y_i \to y$ such that $y_i \in (Y_s)_{sp}$ and $\lim_{i\to\infty} T_{y_i}(Y_{s_i}) = T_y(Y_0)$.

Proof of the Theorem. Let $Q \in L(P)$ be some polynomial and let $\mu : x = \mu(x)$, $s \in (0, 1]$, be a semi-algebraic curve such that $\lim_{s\to 0} |\mu(s)| = \infty$, $\lim_{s\to 0} P(x + \mu(s))/\tilde{P}(\mu(s)) = Q(x)$. Assume that for $s \to 0$ the ray $R_{\mu(s)}$ tends in direction to that of some vector x_0, $\|x_0\| = 1$. Let $(y' = (y_1, \ldots, y_{n-1}), y_n)$ be coordinates in C^n, such that $x_0 = (0, \ldots, 0, 1)$. Let $\mu' : y' = \mu'(s)$ be the projection of the curve μ into the sub-subspace $\{y_n = 1\}$ with center at 0. Clearly, $\lim_{s\to 0} \mu'(s) = x_0$. Let $x_0 \in X_{l,j}$. We will show that for any simple point y the set $Y = \{z \in C^n | Q(z) = 0\}$ of the space $T_y(Y)$ contains $T_{x_0}(X_{l,j})$. Hence, it will at once follow that $T_{x_0}(X_{l,j}) \subset \Lambda(Q)$ and hence $\Lambda'(Q) \subset NX_{l,j} \subset X^*(P)$.

Let $T = T_y(Y)$. By virtue of Lemma 3.2 it is possible to find some sequences $s_i \to 0$, $y^i \to y$, where y^i is a simple point of the set $Y_i = \{z \in C^n | P(z + \mu(s_i))/\tilde{P}(\mu(s_i)) = 0\}$ such that $\lim_{i\to\infty} T_{y^i}(Y_i) = T$. Let x_i be the projection of the point $\mu(s_i) + y^i$ onto $\{y_n = 1\}$ with center at 0. Then x_i is a simple point of the set X, $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} T_{x_i}(X) = T$. But by the definition of an a-regular stratification $T \supset T_{x_0}(X_j)$, which completes the proof.

4. THE GENERAL CASE; STRATIFICATION AT INFINITY

In this paragraph we denote by G(n, p) the Grassman variety of all p-dimensional linear subspaces of C^n. If $M \in G(n, p)$, $N \in G(n, q)$ is a subspace in C^n, we will denote by $\rho(M, N)$ the term $\max_{x\in M} \min_{y \in N} \sqrt{1 - \frac{|(x, y)|^2}{|x|^2 |y|^2}}$. We note that the graphs of the functions $\rho(M, N)$ are semi-algebraic sets in $G(n, p) \times G(n, q) \times R$ and that $\rho(M, N) = 0$ $M \subset N$. Moreover, if $p = q$, then ρ is the usual metric in G(n, p). An analogous definition is given in the real case.

LEMMA 4.1. Let $R^m = M \oplus N$, and x' (respectively x") be the coordinates in M (respectively in N). Let $\lambda:(x', x'') = (\lambda'(s), \lambda''(s))$, $s \in (0, 1]$ be some semi-algebraic curve in $R^m \setminus M$ such that $\lambda''(s) = 0(|\lambda'(s)|)$ for $s \to 0$ and $\lim_{s\to 0} \lambda''(s) \neq 0$. Let $l_\lambda(s)$ be the tangent line to λ at $(\lambda'(s), \lambda''(s))$. Then $\rho(l_\lambda(s), M) < c\rho(R_{\lambda(s)}, M)$ for $s \to 0$, where $0 < c < 1$.

Proof. The assertion is easily reduced to the case $\dim M = \dim N = 1$. It can be assumed that $\lambda'(s) \to +\infty$ for $s \to 0$. Then for sufficiently large x' we may assume that $\lambda : x'' = \lambda(x')$, where $\lambda(x') = o(x')$ for $x' \to +\infty$ and $\lim_{x'\to+\infty} \lambda(x') \neq 0$. Let $x'' = \sum_{i=i_0}^{\infty} c_i x'^{-i/k}$ be the Poissee expansion of the curve λ for $x' \to +\infty$. Then $-k < i_0 \leq 0$.

We now find that $\rho(R\lambda(x'), M) \sim |\lambda(x')/x'| \sim |c_{i_0}|x'^{-(i_0+k)/k}$, $\rho(l\lambda(x'), M) \sim |d\lambda(x')/dx'| \sim \frac{i_0}{k}|c_{i_0}| \cdot x'^{-(i_0+k)/k}$ for $i_0 \neq 0$, $\rho(l\lambda(x'), M) = o(x'^{-(i_0+k)/k})$ with $i_0 = 0$, whence the assertion of the lemma follows.

Definition 4.2. Let X be an algebraic variety in C^n, and Y' a non-singular sub-variety in p(X), Y = k(Y'). The pair (X, Y) is called a_∞-regular if the following condition is valid:

(a_∞). Let y_0 be any point in Y and let x_i be some sequence of simple points of the set X, such that $\lim_{i\to\infty}|x_i| = \infty$, $\lim_{i\to\infty} C_{x_i} = C_{y_0}$, and $\lim_{i\to\infty} T_{x_i}(X) = T$. Then $T \supset T_{y_0}(Y)$.

We note that condition (a_∞) is weaker than the requirement of Whitney regularity of the pair (X, Y') in CP^n. Example: $X = \{x-xy = 0\} \subset C^3$, $Y' = \{x = 0\} \subset CP^2$. Here the pair (X, Y') is Whitney regular but not a_∞-regular.

THEOREM 4.3. Let X be an algebraic variety in C^n, and Y' an algebraic sub-variety in p(X), Y = k(Y'). Then there exists some algebraic sub-variety $S' \subset Y'$, dim S' < dim Y', such that the pair (X, Y\k(S')) is a_∞-regular.

Proof. Let dim X = r and let $Z = \{x, T | x \in X_{sp}, T = T_x(X)\}$ be a set in $C^n \times G(n, r)$. It is easily proved that the closure $\overline{Z}$ of the set Z in $CP^n \times G(n, r)$ is an algebraic set. Let $Z' = \overline{Z} \cap (CP^{n-1} \times G(n, r))$. Let $Z'' = \{y, T | y \in Y'_{sp}, T \supset T_y(Y)\} \subset CP^{n-1} \times G(n, r)$ (here $T_y(Y)$ denotes the tangent plane to Y at the point $z : C_z = y$).

It is easily proved that $\overline{Z''}$ is an algebraic set. Let $\pi: CP^{n-1} \times G(n, r) \to CP^{n-1}$ be a projection operator and $S' = (\overline{\pi(Z'\setminus Z'')} \cup Y'_{sg}) \cap Y'$. Then S' is an algebraic sub-variety in Y', the pair (X, Y\k(S')) is a_∞-regular, and it suffices to prove that dim S' < dim Y'.

The standard method of hyperplane cross sections leads to a reduction of the problem to the case dim Y' = 1. We will assume now that dim Y' = dim S' = 1. Then there exists a semi-algebraic curve $\eta \subset Z'\setminus Z''$, which under the projection π is mapped in a one-to-one fashion onto some curve $\mu \subset Y'_{sp}$. For $y \in \mu$ we will write $T_y : (y, T_y) \in \eta$.

Let $y_0 \in \mu$ and assume that

$$W^1 = \{x \in X_{sp} \mid \exists y \in \mu : \rho(C_x, y) < (\rho(C_x, y_0))^2, \ \rho(T_x(X), T_y) < \rho(C_x, y_0)\},$$
$$W^2 = \{x \in X_{sp} \mid \rho(C_x, T_x(X)) < (\rho(C_x, y_0))^2\}, \qquad W = W^1 \cap W^2.$$

By the theorem of Tarskii and Zaidenberg it follows that W is a semi-algebraic set. We will show that W is non-empty and that $y_0 \in \overline{W}$ (the closure is taken with respect to CP^n). Let y be any point of the curve μ, $y \neq y_0$. Then the point (y, T_y) is the limit of points of the set Z in $CP^n \times G(n, r)$. Hence in X_{sp} there exists a semi-algebraic curve $\gamma : y = \gamma(s)$, $s \in (0, 1]$, such that $\lim_{s\to 0}(\gamma(s), T_{\gamma(s)}(X)) = (y, T_y)$. We find

1) $\lim_{s\to 0}\rho(C_{\gamma(s)}, y) = 0$, $\lim_{s\to 0}\rho(T_{\gamma(s)}(X), T_y) = 0$, $\lim_{s\to 0}\rho(C_{\gamma(s)}, y_0) = \rho(y, y_0)$, whence if s is sufficiently small $\gamma(s) \in W^1$.

2) From the fact that $l_\gamma(s) \subset T_\gamma(s)$ and $\lim_{s\to 0}\rho(l_\gamma(s), C_{\gamma(s)}) = 0$ it follows that $\lim_{s\to 0}\rho(C_{\gamma(s)}, T_{\gamma(s)}(X)) = 0$, while if s is sufficiently small $\gamma(s) \in W^2$.

Thus, if s is sufficiently small $\gamma(s) \in W$.

Consequently, the set W is nonempty and since y can be chosen arbitrarily close to y_0, $\overline{W} \in y_0$.

Let $\lambda : y = \lambda(s)$, $s \in (0, 1]$, be a semi-algebraic curve in W such that $\lim_{s\to 0}\lambda(s) = y_0$ in CP^n, and let $T'_\lambda(s)$ be the complex plane spanned by $C_{\lambda(s)}$ and $l_\lambda(s)$.

We will show that $\rho(C_{\lambda(s)}, y_0) < K\rho(l_\lambda(s), C_{\lambda(s)})$ for $s \to 0$. With this aim we now set $R^m = C^n$, $M = k(y_0)$ in the conditions of Lemma 4.1. If $\lim_{s\to 0}\lambda''(s) = 0$, then the assertion is obvious. If $\lim_{s\to 0}\lambda''(s) \neq 0$, then by applying Lemma 4.1 we obtain $\rho(l_\lambda(s), y_0) < c\rho(R_{\lambda(s)}, y_0) = c\rho(C_{\lambda(s)}, y_0)$ whence by the triangle inequality $\rho(l_\lambda(s), C_{\lambda(s)}) > (1-c)\rho(C_{\lambda(s)}, y_0)$.

Since $\lambda \in W^2$, it follows that $\rho(C_{\lambda(s)}, T_{\lambda(s)}(X)) < K^2(\rho(l_\lambda(s), C_{\lambda(s)}))^2$ for $s \to 0$, while since $l_\lambda(s) \subset T_{\lambda(s)}(X)$, we obtain $\lim_{s\to 0} T_{\lambda(s)}(X) \supset \lim_{s\to 0} T'_\lambda(s)$. But since $\lambda \in W^1$, $\lim_{s\to 0} T_{\lambda(s)}(X) = \lim_{y\in\mu, y\to y_0} T_y = T_{y_0} \not\supset T_{y_0}(Y)$, whence it suffices to prove that $\lim_{s\to 0} T_{\lambda}'(s) = T_{y_0}(Y)$. We will examine in $\mathbf{C}P^{n-1}$ the curve $\lambda' : \lambda'(s) = C_{x(s)}$. Since $\lambda \in W^1$, $\rho(\lambda'(s), \mu) < (\rho(\lambda'(s), y_0))^2$, whence $\lim_{s\to 0} l_{\lambda'}(s) = l_{\mu}(y_0)$. Since $\lim_{s\to 0} \lambda(s) = y_0$, it follows that $\lim_{s\to 0} T_{\lambda}'(s) = T_{y_0}(Y)$ $T_{y_0}(Y)$. The theorem is proved.

COROLLARY 4.4. Let X be an algebraic set in $\mathbf{C}^n$, $X' = p(X)$. Then there exists a stratification $\{X'_j\}$ of the set X' such that for any j the pair $(X, k(X_j))$ is a_∞-regular.

The corollary is easily obtained from Theorem 4.3.

Now let P be a polynomial in $\mathbf{R}^n$, X the set of its complex roots, $X' = p(X)$ the set of roots of the principal part of the polynomial P. Let $\{X_j\}$ be the a_∞-regular stratification of Corollary 4.4, $Y_{0,j} = X_j \cap \mathbf{R}^n$, $Y_{l,j} = (Y_{l-1,j})_{sg}$ and $X_{l,j} = Y_{l,j} \setminus Y_{l+1,j}$. Then $X^*(P) = (\bigcup_{l,j} \overline{NX_{l,j}})$ is a semi-algebraic set in $\mathbf{R}^n$ and $\dim X^*(P) < n$.

THEOREM 4.5. $X^*(P) \supset K^*(P)$.

The proof is analogous to that of Theorem 3.1.

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