AMBIENT LIPSCHITZ EQUIVALENCE OF REAL SURFACE SINGULARITIES

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Abstract. We present a series of examples of pairs of singular semialgebraic surfaces (germs of real semialgebraic sets of dimension two) in $\mathbb{R}^3$ and $\mathbb{R}^4$ which are bi-Lipschitz equivalent with respect to the outer metric, ambient topologically equivalent, but not ambient Lipschitz equivalent. For each singular semialgebraic surface $S \subset \mathbb{R}^4$, we construct infinitely many semialgebraic surfaces which are bi-lipschitz equivalent with respect to the outer metric, ambient topologically equivalent to S, but pairwise ambient Lipschitz non-equivalent.

1. Introduction

There are three different classification questions in Lipschitz Geometry of Singularities. The first question is the classification of singular sets with respect to the inner metric, where the distance between two points of a set X is counted as an infimum of the lengths of arcs inside X connecting the two points. The equivalence relation is the bi-Lipschitz equivalence with respect to this metric. The second equivalence relation is the bi-Lipschitz equivalence defined by the outer metric, where the distance is defined as the distance in the ambient space. It is well known that the two classifications are not equivalent. For example, all germs of irreducible complex curves are inner bi-Lipschitz equivalent, but the question of the outer classification is much more complicated (see Pham-Teissier [3] and Fernandes [1]). Here we consider another natural equivalence relation. Two germs of semialgebraic

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sets are called ambient Lipschitz equivalent if there exists a germ of a bi-Lipschitz homeomorphism of the ambient space transforming the germ of the first set to the germ of the second one. Two outer bi-Lipschitz equivalent sets are always inner bi-Lipschitz equivalent, but can be ambient topologically non-equivalent (see Neumann-Pichon [2]). The main question of the paper is the following. Suppose we have two germs of semialgebraic sets bi-Lipschitz equivalent with respect to the outer metric. Suppose that the germs are ambient topologically equivalent. Does it imply that the sets are ambient Lipschitz equivalent? In this paper, we present four examples of the germs of surfaces for which the answer is negative. The surfaces in Examples 1, 2 and 3 are ambient topologically equivalent and bi-Lipschitz equivalent with respect to the outer metric, but their tangent cones at the origin are not ambient topologically equivalent. By the theorem, recently proved by Sampaio [4], ambient Lipschitz equivalence of two sets implies ambient Lipschitz equivalence of their tangent cones. Thus the sets in our three examples cannot be ambient Lipschitz equivalent. In Example 4, the tangent cones of the two surfaces at the origin are ambient topologically equivalent. The argument in that case is more delicate and requires a special "broken bridge" construction. The last part of the paper is devoted to the proof of the main theorem of the paper: For any germ of a semialgebraic surface S in $\mathbb{R}^4$ there exist infinitely many semialgebraic surfaces, such that all these surfaces are ambient topologically equivalent to S, bi-Lipschitz equivalent with respect to the outer metric, but any two of them are not ambient Lipschitz equivalent. To prove this theorem, we generalize the broken bridge construction of Example 4.

The question on the relation of these classifications was posed to us by Alexandre Fernandes and Zbigniew Jelonek. We thank them for posing the question. We would like to thank Anne Pichon for her comments and suggestions.

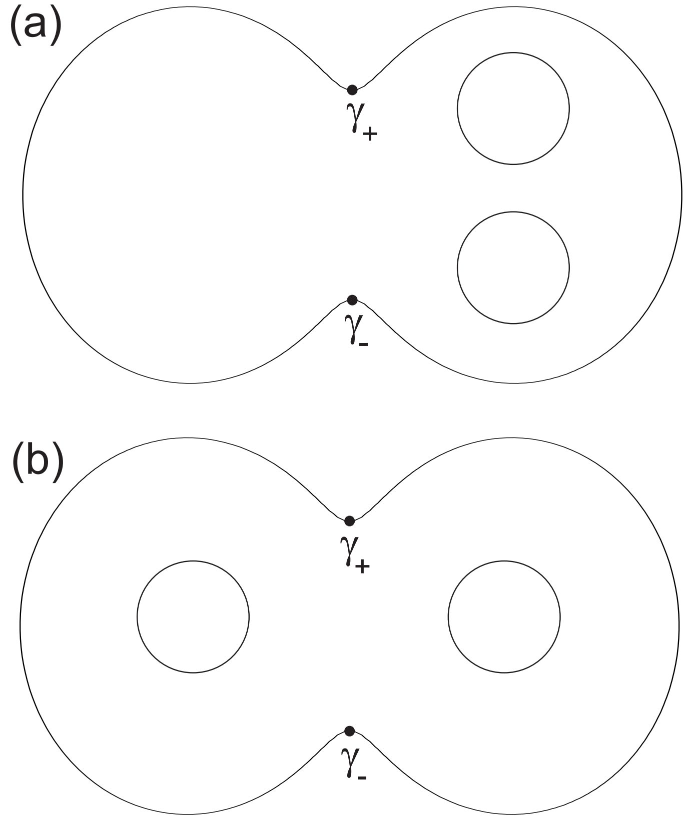

FIGURE 1. Links of the surfaces (a) X_1 and (b) X_2 in Example 1.

2. EXAMPLES IN $\mathbb{R}^3$

Example 1. Consider semialgebraic sets X_1 and X_2 in $\mathbb{R}^3$ (see Fig. 1) defined by the following equations and inequalities:

$$
\begin{gathered}
X_1 = \Big\{ \big((x^2 - 2xt + y^2)\,(x^2 + 2xt + y^2) - t^k\big) \times \\
\Big((x-t)^2 + \big(y - \tfrac{t}{2}\big)^2 - \tfrac{t^2}{16}\Big)\Big((x-t)^2 + \big(y + \tfrac{t}{2}\big)^2 - \tfrac{t^2}{16}\Big) = 0, \\
t \geq 0\Big\}.
\end{gathered} \tag{1}
$$

$$
\begin{gathered}
X_2 = \Big\{ \big((x^2 - 2xt + y^2)\,(x^2 + 2xt + y^2) - t^k\big) \times \\
\Big((x-t)^2 + y^2 - \tfrac{t^2}{16}\Big)\Big((x+t)^2 + y^2 - \tfrac{t^2}{16}\Big) = 0, \\
t \geq 0\Big\}.
\end{gathered} \tag{2}
$$

Here $k > 4$ is an integer.

Theorem 2.1. *The germs at the origin of the surfaces X_1 and X_2 are bi-Lipschitz equivalent with respect to the outer metric, ambient topologically equivalent, but not ambient Lipschitz equivalent.*

Proof. Notice that $X_1 = U_1 \cup U_2 \cup U_3$, where

$$U_1 = \left\{ \left((x-t)^2 + y^2 - t^2\right)\left((x+t)^2 + y^2 - t^2\right) = t^k,\ t \geq 0 \right\}$$

and the sets U_2, U_3 are straight cones over the circles $(x-1)^2 + (y-\frac{1}{2})^2 = \frac{1}{16}$ and $(x-1)^2 + (y+\frac{1}{2})^2 = \frac{1}{16}$. The set X_2 is the union of U_1 and the sets V_2, V_3 which are straight cones over the circles $(x-1)^2 + y^2 = \frac{1}{16}$ and $(x+1)^2 + y^2 = \frac{1}{16}$.

Notice that U_2, U_3, V_2 and V_3 are linearly (thus bi-Lipschitz) equivalent. In particular, there exist invertible linear maps $\varphi : U_2 \to V_2$ and $\psi : U_3 \to V_3$ (one can define $\varphi(x,y,t) = (x, y - \frac{t}{2}, t)$ and $\psi(x,y,t) = (x - 2t, y + \frac{t}{2}, t)$). Observe that $U_2 \cup U_3$ and $V_2 \cup V_3$ are normally embedded. Moreover, there exist positive constants c_1, c_2 such that for any point $p = (x,y,t) \in U_2 \cup U_3 \cup V_2 \cup V_3$ one has $c_1 t < d(p, U_1) < c_2 t$. Thus the map $\phi : X_1 \to X_2$ defined as

$$\phi(p) = \begin{cases} p & \text{if } p \in U_1 \\ \varphi(p) & \text{if } p \in U_2 \\ \psi(p) & \text{if } p \in U_3 \end{cases}$$

is bi-Lipschitz with respect to the outer metric.

The sets X_1 and X_2 are ambient topologically equivalent, each of them being equivalent to a cone over the union of three disjoint circles in the plane $t = 1$, two of them bounding non-intersecting discs inside a disc bounded by the third one.

However, the tangent cones to X_1 and X_2, defined by the homogeneous parts of degree 4 of (1) and (2), are not ambient topologically equivalent. The tangent cone of X_1 at the origin is the union of U_2, U_3 and a straight cone W over two tangent circles $(x-1)^2 + y^2 = 1$ and $(x+1)^2 + y^2 = 1$ in the plane $t = 1$, with the cones U_2 and U_3 inside one of the two circular cones of W, while the tangent cone of X_2 is the

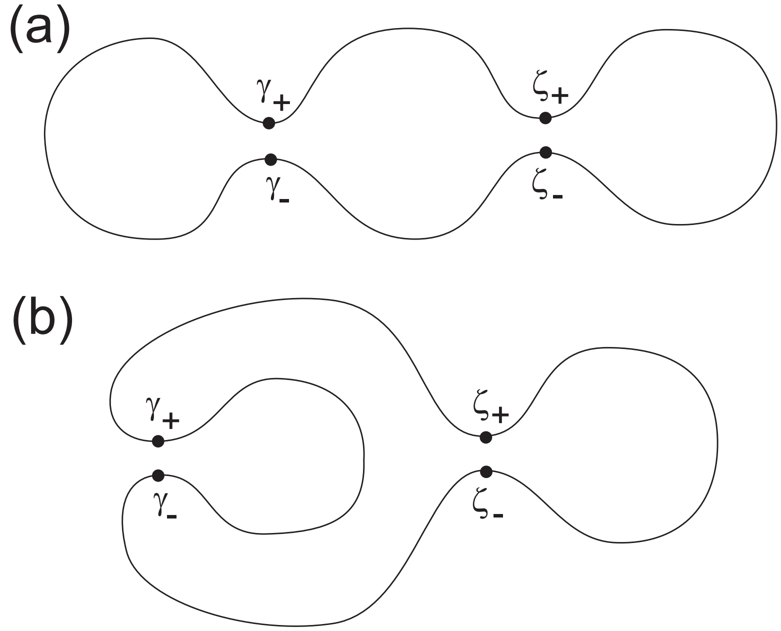

FIGURE 2. Links of the surfaces (a) X_1 and (b) X_2 in Example 2.

union of V_2, V_3 and W, with the cones V_2 and V_3 inside two different cones of W. Thus X_1 and X_2 are not ambient bi-Lipschitz equivalent, by the theorem of Sampaio [4]. This happens, of course, because U_1 is not normally embedded, with the arcs γ_+ and γ_- in $U_1 \cap \{x = 0\}$ (see Fig. 1) having the tangency order $k/4 > 1$. □

Example 2. Let X_1 and X_2 be semialgebraic surfaces in $\mathbb{R}^3$ with the links at the origin shown in Fig. 2, and tangent cones at the origin as in Fig. 3. One can define X_1 and X_2 by explicit semialgebraic formulas, similarly to the method employed in Example 1. Both surfaces X_1 and X_2 are ambient topologically equivalent to a cone over a circle. These surfaces are bi-Lipschitz equivalent with respect to the outer metric, but not ambient Lipschitz equivalent by Sampaio's theorem, since their tangent cones are not ambient topologically equivalent. The arguments are similar to those in Example 1.

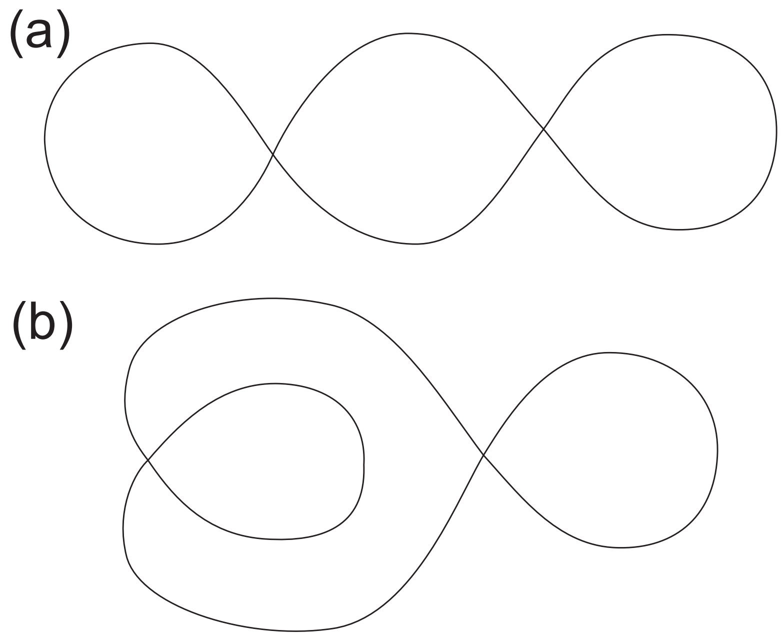

FIGURE 3. Links of the tangent cones at the origin of the surfaces (a) X_1 and (b) X_2 in Example 2.

3. EXAMPLES IN $\mathbb{R}^4$

Example 3. Let $H \subset \mathbb{R}^4$ be a surface defined as follows:

$$\left\{y^2 - x^2 = \left(x^2 + y^2 - 2t^2\right)^2,\ z = 0,\ |y| \le t \le 1\right\}.$$

The surface H has two branches, tangent at the origin. It is bounded by the straight lines

$$l_1 = (z = 0,\ y = x = t), \quad l_2 = (z = 0,\ y = -x = t),$$

$$l_3 = (z = 0,\ y = -x = -t), \quad l_4 = (z = 0,\ y = x = -t).$$

The tangent cone of H at the origin is the surface

$$\{y = \pm x,\ z = 0,\ |y| \le t\}.$$

The link of H (more precisely, the section of H by the plane $\{z = 0,\ t = 1/8\}$) is shown in Fig. 4. The arcs γ_+ and γ_- are tangent at the origin.

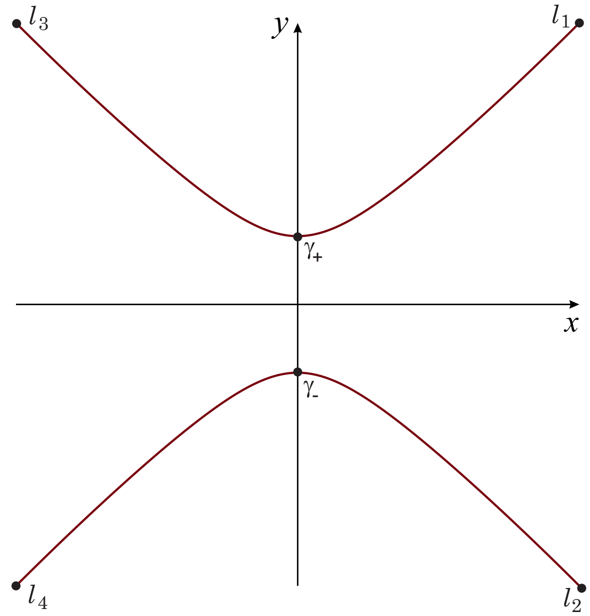

FIGURE 4. Link of the surface H in Example 3.

Let K_1, K_2, K_3 be nontrivial knots in $\mathbb{R}^3$ such that K_3 is a connected sum of K_1 and K_2. Let X_1 be a surface in $\mathbb{R}^4$, obtained as follows.

Consider a smooth semialgebraic embedding $\widetilde{K}_1$ of the knot K_1 to the hyperplane $\{t = 1\}$ in $\mathbb{R}^4_{x,y,z,t}$. Suppose that $\widetilde{K}_1$ contains the points $(1,1,0,1) \in l_1$ and $(1,-1,0,1) \in l_3$, and that $\widetilde{K}_1 \cap H$ contains only these points. Let $s_1 \subset \widetilde{K}_1$ be the segment connecting the points $(1,1,0,1)$ and $(1,-1,0,1)$ such that replacing this segment by a straight line segment does not change the embedded topology of $\widetilde{K}_1$. Let $\widetilde{K}_2$ be a smooth semialgebraic realization of K_2, in the same hyperplane of $\mathbb{R}^4$. Suppose that $\widetilde{K}_2$ contains the points $(-1,1,0,1) \in l_2$ and $(-1,-1,0,1) \in l_4$, and that a segment s_2 of $\widetilde{K}_2$ connecting these points may be replaced by a straight line segment without changing the embedded topology of $\widetilde{K}_2$. Suppose that $\widetilde{K}_2 \cap H$ contains only the points $(-1,1,0,1)$ and $(-1,-1,0,1)$, and that $\widetilde{K}_2 \cap \widetilde{K}_1 = \emptyset$.

Let K_1' be the straight cone over $\widetilde{K}_1 - s_1$ and let K_2' be the straight cone over $\widetilde{K}_2 - s_2$. Let $X_1 = K_1' \cup H \cup K_2'$. The link of the set X_1 is shown in Fig. 5a.

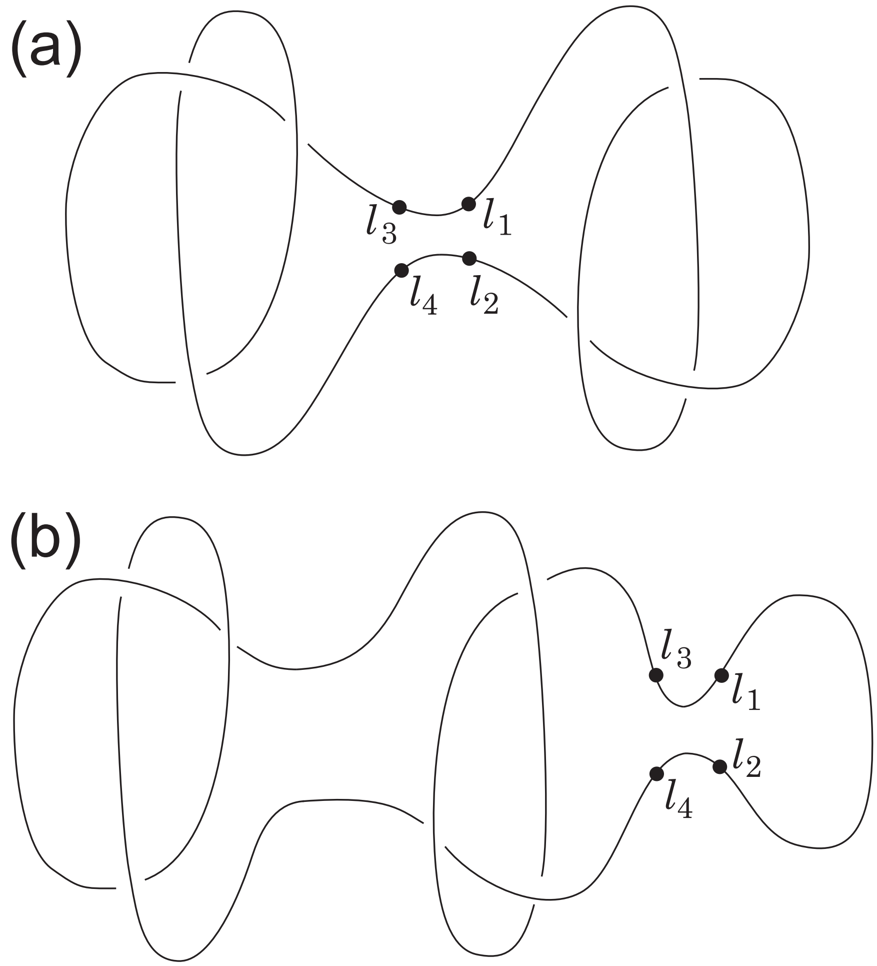

FIGURE 5. Links of the surfaces (a) X_1 and (b) X_2 in Example 3.

Let us define the set X_2 using the same construction as above, with the knot K_1 replaced by K_3 and the knot K_2 by the unknotted circle K_4. We assume, as before, that a smooth semialgebraic realisation $\widetilde{K}_3$ of K_3 contains points $(1, 1, 0, 1)$ and $(1, -1, 0, 1)$, that $\widetilde{K}_3 \cap H$ contains only these points, and that replacing the segment s_3 of $\widetilde{K}_3$ connecting these points by a straight line segment does not change the embedded topology of $\widetilde{K}_3$. Similar assumptions are made about a smooth semialgebraic embedding $\widetilde{K}_4$ of K_4 and its segment s_4 connecting the points

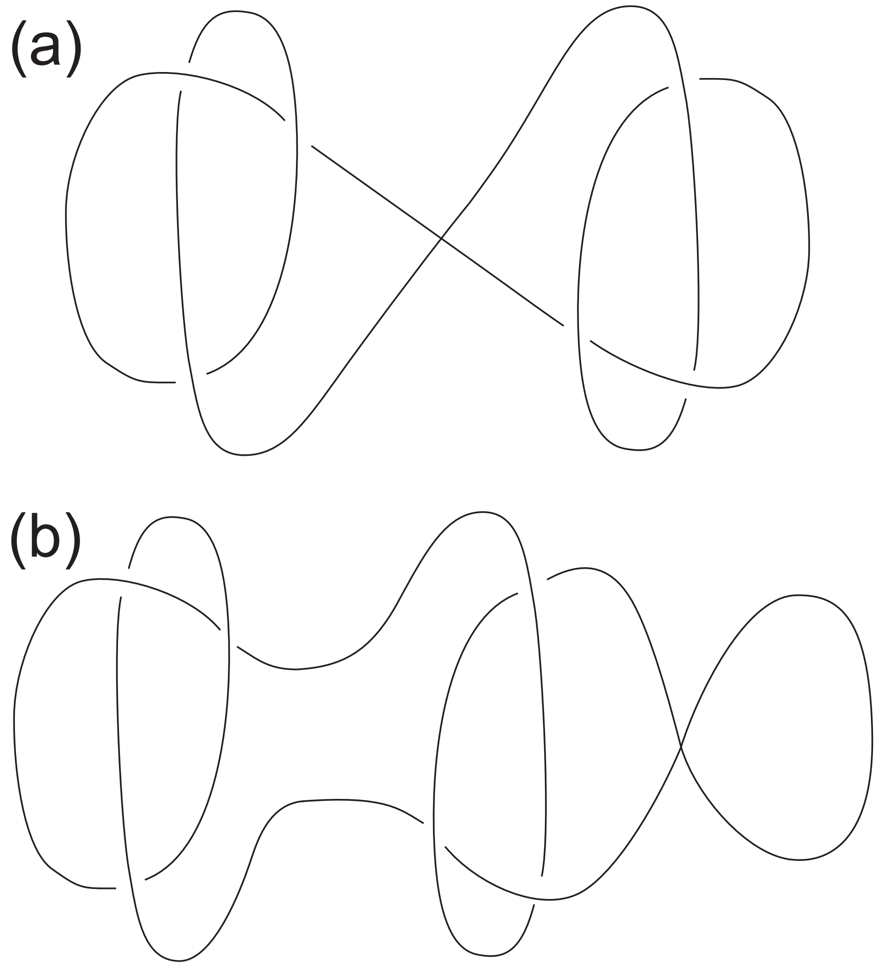

FIGURE 6. Links of the tangent cones at the origin of the surfaces (a) X_1 and (b) X_2 in Example 3.

$(-1,1,0,1)$ and $(-1,-1,0,1)$. Let K_3' and K_4' be the straight cones over $\widetilde{K}_3 - s_3$ and $\widetilde{K}_4 - s_4$. Let $X_2 = K_3' \cup H \cup K_4'$ (see Fig. 5b).

Theorem 3.1. *The germs of the sets X_1 and X_2 at the origin are bi-Lipschitz equivalent with respect to the outer metric, ambient topologically equivalent, but not ambient bi-Lipschitz equivalent.*

Proof. Since $\widetilde{K}_1$, $\widetilde{K}_2$, $\widetilde{K}_3$, $\widetilde{K}_4$ are smooth, the corresponding cones K_1', K_2', K_3', K_4' are normally embedded and bi-Lipschitz equivalent with respect to the outer metric. The bi-Lipschitz maps

$$\phi_1 : K'_1 \to K'_3 \ \text{ and } \ \phi_2 : K'_2 \to K'_4$$

can be chosen in such a way that

$$\phi_1(K'_1 \cap \{t = c\}) \ = \ K'_3 \cap \{t = c\} \quad \text{for all} \ \ c > 0, \ \ \text{and}$$

$$\phi_2(K'_2 \cap \{t = c\}) \ = \ K'_4 \cap \{t = c\} \quad \text{for all} \ \ c > 0.$$

Then one can define the map ϕ as follows:

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if} \quad x \in K'_1 \\ x & \text{if} \quad x \in H \\ \phi_2(x) & \text{if} \quad x \in K'_2 \end{cases}$$

Clearly, the map ϕ is a bi-Lipschitz map. The surfaces X_1 and X_2 are ambient topologically equivalent because their links are knots equivalent to K_3. From the other hand, the corresponding tangent cones at the origin are not ambient topologically equivalent: the tangent cone of X_1 is a straight cone over the union of K_1 and K_2, pinched at one point (see Fig. 6a), while the tangent cone of X_2 is a straight cone over the union of K_3 and the unknotted circle, pinched at one point (see Fig. 6b).

By the theorem of Sampaio [4], the surfaces X_1 and X_2 are not ambient bi-Lipschitz equivalent. □

Example 4.

For $1 \leq \beta < q$, define the set $A_{q,\beta} = T_+ \cup T_- \subset \mathbb{R}^4$, where

$$T_\pm = \left\{0 \leq t \leq 1, \ -t^\beta \leq x \leq t^\beta, \ y = \pm t^q, \ z = 0\right\}$$

are two normally embedded β-Hölder triangles tangent at the origin with the tangency exponent q. The set $A_{q,\beta}$ is called a (q,β)-bridge (see Fig. 7, left). The boundary of $A_{p,q}$ consists of the four arcs

$$\{t \geq 0, \ x = \pm t^\beta, \ y = \pm t^q\}.$$

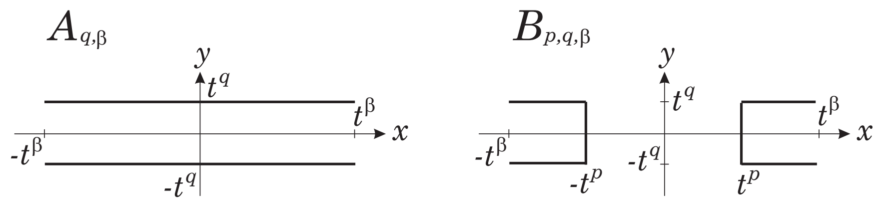

FIGURE 7. Links of a (q, β)-bridge $A_{q,\beta}$ and a broken (q, β)-bridge $B_{p,q,\beta}$.

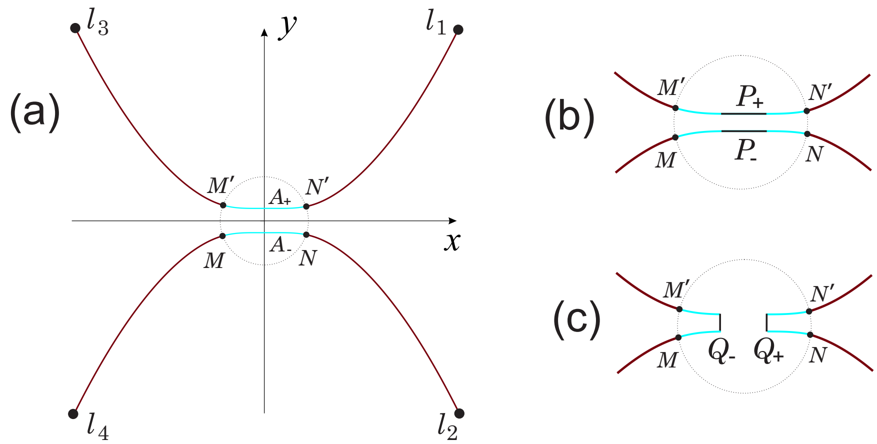

FIGURE 8. (a) Link of the surface G in Example 4. (b) Hölder triangles P_+ and P_-. (c) Broken $(3, 2)$-bridge B with the Hölder triangles Q_+ and Q_-.

For some p such that $\beta < p < q$, let $B_{p,q,\beta}$ be the set obtained from $A_{q,\beta}$ by removing from T_+ the p-Hölder triangle bounded by the arcs $\{t \geq 0,\ x = \pm t^p,\ y = t^q,\ z = 0\}$, and from T_- the p-Hölder triangle bounded by the arcs $\{t \geq 0,\ x = \pm t^p,\ y = -t^q,\ z = 0\}$), and replacing them by two q-Hölder triangles

$$\{0 \leq t \leq 1,\ x = t^p,\ -t^q \leq y \leq t^q,\ z = 0\} \quad \text{and}$$

$$\{0 \leq t \leq 1,\ x = -t^p,\ -t^q \leq y \leq t^q,\ z = 0\}\,.$$

The set $B_{p,q,\beta}$ is called a broken (q, β)-bridge (see Fig. 7, right).

Let $G \subset \mathbb{R}^4$ be a surface defined as follows:

$$\left\{y^2t^2 - x^4 = (x^2 + y^2 - 2t^2)^4,\ z = 0,\ |\, y \,| \leq t \leq 1\right\}.$$

The surface G has two branches, tangent at the origin. It is bounded by the straight lines

$$l_1 = (z = 0,\ y = x = t), \quad l_2 = (z = 0,\ -y = x = t),$$

$$l_3 = (z = 0,\ y = -x = t), \quad l_4 = (z = 0,\ y = x = -t).$$

The tangent cone of G at the origin is the surface

$$\left\{y^2 t^2 = x^4,\ z = 0,\ |y| \leq t\right\}.$$

The link of G (more precisely, the section of G by the plane $\{z = 0,\ t = 1/8\}$) is shown in Fig. 8a. The intersection of G with any surface $\{x = ct^\mu\}$, where $\mu \geq 2$, consists of two arcs having the tangency order 3. Thus G contains a subset A, consisting of two normally embedded 2-Hölder triangles A_+ and A_- (see Fig. 8a where $[M, N]$ is the link of A_- and $[M', N']$ is the link of A_+), which is ambient bi-Lipschitz equivalent to a $(3, 2)$-bridge. It is easy to check that such a subset A is unique up to a bi-Lipschitz homeomorphism of $\mathbb{R}^4$ preserving G.

Consider two trivial knots K_0 and K_1 embedded in the hyperplane $\{t = 1\} \subset \mathbb{R}^4_{x,y,z,t}$ as shown in Fig. 9a and Fig. 9b. Suppose that each of these two knots contains the four points

$$(1, 1, 0, 1) \in l_1,\ (1, -1, 0, 1) \in l_2,\ (-1, 1, 0, 1) \in l_3,\ (-1, -1, 0, 1) \in l_4,$$

and that the intersection of each of the two knots with the ball U of radius $\sqrt{2}$ in $\{t = 1\}$ consists of two unlinked segments s_1 and s_2 connecting $(1, 1, 0, 1)$ with $(-1, 1, 0, 1)$ and $(1, -1, 0, 1)$ with $(-1, -1, 0, 1)$, respectively, as shown in Figs. 9a and 9c, where U is shown as a dotted circle. We assume also that the union of the two segments s_1 and s_2 coincides with $G \cap \{t = 1\}$.

We define the surface X_0 as the union of G and a straight cone over $K_0 \setminus U$, and the surface X_1 as the union of G and a straight cone over $K_1 \setminus U$.

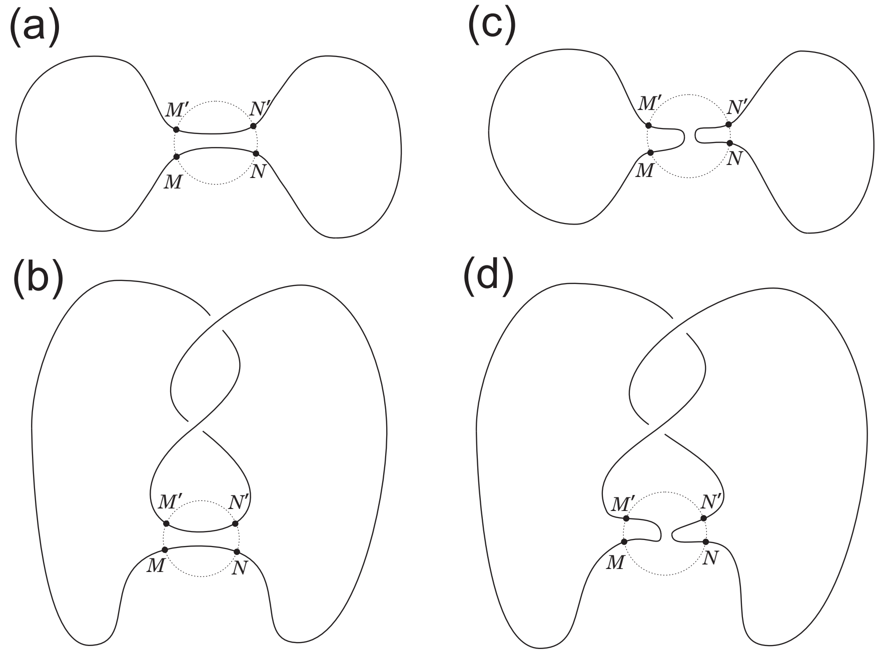

FIGURE 9. Links of the surfaces (a) X_0 and (b) X_1 in Example 4.

Theorem 3.2. *The germs of the surfaces X_0 and X_1 at the origin are bi-Lipschitz equivalent with respect to the outer metric, ambient topologically equivalent, but not ambient Lipschitz equivalent.*

Proof. Suppose that X_0 and X_1 are ambient Lipschitz equivalent. Let $h : \mathbb{R}^4 \to \mathbb{R}^4$ be a bi-Lipschitz homeomorphism such that $h(X_0) = X_1$. The set $A' = h(A)$ is ambient Lipschitz equivalent to a $(3,2)$-bridge. For any arc $\gamma \subset A'$ there is an arc $\gamma' \subset A'$ such that the inner distance in X_1 between γ and γ' has exponent 1, but the outer distance between them has exponent 3. No such arcs exist outside G. Due to the uniqueness of a $(3,2)$-bridge in G up to ambient Lipschitz equivalence, there is a bi-Lipschitz homeomorphism h' of $\mathbb{R}^4$ preserving G and mapping A' to A. Moreover, we may assume h' to be identity outside U, thus $h'(X_1) = X_1$. Combining h with h', we may assume that $h(A) = A$.

For $p \in (2,3)$, let $P \subset A$ be the union of two p-Hölder triangles P_+ and P_- that should be removed from A and replaced by two q-Hölder triangles Q_+ and Q_- to obtain a broken $(3,2)$-bridge B (see Figs. 8b and 8c). Define the surface $\widetilde{X}_0$ (see Fig. 9c) by replacing $A \subset X_0$ with B, and the surface $\widetilde{X}_1 = h(\widetilde{X}_0)$ (see Fig. 9d) by replacing $A = h(A) \subset X_1$ with $h(B)$. Then $\widetilde{X}_0$ and $widetildeX_1$ are not ambient topologically equivalent: the link of $\widetilde{X}_0$ consists of two unlinked circles, while the link of $\widetilde{X}_1$ consists of two linked circles. This contradicts our assumption that X_0 and X_1 are ambient Lipschitz equivalent. □

Remark 3.3. *Notice that the tangent cones of X_0 and X_1 are ambient topologically equivalent to a cone over two unknotted circles, pinched at one point. Thus Sampaio's theorem does not apply, and we need the "broken bridge" construction in this example. Notice also that the broken bridge construction employed in this example allows one to construct examples (both in $\mathbb{R}^3$ and in $\mathbb{R}^4$) of outer bi-Lipschitz equivalent, ambient topologically equivalent but ambient Lipschitz non-equivalent surface germs with the tangent cones as small as a single ray.*

Conjecture 3.4. *Let $(S_0, 0)$ and $(S_1, 0)$ be two normally embedded real semialgebraic surface germs which are ambient topologically equivalent and bi-Lipschitz equivalent with respect to either inner or outer metric (the two metrics are equivalent for normally embedded sets). Then S_0 and S_1 are ambient Lipschitz equivalent.*

4. Main result

Theorem 4.1. *For any semialgebraic surface germ $(S, 0) \subset \mathbb{R}^4$ there exist infinitely many semialgebraic surface germs $(X_i, 0) \subset \mathbb{R}^4$ such that*

1) *For all i, the germs $(X_i, 0)$ are ambient topologically equivalent to $(S, 0)$;*

2) *All germs $(X_i, 0)$ are bi-Lipschitz equivalent with respect to the outer metric;*

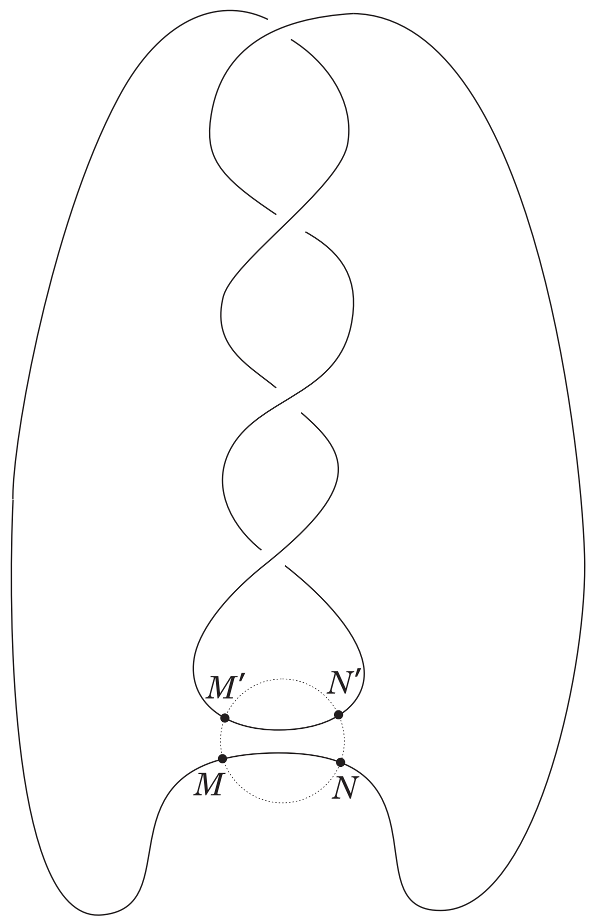

FIGURE 10. Link of the surface X_2 in Step 1.

3) *The tangent cones of all germs* X_i *at the origin are ambient topologically equivalent;*

4) *For* $i \neq j$ *the germs* X_i *and* X_j *are not ambient bi-Lipschitz equivalent.*

Step 1 Consider first the case where the link L of S is an unknotted circle. Our construction would be a modification of Example 4. The germ $(X_i, 0)$ is obtained from the surface G considered in Example 4 (see Fig. 8a) by attaching to it the straight cone over two segments in such a way that the braid connecting the pair of points (M, N) with the pair of points (M', N') has i twists. The germs $(X_0, 0)$ and $(X_1, 0)$ are

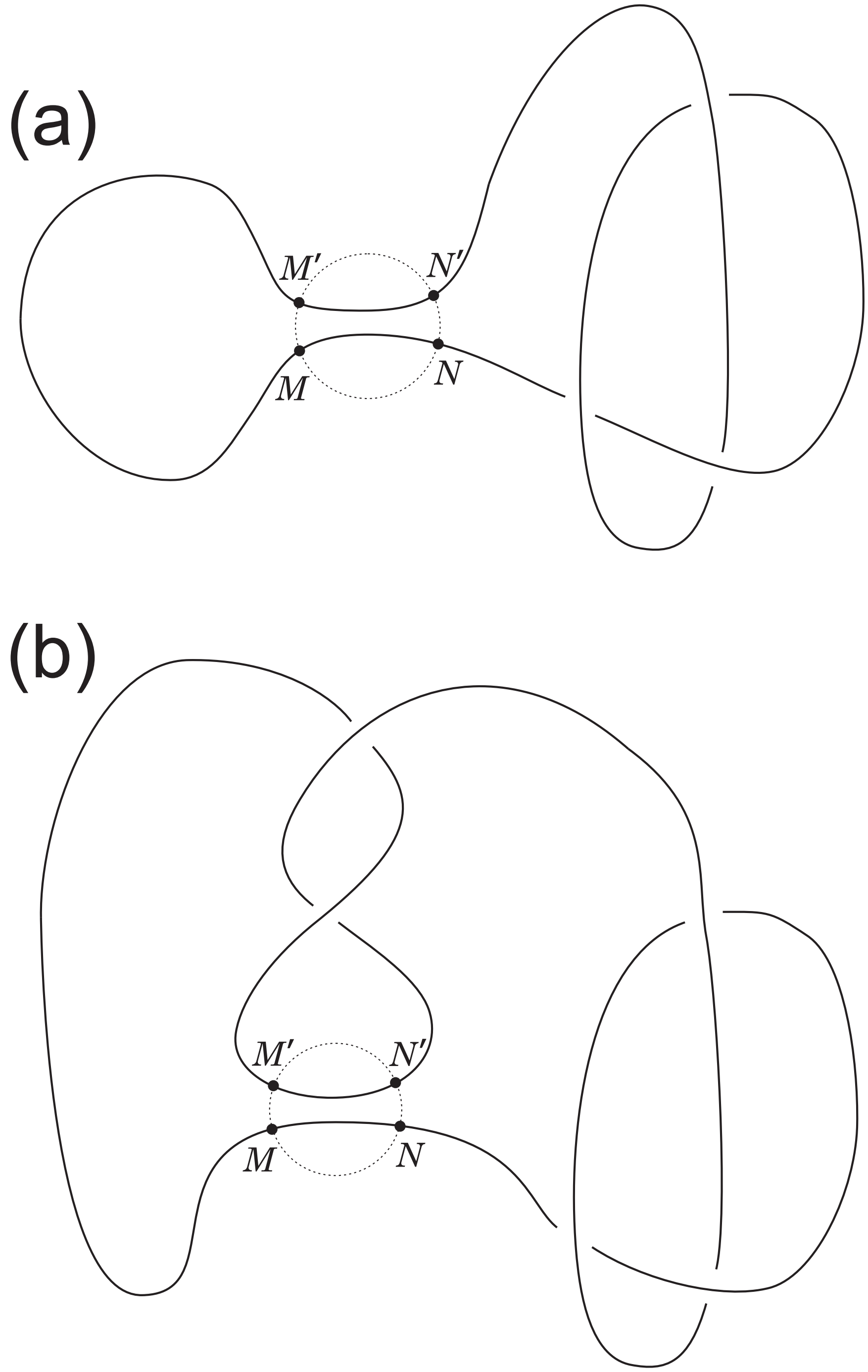

FIGURE 11. Links of the surfaces Y_0 and Y_1 in Step 2.

exactly those considered in Example 4. Their links are shown in Fig. 9a and Fig. 9b. The link of the germ $(X_2, 0)$ is shown in Fig. 10. All these links are unknotted, thus all the surfaces X_i are ambient topologically equivalent to the straight cone over the unknotted circle.

The same arguments as in the proof of Theorem 3.2 show that all germs $(X_i, 0)$ are bi-Lipschitz equivalent with respect to the outer metric. We now construct a bi-Lipschitz map $f_{ij} : (X_i, 0) \to (X_j, 0)$ which is the identity on the surface G. Since the complements of G in X_i and X_j are straight cones, the map f_{ij} on the cones can be defined as the conical extension of a bi-Lipschitz map on the links. The tangent cone of X_i at the origin is ambient topologically equivalent to the cone over two unknotted circles pinched at a point.

From the other hand, the broken bridge construction described in Example 4 transforms X_i into a set $\widetilde{X}_i$ with the link consists of two circles having the linking number i, same as the number of twists of the braid connecting (M, N) with (M', N'). This implies that X_i is not ambient bi-Lipschitz equivalent to X_j for $i \neq j$.

Step 2 Consider the case where the link L of the surface S has a subset L' which is a non-trivial knot. Consider the link L_i of the surface X_i constructed in Step 1. Since it is unknotted, its connected sum with L' is ambient topologically equivalent to L'. We define the surface Y_i so that its link is the link L of S with L' replaced by the connected sum of L' and L_i. The surface X' (except the cone over a segment of its link where L' is attached) is a subset of Y_i, and its complement in Y_i is a cone over the link L of S (except the cone over a segment of L' where it is attached to L_i). The links L_0 and L_1 from Example 4 with the trefoil knot attached are shown in Fig. 11a and Fig. 11b.

By the same arguments as in Step 1 we obtain the conclusion of Theorem 4.1 in this case.

Step 3 Consider the case where the link L of S is homeomorphic to a segment. Consider a germ $(X_i, 0)$, defined in Step 1. Let $T_\beta \subset X_i$ be a Hölder triangle with the vertex at the origin, a subset of the conical part of X_i.

Let $(Z_i, 0) = (X_i, 0) - T_\beta$. We claim that the germs $(Z_i, 0)$ satisfy the conditions 1, 2, 3, 4 of Theorem 4.1. The conditions 1 and 2 are evidently satisfied. The condition 3 is true because the tangent cones of $X_i$and Z_i at the origin are the same.

Suppose that $h_{ij} : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ is a bi-Lipschitz map, such that $h_{ij}(Z_i, 0) = (Z_j, 0)$, where $i \neq j$. Let us apply the map h_{ij} to $(X_i, 0)$. Notice that the set $h_{ij}(X_i)$ is different from X_i only in a small β-horn containing the Hölder triangle T_β.

If we apply the broken bridge construction to $h_{ij}(X_i)$, we get the same linking number for the two components as in Theorem 3.2.

Since the link of any surface germ either has a connected component homeomorphic to a segment, or contains a subset which is a (possibly trivial) knot, this completes the proof of Theorem 4.1.

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