Zeros of eigenfunctions of some anharmonic oscillators

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Abstract

We study complex zeros of eigenfunctions of second order linear differential operators with real even polynomial potentials. For potentials of degree 4, we prove that all zeros of all eigenfunctions belong to the union of the real and imaginary axes. For potentials of degree 6, we classify eigenfunctions with finitely many zeros, and show that in this case too, all zeros are real or pure imaginary.

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1. Introduction

We consider eigenvalue problems of the form

$$-y'' + P(z)y = \lambda y, \quad y(-\infty) = y(\infty) = 0, \tag{1}$$

where P is a real even polynomial with positive leading coefficient, which is called a potential. The boundary condition is equivalent to $y \in L^2(\mathbf{R})$ in this case. It is well-known that the spectrum is discrete, and all eigenvalues λ are real and simple, see, for example [2, 14]. The spectrum can be arranged in an increasing sequence $\lambda_0 < \lambda_1 < \dots$.

Eigenfunctions y are real entire functions of order $(\deg P + 2)/2$ and each of them has finitely many real zeros. The number of real zeros of an eigenfunction is equal to the subscript of the corresponding eigenvalue λ_k. Asymptotic behavior of complex zeros of eigenfunctions is well-known, in particular, their arguments accumulate to finitely many directions, the so-called Stokes' directions, [1]. Using this one can show that for a real even potential P of degree 4 with positive leading coefficient, *all but finitely many* zeros of each eigenfunction lie on the imaginary axis. See also [17, 18] where a similar result was obtained for some cubic potentials.

Theorem 1. *Let P be a real even polynomial of degree 4 with positive leading coefficient. Then all non-real zeros of eigenfunctions y of the problem (1) belong to the imaginary axis.*

Under the assumptions of Theorem 1, every eigenfunction has infinitely many imaginary zeros.

For the special case that $P(z) = cz^4 + d$, $c > 0, d \in \mathbf{R}$, Theorem 1 was conjectured by Titchmarsh [16, p. 147] and proved by Hille [7, p. 617-618], see also [8, p. 188-190]. Operator considered in Theorem 1 with $P(z) = cz^4 + bz^2$, $c > 0, b \geq 0$ plays an important role in physics. It is usually called a quartic anharmonic oscillator. If $b < 0$, the potential P is also called a double well potential. A very brief survey of known results about this operator is contained in Chapter I of [23].

Once Theorem 1 is established, one can obtain additional information on the location of zeros by the standard methods.

Corollary 1. *For the potential $P(z) = z^4 + bz^2$, $b \in \mathbf{R}$, all eigenvalues satisfy $\lambda > -b^2/4$, so the equation*

$$P(z) - \lambda = 0 \qquad (2)$$

has real roots. All real roots of the eigenfunction of (1) belong to the interval $(-x_1, x_1)$ where x_1 is the largest positive root of (2). If ix_2 is a pure imaginary root of (2), then all non-real roots of y belong to the union of the rays $(-i\infty, -ix_2) \cup (ix_2, i\infty]$.

One can obtain a density of the asymptotic distribution of zeros of the n-th eigenfunction as $n \to \infty$. We will do this in a forthcoming paper [4].

For some potentials P, there exist eigenfunctions with finitely many zeros.

Such eigenfunctions have the form

$$y(z) = Q(z) \exp T(z), \tag{3}$$

with polynomials Q and T. For example, this is the case when P is of degree 2; then *all* eigenfunctions are of the form (3), and Q's are the Hermite polynomials. Eigenfunctions with finitely many zeros can also occur for polynomials P of higher degree, and these situations are of interest to physicists and mathematicians [9, 13, 19, 20, 22, 23]. It is easy to see that eigenfunctions of the form (3) can exist only in the case that $\deg P \equiv 2 \,(\mathrm{mod}\ 4)$. Moreover, for every d such that $d \equiv 2 \,(\mathrm{mod}\ 4)$ there exist real polynomials P of degree d such that some eigenfunctions have the form (3), see [5].

Theorem 2. *Let P be a real even polynomial of degree 6 with positive leading coefficient. If (3) is an eigenfunction of (1), then all non-real zeros of Q belong to the imaginary axis.*

Since the union of the real and imaginary axes does not contain any Stokes directions for a sextic potential P (see Section 2), we conclude that eigenfunctions of a sextic potential with infinitely many zeros cannot have all zeros in the union of the real and imaginary axes.

In the proof of Theorem 2 we obtain a classification of eigenfunctions (3) which can occur in operators (1) with even sextic potential. It turns out that this classification fits the classification of the so-called "quasi-exactly solvable" sextic potentials [9, 23]. As a corollary we obtain in Section 6 that for even sextic potentials, eigenfunctions (3) can occur only for Lie-algebraic quasi-exactly solvable sextic potentials listed in [20, 9, 23]. More precisely, Theorem 2 combined with the results of Turbiner and Ushveridze [22], [23] gives the following

Corollary 2. *Let P be a real even polynomial of degree 6, and suppose that problem (1) has at least one solution y of the form (3). Then*

$$P(z) = c^2 z^6 + 2bc z^4 + \{b^2 - c(4m + 2p + 3)\} z^2 + \mathrm{const},$$

where $c \in \mathbf{R} \backslash \{0\}$, $b \in \mathbf{R}$, $p \in \{0, 1\}$ and m is a non-negative integer.

It was shown by Turbiner and Ushveridze [22] that these potentials have exactly $m + 1$ linearly independent eigenfunctions of the form (3). They correspond to the first $m + 1$ even numbered eigenvalues if $p = 0$ and to the first $m + 1$ odd-numbered eigenvalues if $p = 1$.

The proofs of theorems 1 and 2 are of purely topological nature, they are based on the study of the action of the symmetry group $\mathbf{Z}_2 \times \mathbf{Z}_2$ of the problem (1) on certain partitions of the complex z-plane associated with the eigenfunctions.

Phenomenon described in Theorems 1 and 2 occurs only for potentials of degrees 4 and 6:

Theorem 3. *For every $k \geq 2$, there exists a real even polynomial P of degree $4k + 2$ with positive leading coefficient, such that the problem (1) has an eigenfunction of the form (3), but the zero set of Q is not a subset of the union of the real and imaginary axis.*

Polynomial P in Theorem 3 does not belong to the classification of quasi-exactly solvable potentials that arise from finite-dimensional Lie algebras of differential operators in [9].

2. Preliminaries

From now on, we always assume that P is real, even and has positive leading coefficient. We denote $d = \deg P$.

Making the change of the independent variable $z \mapsto -z$ we conclude that every eigenfunction is either even or odd. We normalize even eigenfunctions by the condition
$$y(0) = 1,$$
and the odd ones by the condition

$$y'(0) = 1.$$

Consider another solution y_1 of the differential equation in (1) normalized by

$$y_1(0) = 0, \quad y_1'(0) = 1$$

in the case that the eigenfunction y is even, and

$$y_1(0) = 1, \quad y_1'(0) = 0$$

4

in the case that the eigenfunction y is odd. Then y_1 is even or odd, and its parity is opposite to the parity of y. Thus the meromorphic function $f = y/y_1$ is real and odd, in particular it is symmetric with respect to both real and imaginary axes: if we denote the reflections with respect to the coordinate axes by

$$R(z) = \overline{z} \quad \text{and} \quad I(z) = -\overline{z},$$

then

$$f \circ R = R \circ f \quad \text{and} \quad f \circ I = I \circ f. \tag{4}$$

The following facts are well-known [11, 12, 14]. The rays

$$\rho_j = \{t \exp\{\pi i(2j-1)/(d+2)\} : 0 < t < \infty\}, \quad 0 \le j \le d+1$$

are the Stokes' directions. They divide the plane into $d+2$ sectors S_j, where S_j is bounded by ρ_j and ρ_{j+1}. In each sector, each non-zero solution of the differential equation in (1) exponentially tends either to 0 or to ∞, (on every ray from the origin in this sector), in particular, $y(z) \to 0$ in S_0 and $S_{d/2+1}$ in view of the boundary conditions in (1), while y_1 tends to ∞ in these two sectors.

Notice that the set of zeros and the set of poles of f are both invariant with respect to R and I.

Meromorphic function f is of order $(d+2)/2$, has no critical points (which means that $f'(z) \neq 0$ and all poles are simple) and the set of its asymptotic values is finite. Such functions have been studied in great detail in [5, 6, 10, 11, 12, 14]. If A is the set of asymptotic values, then the restriction

$$f : \mathbf{C} \backslash f^{-1}(A) \to \overline{\mathbf{C}} \backslash A$$

is an (unramified) covering, and also f is unramified at preimages of A.

In each sector S_j, the function f tends to an asymptotic value a_j exponentially, and the asymptotic values in adjacent sectors are distinct. More precisely, for every sufficiently small $\epsilon > 0$,

$$f(re^{i\theta}) \to a_j, \quad r \to \infty,$$

uniformly with respect to $\theta \in [\rho_j + \epsilon, \rho_{j+1} - \epsilon]$. We have $a_0 = a_{d/2+1} = 0$, and the symmetry properties (4) imply that $a_j = R(a_{-j})$, and $a_j = I(a_{d/2+1-j})$. Here we understand the index j as a residue modulo $d + 2$.

The only singularities of the inverse function f^{-1} are logarithmic branch points; they all lie over the asymptotic values. The total number of the logarithmic branch points is $d+2$, and they correspond to the $d+2$ sectors S_j.

As an example, consider the situation in Theorem 1, where $d + 2 = 6$. If we denote $a = a_1$ then the symmetry relations (4) imply that the asymptotic values are

$$(a_0, a_1, a_2, a_3, a_4, a_5) = (0, a, I(a), 0, -a, R(a)).$$

Now, the condition that $a_1 \neq a_2$ implies that a cannot belong to the imaginary axis. We will later see in the course of the proof of Theorem 1 that a cannot be real.

In Theorem 2, we have $d + 2 = 8$ and the form of the eigenfunction (3) shows that $a_{2k} = 0$, $k = 0, 1, 2, 3$. Denoting $a_1 = a$ again, we obtain from the symmetry relations (4) that

$$(a_1, a_3, a_5, a_7) = (a, I(a), -a, R(a)).$$

We will see in the course of the proof of Theorem 2 that a can be neither real nor pure imaginary. We conclude that in both theorems 1 and 2 f has five asymptotic values,

$$a, I(a), -a, R(a) \quad \text{and} \quad 0. \tag{5}$$

To study topological properties of the function f one considers the pull-back by f of an appropriate cell decomposition of the Riemann sphere. The usual choice of this cell decomposition leads to an object which is called the line complex [3, 5, 6, 10, 11, 12]. However classical line complexes are not convenient for our purposes because they do not reflect the symmetry relations (4), see, for example [5]. So in the main part of the proofs of theorems 1 and 2 (Sections 3 and 4) we use slightly different approach.

However we find it more convenient to use the standard line complexes in the proof of Theorem 3. So we recall the definition of a line complex in the beginning of Section 5.

3. Common part of the proofs of Theorems 1 and 2

In this Section, f is a meromorphic function of finite order, with no critical points and five asymptotic values as in (5), where a is neither real

6

nor imaginary, and satisfies the symmetry conditions (4). We will treat the simpler case of real or imaginary a separately, in the end of Section 4.

We will work with partitions of a topological space X which can be either the plane $\mathbf{C}$ or the Riemann sphere $\overline{\mathbf{C}}$ into subsets which we call vertices, edges and faces. All our partitions are locally finite, that is every point in X has a neighborhood that intersects only finitely many edges, faces and vertices.

A *vertex* is just a point in X. An *edge* in X is the image of the interval $(0,1)$ under a continuous map $\phi : [0,1] \to X$ whose restriction on $(0,1)$ is injective. The points $\phi(0)$ and $\phi(1)$ are the endpoints of the edge (they do not belong to the edge but always belong to X). The endpoints may be equal. We also say that the edge connects $\phi(0)$ with $\phi(1)$. The degree of a vertex x of a partition is defined as the total number of ends of edges whose endpoints coincide with this vertex. Thus an edge with $x \in \{\phi(0), \phi(1)\}$, may contribute one or two units to the degree of x, one if $\phi(0) \neq \phi(1)$ and two if $\phi(0) = \phi(1)$. A *face* is a simply connected domain in X whose boundary is locally connected.

A *partition* is a representation of X as a locally finite disjoint union of faces, edges and vertices, such that all endpoints of all edges are vertices, and the boundary of every face consists of edges and vertices. We *do not* require that the closure of a face be homeomorphic to a closed disc.

We begin with the partition C of the Riemann sphere $\overline{\mathbf{C}}$ which consists of:

- One vertex, ∞,

- Four edges L_k, $1 \leq k \leq 4$, each beginning and ending at ∞, and such that L_k separates one of the asymptotic values $a, I(a), -a, R(a)$ from all other asymptotic values. Moreover, we require that the union of these edges be invariant with respect to R and I, more precisely, $L_2 = I(L_1)$, $L_3 = -L_1$ and $L_4 = R(L_1)$.

- Five faces D_k, $0 \leq k \leq 4$ which are the components of $\mathbf{C} \backslash \cup_{k=1}^{4} L_k$. We enumerate them so that $0 \in D_0$, and $\partial D_k = L_k \cup \{\infty\}$, $1 \leq k \leq 4$.

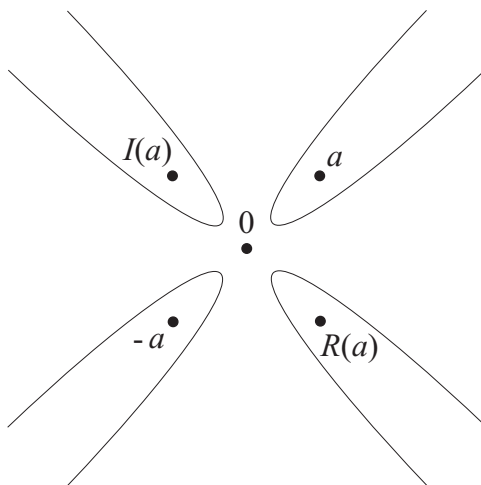This partition is shown in Figure 1. In this illustration, a belongs to the first quadrant.

Fig. 1. Partition C of the Riemann sphere.

Now we consider the preimage of C under f. This is a partition Φ of the plane. Faces, edges and vertices of Φ are defined as components of the f-preimages of faces, edges and vertices of C.

So the vertices of Φ coincide with the poles of f. The degree of each vertex is 8, the same as the degree of the vertex in the partition C.

The edges of Φ are disjoint curves connecting vertices. The edges can be of two types: an edge connecting a vertex to itself is called a *loop*, and an edge connecting two distinct vertices is called a *simple edge*.

Let F be a face of Φ which is mapped to some D_k. Then

$$f : F \backslash f^{-1}(a_k) \to D_k \backslash \{a_k\}$$

is a covering map. Since f has no critical points, this covering can have degree either 1 or ∞. It is easy to see that F is bounded if and only if the degree of this covering is 1 ([12, XI,§1]). Accordingly, we can classify the faces of Φ into three types:

(i) Bounded faces whose boundaries consist of a loop and a vertex. We call them *loop-faces*. The closure of each loop-face is mapped by f homeomorphically onto the closure of one of the D_k with $1 \le k \le 4$.

(ii) Bounded faces which are mapped by f homeomorphically onto D_0. (This homeomorphism does not necessarily extend to a homeomorphism between the closures!) Each of these faces contains exactly one zero of f, and each zero of f belongs to one of these faces. We call them *zero-faces*.

8

(iii) Unbounded faces. The restriction of f onto an unbounded face is a universal covering over $D_k \backslash b_k$ for some $k \in \{0, \ldots, 4\}$. Here b_k is the asymptotic value which is contained in D_k. The boundary of each unbounded face consists of countably many edges and countably many vertices.

We label all faces by the corresponding asymptotic values: if a face is a component of the preimage of D_k then its label is the asymptotic value $b_k \in D_k$. Bounded faces labeled by b_k with $1 \le k \le 4$ are the loop-faces, while bounded faces labeled by 0 are the zero-faces.

We will need the following properties of the partition Φ.

1. Every edge belongs to the boundaries of two faces, one of them labeled by 0 and another has a non-zero label.

2. The 1-skeleton[1] of Φ is connected. Indeed, the loops L_k, $1 \le k \le 4$ generate the fundamental group of $\overline{\mathbf{C}} \backslash \{b_0, \ldots, b_4\}$. As

$$f : \mathbf{C} \backslash f^{-1}(\{b_0, \ldots, b_4\}) \to \overline{\mathbf{C}} \backslash \{b_0, \ldots, b_4\}$$

is a covering, we conclude that every pair of poles of f can be connected by a curve which belongs to the 1-skeleton of Φ.

3. Every edge connecting two different vertices belongs to the boundary of some unbounded face. Indeed, suppose that an edge e connecting two different vertices belongs to the boundaries of two bounded faces F_1 and F_2. As the labels of these two faces are distinct (by property 1 above), one of these labels is not 0. But a bounded face whose label is not 0 has to be a loop-face (see (i)). So e has to be a loop, which contradicts the assumption.

Now we transform our partition Φ into a new one whose 1-skeleton is a tree. By a tree we mean a closed subset of the plane which is contractible and connected. A union of disjoint trees is called a forest.

This is done in two steps.

Step 1. Remove all loop edges. The resulting partition of the plane is called Φ'. Each face F' of Φ' is a union of a face F of Φ with some loops and loop-faces of Φ. We label F' in Φ' by the same label as F had in Φ. So all bounded faces of Φ' are now labeled by 0. It is easy to see that the faces of Φ' are Jordan regions with at least two boundary edges and at least two boundary vertices. It easily follows from the property 2 of Φ that Φ' has connected

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[1] The union of edges and vertices

1-skeleton. Moreover, it is evident that Φ' is invariant with respect to both I and R, because Φ had this property, and the union of all loop faces and loop edges is also R- and I-invariant.

Step 2. Every bounded zero-face F' of Φ' contains a unique zero of f. We call this zero an o-vertex and define a new partition Φ'' of the plane in the following way. The vertices of the new partition are the vertices of Φ', which we call now x-vertices, and the new vertices which are called o-vertices. In other words, o-vertices are the zeros of f and x-vertices are the poles of f. To define the edges of Φ'' we connect each o-vertex in a zero-face F' of Φ' to each x-vertex on the boundary of F' by a new edge inside F', so that these new edges are disjoint. This is possible to do because the closure of F' is locally connected. Then we remove all edges of Φ' on the boundary of F'. We perform this operation on every zero-face F' of Φ'.

Let us show that on Step 2, we can choose the new edges of Φ'' in such a way that Φ'' is symmetric with respect to both R and I. Indeed, there are three possibilities for the orbit of a zero-face F' of Φ' under the action of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ generated by I and R: the orbit of F' can consist of one, two or four faces. We consider these possibilities separately.

If the orbit consists of 4 elements then F' is neither R- nor I-invariant. We choose the edges connecting the o-vertex in F' to the boundary x-vertices arbitrarily (with the only condition that they are disjoint), and then in other faces of the orbit of F' we use the images of these edges under the action of the group.

Suppose now that the orbit of F' consists of two elements, for example, F' is R-invariant but not I-invariant. We first define the new edges in F', so that the union of these new edges is R-invariant. This can be done if we notice that an R-invariant simply connected region intersects the real line by an interval. If some endpoint x of this interval is a x-vertex, we connect x to o by an interval of the real line. If $x \in \partial F'$ is a x-vertex in the upper half-plane, we connect it with o by a curve in the intersection of F' with the upper half-plane, so that these curves for different x-vertices are disjoint. (We use here the fact that an intersection of a simply connected R-symmetric region with the upper half-plane is always connected, and its boundary is locally connected if the boundary of F' is.) Finally if $x \in \partial F'$ is in the lower half-plane, we use the R-image of the edge connecting o with $R(x)$. Then we define the new edges in the other face $I(F')$ of the orbit of F' as the I-images of the edges in F'. The procedure for an I-invariant but

not R-invariant face is the same. There are no other possibilities for an orbit of two elements: if $F' = -F'$ then F' has to be invariant with respect to the whole group, since F' is simply connected and a centrally symmetric simply connected region has to contain 0.

The remaining case of F' which is both R- and I- invariant is treated similarly. The intersection of such face with the coordinate cross $\mathbf{R} \cup i\mathbf{R}$ consists of the union of two symmetric intervals, one on the real axis, another on the imaginary axis. To the x-vertices at the endpoints of these intervals (if there are any such vertices) we draw straight edges from o. Then we notice that the intersection of F' with the first quadrant is connected and has locally connected closure. So we can draw the edges from o to the x-vertices in the first quadrant so that these edges are contained in the first quadrant. The remaining edges in F' are the images of those in the first quadrant under the symmetry group action.

The following Proposition summarizes the needed properties of Φ''

Proposition 1. *The partition Φ'' has the following properties.*
a) Its 1-skeleton is an infinite tree properly embedded in the plane.
b) Every edge belongs to the boundaries of two faces with distinct labels.
c) An o-vertex cannot belong to the boundary of a face labeled 0.
d) Each edge either belongs to the boundary of a face with label 0 or has an o-vertex as one of the extremities.
e) Φ'' has $d+2$ ends and $d+2$ faces.
f) Faces F_0 and $F_{d/2+1}$ in a counter-clockwise order have labels 0 and these faces are interchanged by I.

Proof. First we prove that Φ'' has no bounded faces. Suppose that F' is a bounded face of Φ'. It is a Jordan region with some number $k \geq 2$ boundary edges (see the description of Step 2). On Step 2 we replaced F' by certain number of triangles (=regions bounded by three edges), one triangle for each boundary edge of F', and then glued this triangle to some face F_0' of Φ', exterior to F', along this edge. This face F_0' was unbounded by property 3 above. So it remains unbounded after adding triangles along some edges. So step 2 destroys all bounded faces and does not create new ones. Thus the 1-skeleton of Φ'' is a forest. That it is connected, follows from connectedness of the 1-skeleton of Φ'; evidently Step 2 does not destroy connectedness.

It follows that the 1-skeleton of Φ'' is a tree. As the vertices and edges accumulate only to infinity, the tree is properly embedded in $\mathbf{C}$. This proves a).

11

To prove b), we first notice that each edge of Φ'' that comes from Φ' belongs to the boundaries of two faces with distinct labels (property 1 above). So it remains to prove b) for the new edges added on Step 2. Let e be an edge added on step 2. It connects an o and x inside a zero-face F' of Φ'. Let e_1 and e_2 be the two edges on the boundary of F' incident to the common vertex x. Then e_1 and e_2 are f-preimages of two different edges L_k and L_j of C, and thus the faces F_1' and F_2' exterior to F' that have e_1 and e_2 on their boundaries have distinct labels. Thus e is a common boundary edge of two faces with distinct labels. This proves b).

To prove c), we consider an o-vertex and the bounded face F' of Φ' which contains this vertex. This face F' is labeled by 0 (as all bounded faces of Φ' are), and thus there cannot be an edge in Φ' in the common boundary of F' and another face labeled by 0. It follows that Step 2 cannot produce an o-vertex on the boundary of a face with label 0.

To show d), we recall property 1 of the partition Φ. iIt is clear that Φ' also has this property. On Step 2, all edges which belong to the boundaries of zero-faces were removed and new edges added, each having a o-vertex as an extremity.

To prove e) and f), we notice that the number of unbounded faces and the labeling of these faces do not change when we perform steps 1 and 2. Now statement e), as well as f) follow from the asymptotic properties of f stated in the Preliminaries: each unbounded face (of any partition Φ, Φ' or Φ'') is asymptotic to one of the sectors S_j where $f(z)$ tends to the label of this face. The total number of sectors is $d+2$, and the bisectors of two of them are the positive and negative rays of the real line.

This completes the proof of Proposition 1.

In the next Section we will classify all trees satisfying the conclusions of the Proposition 1 with $d = 4$. For $d = 6$ we will use an additional property that arises from (3): that every other face of Φ'' is labeled by 0.

It will result from this classification that all trees arising from the eigenfunctions in theorems 1 and 2 as 1-skeletons of Φ'' have the following property: *each o-vertex is either fixed by R or fixed by I.* As the o-vertices are the zeros of f and thus the zeros of the eigenfunction y, this will prove our theorems 1 and 2.

4. Completion of the proofs of theorems 1 and 2

We classify the embedded trees up to homeomorphisms of the plane that commute with both R and I, send vertices to vertices and preserve the labels of faces and vertices (o and x).

Proposition 2. *For $d = 4$, there are only three types of embedded trees satisfying the conclusions of Proposition 1; they are shown in Fig. 2. All o-vertices lie in the union of the real and imaginary lines. All trees of these three types are parametrized by a non-negative integer n: the number of o-vertices on the real line. First type occurs when $n = 0$, second for even $n \geq 2$ and third when n is odd.*
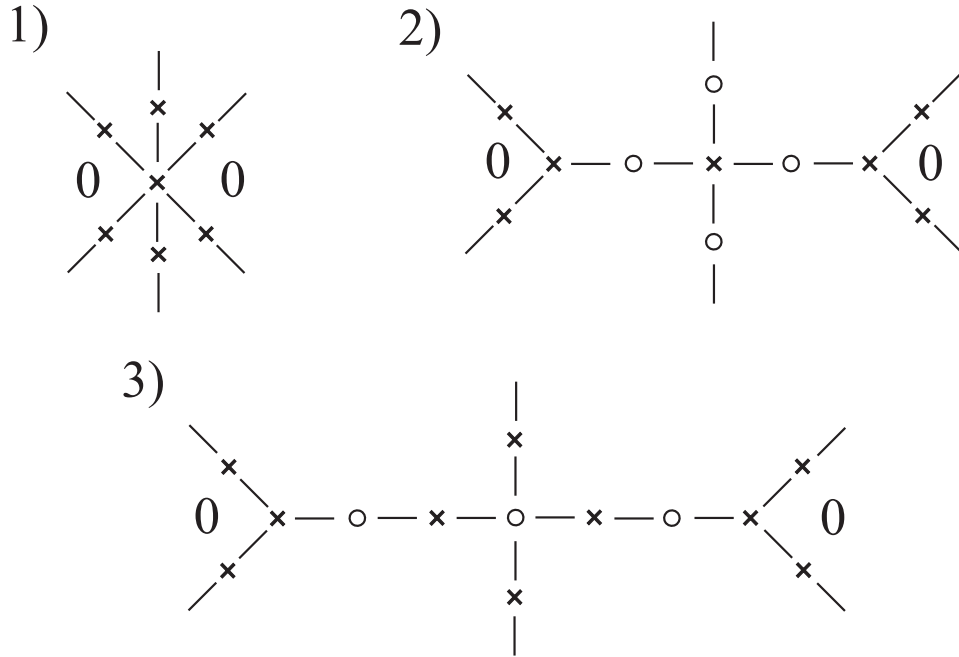

Fig. 2. Types of possible trees in Theorem 1.

Proof. If vertices of order 2 are ignored, there are two topological types of properly embedded trees with 6 ends, both I and R symmetries and satisfying b) and f) of Proposition 1:
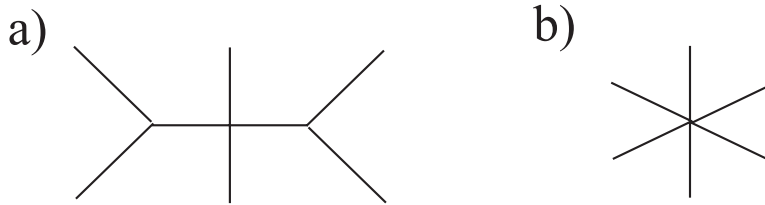
Fig. 3. Topological types of possible trees in Theorem 1 (ignoring vertices of order 2).

This simple fact can be proved along the same lines as Proposition A in the Appendix and it is left to the reader. Now statements c) and d) of Proposition 1 imply that the o-vertices lie on the coordinate cross.

For Theorem 2, we have one property in addition to those stated in Proposition 1: every even-numbered face (in the natural cyclic order) is labeled by 0.

Proposition 3. *For $d = 6$, there are only five types of embedded trees satisfying the conclusions of Proposition 1 and the additional property that every even-numbered face is labeled by 0. They are shown in Figure 4. These trees are parametrized by two integers: the total number m of o-vertices and the number n of o-vertices on the real line. These integers satisfy the following evident restrictions: $0 \leq n \leq m$ and $n - m$ is even.*
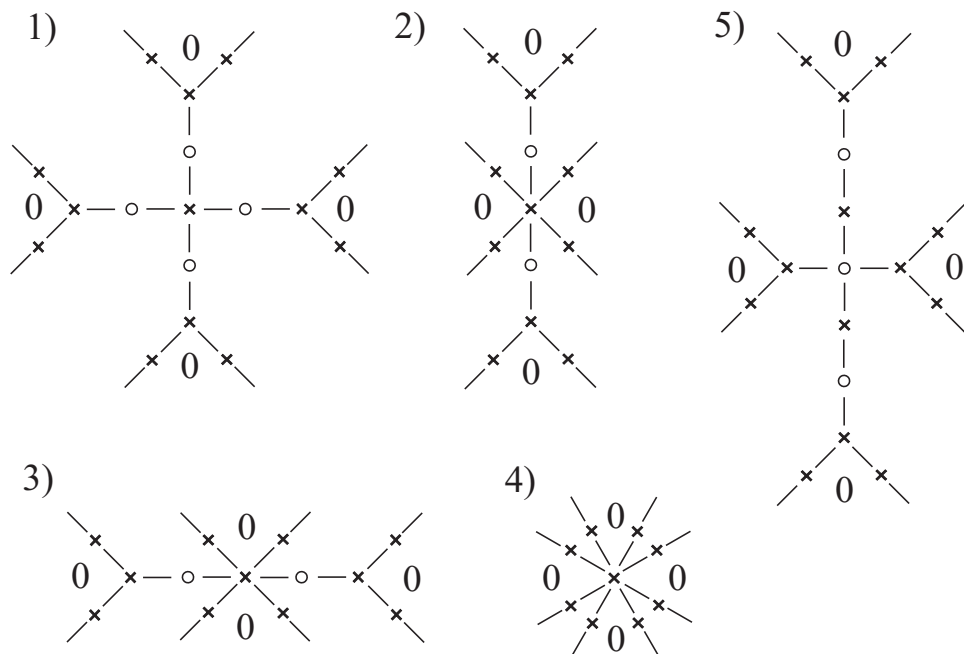
Fig. 4. Possible types of trees in Theorem 2.

Proof. Only topological types a), c), f) and j) of Proposition A in the Appendix satisfy conditions b) and f) of Proposition 1. Topological type a) gives two types of trees, depending on the type of vertex in the center.

One can show, using a result of Nevanlinna cited in the next Section that all types of trees described in propositions 2 and 3 can actually occur for the eigenfunctions in theorems 1 and 2.

To complete the proof of Theorem 1, it remains to prove that a cannot be real. The proof is by contradiction. If a is real, we have only three asymptotic values, $0, a, -a$. We repeat with simplifications the construction in Section 3. The partition C of the Riemann sphere has two edges and three faces now:
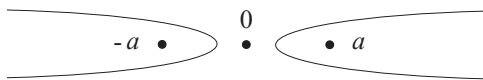

Fig. 5. Partition C in the case of real a.

The degree of each vertex of Φ is 4. The boundary of each zero-face in Φ contains two edges and two vertices, and thus it contains no loops.

15

Proposition 1 remains true for the partition Φ''. But now an analog of the Propositions 2 and 3 shows that no admissible graphs exist. Indeed, graphs of the types 2 and 3 in Fig. 2 are excluded because two faces labeled a have common edges in them. Graph of the type 1 is excluded because it has a vertex of degree 6. For the same reasons, all graphs in Fig. 4 are excluded. Thus a cannot be real.

To complete the proof of Theorem 2, we have to show that a can be neither real nor imaginary. The argument is the same as we just used to show that a cannot be real in Theorem 1, and it is left to the reader.

Proof of Corollary 1. We recall that all eigenfunctions of (1) are either even or odd. We use the formula

$$\overline{y(x)}y'(x)|_a^b = \int_a^b |y'(x)|^2 dx + \int_a^b (P(x) - \lambda)|y(x)|^2 dx, \tag{6}$$

which is called the Green transform of the equation (1), see, for example, [7, 8.1]. From this formula, we infer that on every open interval $J \subset \mathbf{R}$ where $P(x) - \lambda \geq 0$, the eigenfunction y can have at most one zero. Moreover, as $y(\pm\infty) = 0$, we can also take into account the zeros at $\pm\infty$. If $P(z) = z^4 + bz^2$, we first apply (6) to the interval $(a, b) = (-\infty, +\infty)$ to conclude that $P - \lambda$ changes sign on the real line, that is $\lambda > -b^2/4$. Now $P - \lambda$ has either two or four real roots, and both possibilities actually can occur [21]. If

$$x_1 = \sqrt{-b/2 + \sqrt{b^2/4 + \lambda}}$$

is the largest real root, then an application of (6) to intervals $(a, +\infty)$ with $a \geq x_1$ shows that all real zeros of y belong to $(-x_1, x_1)$.

If $P - \lambda$ has an imaginary root ix_2, then a similar argument shows that the interval $[-ix_2, ix_2]$ is free of zeros of even eigenfunctions y and contains exactly one zero at 0 of each odd eigenfunction y.

16

5. Proof of Theorem 3

We begin by recalling the definition of a line complex. Let f be a meromorphic function of finite order without critical points and with finitely many asymptotic values.

Let Γ be an oriented Jordan curve in the Riemann sphere $\overline{\mathbf{C}}$, which passes once through each asymptotic value. Orientation of Γ induces a cyclic order on the set of asymptotic values. We denote the asymptotic values in this order by $(b_1, \ldots, b_q)$. The subscripts $1, \ldots, q$ in b_k are considered as residues modulo q, so that $b_{q+1} = b_1$, etc. The curve Γ with marked points b_k is called the *base curve*, and the points b_k are called the *base points*.

Choose one point o inside Γ and one point x outside Γ. Then connect o and x by q disjoint edges $\Gamma_1, \ldots, \Gamma_q$, so that each Γ_k intersects Γ exactly once, and this intersection happens on the open arc (b_k, b_{k+1}) of Γ.

The points o and x, the curves Γ_k and the components of the complement to the union of these curves and points form a partition of the Riemann sphere with two vertices, q edges and q faces.

The f-preimage of this partition of the plane $\mathbf{C}$ is traditionally called the *line complex L*. The faces, edges and vertices of the line complex can be naturally *labeled* by the names of their images. A line complex is considered as a topological object: two line complexes are equivalent if they can be mapped one onto another by an orientation preserving homeomorphism of the plane.

The following properties of a line complex are known [3, 6, 10, 11, 12] and easy to prove.

The 1-skeleton of L is a bi-partite graph (in particular it has no loops). Every vertex belongs to the boundaries of exactly q faces, and the labels of these q faces are all distinct; the labels $(a_1, \ldots, a_q)$ of these faces follow anti-clockwise around an o-vertex and clockwise around a x-vertex.

The faces are of two types: 2-gons and ∞-gons. The 2-gons are mapped by f homeomorphically onto neighborhoods of the base points. So each 2-gon contains exactly one simple solution of the equation $f(z) = b_k$, where b_k is the label of this 2-gon. ∞-gons correspond to the logarithmic branch points of f^{-1}, and the restriction of f onto an ∞-gon is a universal cover of a punctured neighborhood of the base point b_k, where b_k is the label of this ∞-gon. In particular, the equation $f(z) = b_k$ has no solutions in an ∞-gon labeled b_k.

An unlabeled line complex can be defined intrinsically, without any reference to f. It is a partition of the plane into vertices, edges and faces with the following property:

The 1-skeleton is a connected bi-partite properly embedded graph whose all vertices have the same degree q.

If such partition of the plane is given, there are two choices of labeling the vertices with x and o. Furthermore, if a cyclically ordered set of points $(b_1, \ldots, b_q)$ is given, we can always label the faces of our partition with elements of this set, so that the subscripts of the labels increase counter-clockwise around each o-vertex and clockwise around a x-vertex. Notice that such labeling is uniquely defined once the label of one face is specified, and the label of one face can be prescribed arbitrarily.

A fundamental theorem which is due to Nevanlinna [11], see also [6], says the following.

Suppose that a labeled line complex is given with $d+2$ unbounded faces and all bounded faces are 2-gons. Choose a base curve passing through the labels according to their cyclic order. Then there exists a meromorphic function f in the plane of order $(d+1)/2$ with no critical points and whose line complex with respect to this base curve is equivalent to the given one.

This function is unique up to a change of the independent variable $z \mapsto cz + b$, $c \neq 0$.

Each f given by this theorem is a ratio of two linearly independent solutions of the differential equation $-y'' + Py = 0$ where

$$-2P = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is a polynomial of degree d.

We use this result to prove Theorem 3. For simplicity of illustrations, we consider only the case $d = 10$. Then we should have 12 sectors S_j, and we choose the asymptotic values in these sectors to be $a_0 = a_2 = a_4 = a_6 = a_8 = a_{10} = 0$ and

$$(a_1, a_3, a_5, a_7, a_9, a_{11}) = (a, b, a, R(a), R(b), R(a)),$$

where a and b are distinct and belong to the positive imaginary axis, for example one can take $a = 2i$ and $b = i$ as we do in our pictures. So we have

5 asymptotic values. Then we choose the imaginary axis (oriented "up") as our base curve, the o point at -1 and the x at $+1$.

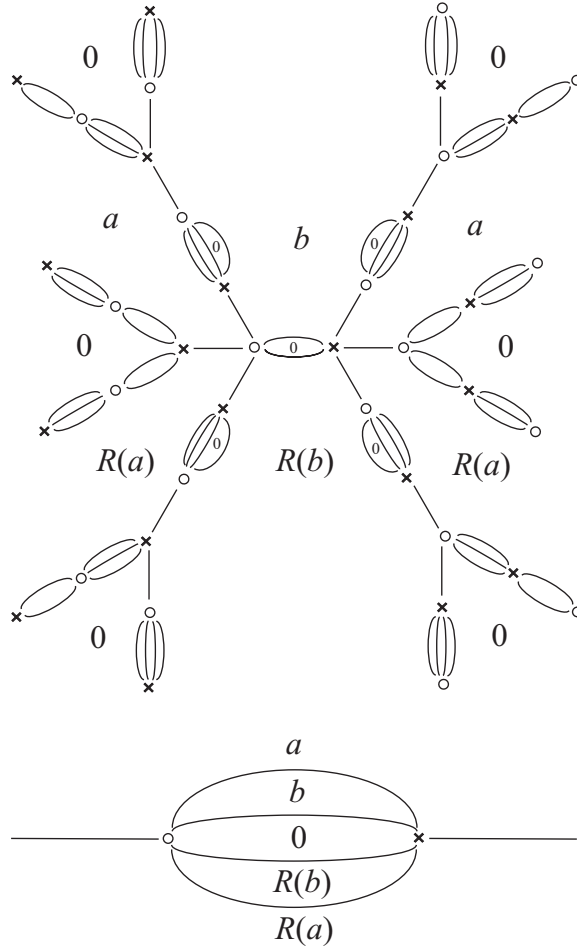Now we consider the following line complex:

Fig. 6. Line complex for Theorem 3. Curves Γ_k are shown in the bottom.

Meromorphic function f corresponding to this complex by Nevanlinna's theorem will be normalized as follows: $o^* = -1$ and $x^* = 1$, where o^* and x^* are the two vertices closest to the center of the picture. Then it is clear that f has the necessary symmetry properties (4). Four zeros shown in the picture form a single orbit under the $\mathbf{Z}_2 \times \mathbf{Z}_2$ action. So these zeros are neither real nor imaginary.

6. Quasi-exactly solvable sextic potentials

Here we prove Corollary 2 stated in the Introduction.

In 1987, Turbiner and Ushveridze [22] (see, a detailed exposition in [23]) discovered the family of sextic potentials

$$P_{m,p,b}(z) = z^6 + 2bz^4 + (b^2 - 4m - 2p - 3)z^2,$$

where m is a non-negative integer, $p \in \{0, 1\}$, and $b \in \mathbf{R}$, with the following remarkable property. Problem (1) with $P = P_{m,p,b}$ has $m + 1$ eigenfunctions of the form

$$y(z) = z^p Q_{k,m,p,b}(z^2) \exp\left(-\frac{z^4}{4} - b\frac{z^2}{2}\right), \quad 0 \le k \le m$$

where $Q_{k,m,p,b}$ are polynomials of degree m. These eigenfunctions correspond to the eigenvalues λ_{2k+p} of the Schrödinger operators (1) with $P = P_{m,p,b}$. Such potentials, having several first eigenfunctions of the form (3), are sometimes called quasi-exactly solvable [23].

Using the method which goes back to Stieltjes [15], Ushveridze proved that all zeros of the polynomials $Q_{k,m,p,b}$ are real [23], pp. 53–56, so all zeros of the eigenfunctions belong to the union of the real and imaginary axes. More precisely, $Q_{k,m,p,b}$ has k positive and $m - k$ negative zeros.

On the other hand, in the proof of our Theorem 2 we obtained the classification of *all* meromorphic functions $f = f_{k,n,p,a}$ arising from eigenfunctions of the form (3) for all problems (1) with even polynomials P of degree 6. Here $f_{k,n,p,a}$ is the function f from Section 2 with $2n+p$ zeros, $2k+p$ of them real, and asymptotic value a in the first quadrant.

Proportional functions f belong to the same potential P and the same eigenvalue, hence we can normalize so that $a = \exp(i\alpha)$ where $\alpha \in (0, \pi/2)$ when $p = 1$ and $\alpha \in (-\pi/2, 0)$ when $p = 0$.

Thus we obtain a family of functions $\alpha = g_{k,m,p}(b)$. To each potential $P_{m,p,b}$ and each $k \in [0, m]$ this function g puts into correspondence the argument of the asymptotic value a of the corresponding function $f = y/y_1$, where y is the $2k+p$-th eigenfunction of (1), and y_1 is a second linearly independent solution of the differential equation in (1), with $P = P_{m,p,b}$ and $\lambda = \lambda_{2k+p}$. This function f is $f_{k,m,p,a}$ for some a, and we define $g_{k,m,p}(b) = \text{Arg } a$. Moreover, f is normalized so that $|a| = 1$ and $0 < |\text{Arg } a| < \pi/2$.

Thus each function g maps the real line into an open interval $J \in \mathbf{R}$, where $J = (0, \pi/2)$ for $p = 1$, and $J = (-\pi/2, 0)$ for $p = 0$. We have to prove that all these functions g are surjective.

It is well-known that g is continuous and real analytic (see, for example, [14]). On the other hand, for each $\alpha \in J$, there is a neighborhood $V \subset J$ of α, such that a (right) inverse branch $h : J \to \mathbf{R}$ of g can be defined, that is $h \circ \phi = \mathrm{id}_J$, and moreover, this h is real analytic [14]. We conclude that g is surjective and this proves Corollary 2.

Appendix. Classification of trees

In this Appendix we classify all trees with 8 ends, that have no ends on the real and imaginary axes, and are both R- and I-symmetric. In what follows we call such trees *double-symmetric*. We say that two double-symmetric trees are isomorphic if there exists an orientation-preserving and commuting with both R and I homeomorphism of the complex plane sending one to the other. We do not require that this homeomorphism sends vertices to vertices.

Proposition A *There exist* 11 *non-isomorphic double-symmetric trees with no ends fixed by either R or I, see Figure 7.*
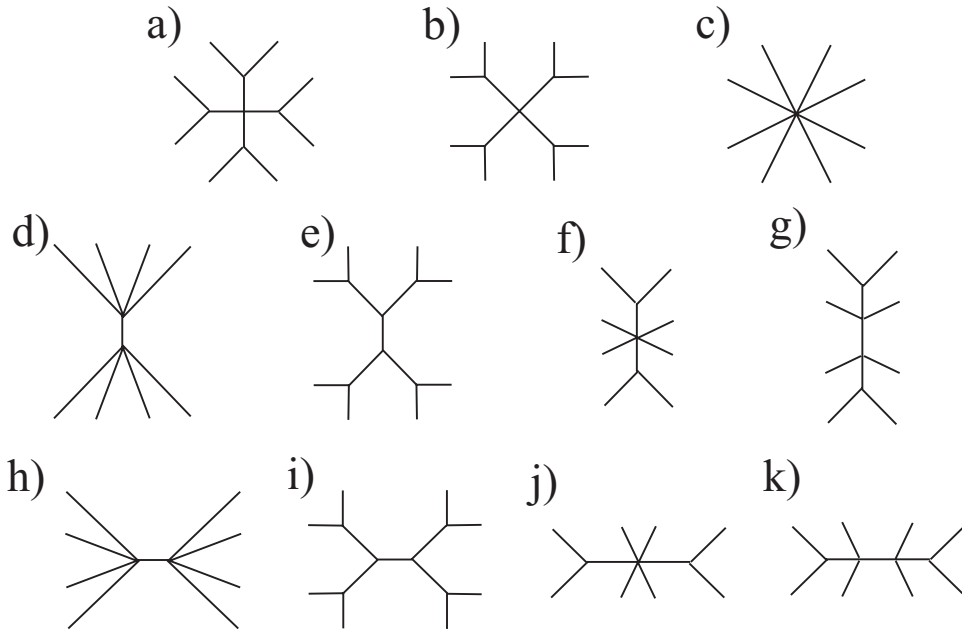

Fig. 7. Double symmetric trees with 8 ends none of which is on the axes.

The proof uses the following simple lemma classifying symmetric planar rooted trees with 4 ends whose proof is left to the reader.

Lemma. *There exist 6 non-isomorphic planar rooted trees with 4 ends (not counting the root) and no vertices of degree 2, symmetric with respect to a reflection in a line passing through the root, see Figure 8.*
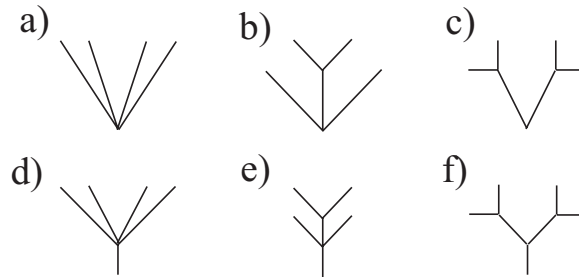

Fig. 8. I-symmetric rooted trees with 4 ends.

To prove Proposition A we argue as follows. Any double-symmetric tree contains the origin (since a doubly symmetric 1-complex embedded in the plane and not containing the origin either is either disconnected or contains a cycle). Consider the intersection of such a tree with the coordinate axes. If it intersects both axes non-trivially, i.e., not only at the origin, then the only possibility is the tree marked a in Figure 7. Assume now that a double-symmetric tree intersects the real axis only at the origin. Then the intersection of that tree with the closed upper half-plane is a planar tree rooted at the origin with 4 ends, symmetric with respect to I. By the Lemma, there are exactly 6 such trees. They correspond to the trees marked b through g in Figure 7. To get all the remaining double-symmetric trees we have to rotate the latter 6 trees by $\pi/2$. Notice that trees marked b and c are invariant under this rotation, producing just 4 new trees, namely, trees marked h through k in Figure 7. Thus the total number of trees equals $1 + 6 + 4 = 11$.

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