Quasi-exactly solvable quartic: elementary integrals and asymptotics

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> Littlewood, when he makes use of an algebraic identity, always saves himself the trouble of proving it; he maintains that an identity, if true, can be verified in few lines by anybody obtuse enough to feel the need of verification.
>
> *Freeman Dyson* [7]

Abstract

We study elementary eigenfunctions $y = pe^h$ of operators $L(y) = y'' + Py$, where p, h and P are polynomials in one variable. For the case when h is an odd cubic polynomial, we investigate the real level crossing points and asymptotics of eigenvalues. This study leads to an interesting identity with elementary integrals.

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1 Introduction

Following Bender and Boettcher [3], we consider the eigenvalue problem

$$w'' + (\zeta^4 + 2b\zeta^2 + 2iJ\zeta + \lambda)w = 0, \quad w(te^{-i(\pi/2 \pm \pi/3)}) \to 0, \quad t \to +\infty, \quad (1)$$

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with real parameters b and J.

When J is a positive integer, this problem is quasi-exactly solvable (QES) which means that for every b there are J eigenvalues $\lambda_{J,k},\ k = 1, \ldots, J$ with *elementary* eigenfunctions

$$y_{J,k}(\zeta) = p_{J,k}(\zeta) \exp(-i\zeta^3/3 - ib\zeta),$$

where $p_{J,k}$ are polynomials of degree $J - 1$. These J eigenvalues are found from algebraic equations

$$Q_J(b, \lambda) = 0, \quad J = 1, 2, \ldots, \tag{2}$$

where Q_J are real polynomials of degree J in λ.

For every J and b, problem (1) has infinitely many eigenvalues λ tending to infinity, which satisfy an equation

$$F_J(b, \lambda) = 0, \tag{3}$$

where F_J is a real entire function of two variables which is called the *spectral determinant*.

We denote by $Z_J(\mathbf{R})$ and $Z_J^{QES}(\mathbf{R})$ the sets of real solutions (b, λ) of equations (3) and (2), respectively. These are certain analytic curves in $\mathbf{R}^2$. A computer-generated picture of these curves is given in [3]. The symbols Z_J and Z_J^{QES} denote the sets of all complex solutions of these equations.

The most conspicuous feature seen in the picture in [3] is a set of real crossing points where the curves of $Z_J^{QES}(\mathbf{R})$ intersect the rest of $Z_J(\mathbf{R})$.

In this paper, we prove that infinitely many such points exist for each odd J (Proposition 1). We characterize real and complex crossing points as intersection points of Z_J^{QES} with Z_{-J} (Theorem 1). We prove that all non-QES eigenvalues are real when b is real and J is a positive integer (Theorem 2.)

In sections 8, 9, we study the asymptotic behavior of polynomials Q_J as $b \to \pm\infty$, using singular perturbation methods from [10]. Appendix A with a description of the relevant results of [10] is included.

Our proof of Theorem 1 is related to a challenging conjecture on certain elementary integrals. We rigorously verified this conjecture for integers $J \leq 5$ using symbolic computation. The conjecture is discussed in sections 3, 5.

2 Elementary solutions

Let h and p be two polynomials in one variable. When $y = p(z)e^{h(z)}$ satisfies a second order differential equation

$$y'' + Py = \lambda y, \tag{4}$$

where P is a polynomial? Substitution gives

$$\frac{p''}{p} + 2\frac{p'}{p}h' + h'' + h'^2 + P - \lambda = 0. \tag{5}$$

Such P exists if and only if

$$p'' + 2p'h' \quad \text{is divisible by} \quad p. \tag{6}$$

Another criterion is obtained if we consider the second solution y_1 of (4) which is linearly independent of y. This second solution can be found from the condition

$$yy_1' - y'y_1 = 1. \tag{7}$$

Solving (7) with respect to y_1 we obtain

$$y_1 = pe^h \int p^{-2}e^{-2h}. \tag{8}$$

As all solutions of (4) must be entire functions, we conclude that

$$\text{all residues of} \quad p^{-2}e^{-2h} \quad \text{vanish.} \tag{9}$$

This condition is *necessary and sufficient* for $y = pe^h$ to satisfy equation (4) with some P. Indeed, if (9) holds, then y_1 defined by (8) is an entire function, so (y, y_1) is a pair of entire functions whose Wronski determinant equals 1, so this pair must satisfy a differential equation (4) with entire P, and asymptotics at infinity show that P must be a polynomial.

Thus conditions (6) and (9) are equivalent. One can give another equivalent condition in terms of zeros of p, as in [19]. Let

$$p(z) = \prod_{j=1}^{n}(z - z_j), \quad p_k(z) = p(z)/(z - z_k).$$

Then (6) is equivalent to

$$p''(z_k) + 2p'(z_k)h'(z_k) = 0,$$

for all $k = 1, \ldots, n$. We have $p'(z_k) = p_k(z_k)$ and $p''(z_k) = 2p'_k(z_k)$, so the condition

$$\sum_{j \neq k} \frac{1}{z_k - z_j} = -h'(z_k), \quad 1 \leq k \leq n, \tag{10}$$

is equivalent to (6) and (9). Equation (10) is the equilibrium condition for n unit charges at the points z_k in the plane, repelling each other with the force inversely proportional to the distance, and in the presence of external field $\overline{h'(z)}$. Equations (10) express the fact that $(z_1, \ldots, z_n)$ is a critical point of the "master function"

$$\Psi(z_1, \ldots, z_n) = \prod_{(j,k):k<j} (z_k - z_j) \prod_k e^{h(z_k)}.$$

3 Remarkable identity

From now on we suppose that h is an odd polynomial of degree 3, which we write in the form

$$h(z) = z^3/3 - bz. \tag{11}$$

Suppose that all residues of $p^{-2}e^{-2h}$ vanish. Then the integral $\int p^{-2}e^{-2h}$ is a meromorphic function in the plane. Surprisingly, the integral of some linear combination

$$\int \left(p^2(-z)e^{-2h(z)} - Cp^{-2}(z)e^{-2h(z)} \right)$$

is not only meromorphic but is an *elementary function*! Here C is a constant depending on b and p.

Conjecture. *Let h be given by* (11). *Let p be a polynomial. All residues of $p^{-2}e^{-2h}$ vanish if and only if there exist a constant C and a polynomial q such that*

$$\left(p^2(-z) - \frac{C}{p^2(z)} \right) e^{-2h(z)} = \frac{d}{dz} \left(\frac{q(z)}{p(z)} e^{-2h(z)} \right). \tag{12}$$

In other words:

$$p^2(z)p^2(-z) - C = q'(z)p(z) - q(z)p'(z) - 2q(z)p(z)h'(z).$$

It is known [3] that for given h of the form (11) there exist polynomials p of any given degree such that all residues of $p^{-2}e^{-2h}$ vanish. These polynomials

p have simple roots. We verified the conjecture for $\deg p \leq 4$ by symbolic computation with Maple. We don't know whether there is any analog of the Conjecture for other polynomials h.

Substituting $p_n(z) = z^n + az^{n-1} + \ldots$ into (5) and using (11), we conclude that

$$P(z) - \lambda = -h'^2(z) - h''(z) - 2nz + 2a = -z^4 + 2z^2 b - 2(n+1)z - b^2 + 2a. \quad (13)$$

We choose $\lambda = b^2 - 2a$ so that $P(0) = 0$. Equation (5) now becomes

$$p_n'' + 2(z^2 - b)p_n' - (2nz - 2a)p_n = 0. \quad (14)$$

Coefficients of p_n can be now determined by a linear recurrence formula. Putting

$$p_n(z) = \sum_{j=0}^{n} a_j z^{n-j}, \quad a_{-1} = 0, \quad a_0 = 1, \quad a_1 = a,$$

we obtain the recurrence

$$ja_j = aa_{j-1} - b(n-j+2)a_{j-2} + \frac{(n-j+2)(n-j+3)}{2} a_{j-3}. \quad (15)$$

Coefficients a_j are found from this formula one by one beginning from $a_1 = a$. Vanishing of the constant term in (14) gives a polynomial equation $Q^*_{n+1}(b, a) = 0$ in which we can substitute $a = (b^2 - \lambda)/2$ and write it as

$$Q_{n+1}(b, \lambda) = 0. \quad (16)$$

Polynomials p_n, Q^*_{n+1} for small n are given in Appendix B.

We have $\deg_\lambda Q_{n+1} = n + 1$, [3]. For every b and every λ satisfying this equation, the differential equation (4), with P as in (13), has a unique solution $y = p_n e^h$ where p_n is a monic polynomial of degree n. Coefficients of p_n are polynomials in b and λ.

Polynomials Q_{n+1} are fundamental for our subject, but little is known about them. It seems hard to investigate them algebraically. In section 8, we will use analytic tools to establish some properties of these polynomials, in particular we will find the terms of top weight and asymptotics of λ as $b \to \infty$.

Functions $y = p_n e^h$ are eigenfunctions of the operator

$$L_J(y) = y'' - (z^4 - 2bz^2 + 2Jz)y, \quad J = n + 1. \quad (17)$$

This operator maps the space $\{pe^h : \deg p \leq n\}$ of dimension $n+1$ into itself. For each non-negative integer n, and generic b, the operator (17) has $n+1$ eigenfunctions of the form $p_n e^h$ with eigenvalues λ which are solutions of (16).

We assume without loss of generality that Q_{n+1} is monic as a polynomial in λ, and p_n is a monic polynomial in z. Then the constant C in the Conjecture turns out to be

$$C(b,\lambda) = \alpha_n \frac{\partial}{\partial \lambda} Q_{n+1}. \tag{18}$$

Symbolic computation for small n shows that $\alpha_n = (-1)^n 2^{-2n}$.

4 Boundary value problem

Eigenfunctions pe^h do not belong to $L^2(\mathbf{R})$, but they satisfy the boundary conditions

$$y(te^{\pm \pi i/3}) \to 0 \quad \text{as} \quad t \to \infty. \tag{19}$$

With these boundary conditions, the operator (17) is not Hermitian but PT-symmetric [3, 14, 18].

Physicists write the boundary value problem for the operator L_J with the boundary conditions (19) in the equivalent form (1), which corresponds to the rotation of the independent variable $i\zeta = z$, $w(\zeta) = y(i\zeta)$. We also find this form convenient in certain arguments, and will use it in sections 7, 8. We keep the notation $y(z)$ for an eigenfunction of (17), (19), while $w(\zeta)$ stands for an eigenfunction of (1).

It is known that the boundary value problem (17), (19) has an infinite sequence of eigenvalues tending to infinity [16]. Eigenvalues λ are solutions of the equation

$$F_{n+1}(b,\lambda) = 0, \tag{20}$$

where F_{n+1} is a real entire function on $\mathbf{C}^2$ which is called the *spectral determinant* [14]. The set of all solutions of (20) in $\mathbf{C}^2$ is called the *spectral locus* and we denote it by Z_{n+1}. As F_{n+1} is real, the set of eigenvalues is symmetric with respect to the real line when b is real. For each real b, all sufficiently large eigenvalues (how large, depends on b) are real [14].

Equation (20) is reducible: F_{n+1} is evidently divisible by Q_{n+1}. On the other hand, equation (16) is irreducible, as follows from [9] or [2], and it defines a smooth algebraic curve in $\mathbf{C}^2$. This algebraic curve will be denoted by Z_{n+1}^{QES}.

5 Theorem 1

Now we discuss a corollary of our conjecture that we can prove. Let us fix a simple curve γ in $\mathbf{C}$ parametrized by the real line, with the properties $\gamma(t) \to \infty,\ \arg\gamma(t) \to \pm\pi/3,\ t \to \pm\infty$. Then (12) implies

$$\int_\gamma p^2(z)e^{2h(z)}dz = C\int_\gamma p^{-2}(-z)e^{2h(z)}dz. \tag{21}$$

To obtain this we replace $z \mapsto -z$ in (12) then integrate along γ; the integral in the right hand side of (12) vanishes because $\Re h(z) \to -\infty$ as $z \to \infty$ on γ. Let γ_z be a curve consisting of the piece $\{\gamma(t) : -\infty < t \leq 0\}$ followed by a curve from $\gamma(0)$ to z. Put

$$g(z) = p(-z)e^{-h(z)}\int_{\gamma_z} p^{-2}(-\zeta)e^{2h(\zeta)}d\zeta.$$

Then $g^-(z) := g(-z)$ satisfies $L_{n+1}(g^-) = \lambda g^-$ so g satisfies $L_{-n-1}(g) = \lambda g$. To check this, we make a substitution $z \mapsto -z$ in (17). Moreover, if the integral in the right hand side of (21) is zero, then g also satisfies the boundary condition (19). Therefore our conjecture (12), (18) has the following corollary:

Theorem 1. *The points $(b,\lambda) \in Z_{n+1}^{QES}$ where the eigenfunction $y = pe^h$ satisfies*

$$\int_\gamma y^2(z)dz = 0 \tag{22}$$

are either zeros of $C(b,\lambda)$ or points of intersection of Z_{n+1}^{QES} with Z_{-n-1}.

We will prove Theorem 1 in the next section.

Equation (22) is the well-known condition of level crossing, which we discuss in section 7.

Thus the Corollary says that the eigenvalues at the points on Z_{n+1}^{QES} which are singular points of Z_{n+1} are eigenvalues of two spectral problems, one for L_{n+1} another for L_{-n-1}.

6 Darboux transform

Proof of Theorem 1. Assuming that all residues of $p^{-2}e^{-2h}$ vanish, we will prove that the right and left sides of (21) with C as in (18) have the same zeros on Z_{n+1}^{QES}. Fix the integer $n \geq 0$. Let $\psi_k = p_k e^h, \quad k = 0, \ldots, n$ be all elementary eigenfunctions of L_{n+1}. They are linearly independent, and they span a space V invariant under L_{n+1}. As V is a subspace of $U = \{pe^h : \deg p \leq n\}$, we conclude that $V = U$. So the Wronski determinant $W = W(\psi_0, \ldots, \psi_n)$ is proportional to the Wronski determinant

$$W(e^h, ze^h, \ldots, z^n e^h) = \left(\prod_{k=0}^{n} k! \right) e^{(n+1)h}.$$

Now let us perform the Darboux transform of L_{n+1} killing these $n+1$ eigenfunctions. We recall that Darboux transform [5, 13, 4, 8]) applies to any operator $-D^2 + V$ with eigenfunctions $\phi_0, \ldots, \phi_n$ and corresponding eigenvalues $\lambda_0, \ldots, \lambda_n$. The transformed operator is

$$-D^2 + V - 2\frac{d^2}{dz^2} \log W(\phi_0, \ldots, \phi_n),$$

and its eigenvalues are those eigenvalues of $-D^2 + V$ which are distinct from $\lambda_0, \ldots, \lambda_n$. As in our case $2(\log W)'' = 2(n+1)h'' = 4(n+1)z$, the result of application of the Darboux transform to L_{n+1} and eigenfunctions $\psi_k,\ k = 0, \ldots, n$ is L_{-n-1}.

If the left hand side of (21) is zero at some point $(b, \lambda) \in Z_{n+1}^{QES}$ then $\partial F_{n+1}/\partial\lambda = 0$ at this point (see, for example [18]). As Z_{n+1}^{QES} is smooth, this is possible in exactly two cases: either (b, λ) is a smooth point of Z_{n+1} and $\partial Q_{n+1}(b, \lambda)\partial\lambda = 0$, or (b, λ) is a self-intersection point of Z_{n+1}.

In the second case, (b, λ) belongs to the spectral locus of the Darboux transform L_{-n-1}. This means that the equation

$$L_{-n-1}(y^*) = \lambda y^*$$

with these parameters (b, λ) has a solution y^* that tends to 0 on both ends of γ. Then $y_1 = y^*(-z)$ tends to 0 on both ends of $-\gamma$ and satisfies $L_{n+1}(y_1) = \lambda y_1$. So y_1 satisfies the same differential equation $L_{n+1}(y) = \lambda y$ as y does, and is linearly independent of y. So $y_1 = y \int y^{-2}e^{-2h}$. As this tends to 0 on both ends of $-\gamma$, we conclude that $\int y^{-2}e^{-2h}$ tends to 0 on both ends of $-\gamma$.

So $y^*(z) = y_1(-z)$ tends to 0 on both ends of γ, and this means that the right hand side of (21) is 0.

The argument is evidently reversible. This proves that the right and the left hand sides of (21) have the same zeros on Z_{n+1}^{QES}, that is (21) with $C = \alpha_n \partial Q_n / \partial \lambda$, where $\alpha_n(b, \lambda) \neq 0$ on Z_{n+1}^{QES}.

According to the theorem of Shin [15], all eigenvalues of L_J for $J \leq 0$ are real. Shin's proof of this uses the ODE-IM correspondence discovered by Dorey, Dunning and Tateo, [6].

Combining the Darboux transform used in the prof of Theorem 1 with the result of Shin [15], we obtain

Theorem 2. *For every positive integer J, all non-QES eigenvalues of L_J with boundary conditions (19) are real.*

Proof. These eigenvalues are also eigenvalues of L_{-J} with boundary conditions (19), and the eigenvalues of L_{-n-1} are all real by Shin's theorem [15].

For $J > 1$ there are always some non-real eigenvalues.

7 Level crossing

As Q_{n+1} and F_{n+1} are real functions, it is reasonable to consider real solutions of equations (20) and (16). Eigenfunctions $y(z)$ corresponding to these real solutions are real, while eigenfunctions $w(\zeta)$ (see (1)) are PT-symmetric, that is $w(-\overline{\zeta}) = \overline{w(\zeta)}$. These real solutions (b, λ) form curves in $\mathbf{R}^2$ which we call the *real spectral locus* $Z_{n+1}(\mathbf{R})$ and the *QES real spectral locus* $Z_{n+1}^{QES}(\mathbf{R})$, respectively.

Now we discuss (22). First we state a result which describes $Z_{n+1}^{QES}(\mathbf{R})$.

Theorem 3. *For $n \geq 0$, the spectral locus $Z_{n+1}^{QES}(\mathbf{R})$ consists of $[n/2] + 1$ disjoint analytic curves $\Gamma_{n,m}$, $0 \leq m \leq [n/2]$ (analytic embeddings of $\mathbf{R}$ to $\mathbf{R}^2$).*

For $(b, \lambda) \in \Gamma_{n,m}$, the eigenfunction has n zeros, $n - 2m$ of them real.

If n is odd then $b \to +\infty$ on both ends of each curve $\Gamma_{n,m}$. If n is even then the same holds for $0 \leq m < n/2$, but on the ends of $\Gamma_{n,n/2}$ we have $b \to \pm\infty$.

If $(b, \lambda) \in \Gamma_{n,m}$, $(b, \mu) \in \Gamma_{n,m+1}$ and b is sufficiently large, then $\mu > \lambda$.

The proof of this theorem can be found in [11]. It follows the method of [10] where similar results were established for real spectral loci of other families of cubic and quartic potentials. The method is based on singular perturbation and Nevanlinna parametrization of the spectral locus.

Computer generated pictures of $Z_{n+1}(\mathbf{R})$ show an interesting phenomenon: when n is even, the curve $\Gamma_{n,n/2}$ crosses the non-QES part of the spectral locus [3, Fig. 1]. We will prove that infinitely many such crossings exist for even n and negative b.

We say that a level crossing occurs at a point (b, λ) of the spectral locus if $\partial F_{n+1}/\partial\lambda = 0$ at this point. If y is the eigenfunction corresponding to a point (b, λ), then the level crossing occurs if and only if (22) is satisfied [17, II.7], [18, Thm. 8]. There are two types of level crossing points:

a) Critical points of the function λ at non-singular points of Z_{n+1}.

If such a critical point (b_0, λ_0) is simple and belongs to $Z_{n+1}(\mathbf{R})$ then the two eigenvalues that meet at this point are both real for b on one side of b_0 and complex conjugate on the other side.

b) Singular points of Z_{n+1}.

If two eigenvalues collide at a simple self-intersection point of $Z_{n+1}(\mathbf{R})$ with two distinct non-vertical tangents, then these eigenvalues both remain real in a neighborhood of b_0. Operator L_{n+1} with $b = b_0$ contains a Jordan cell in this case.

We recall that Z_{n+1}^{QES} is a smooth curve. Thus the crossing points on Z_{n+1}^{QES} where only QES eigenvalues collide are all of type a), and they satisfy

$$Q_{n+1}(b, \lambda) = 0, \quad \frac{\partial}{\partial\lambda} Q_{n+1}(b, \lambda) = 0.$$

For each n, there are finitely many such points on Z_{n+1}^{QES}.

We will show that there are always infinitely many crossing points of type b) where QES eigenvalues collide with non-QES eigenvalues. So the curve defined by (20) is not smooth: it has infinitely many self-intersections.

We don't know whether more complicated singularities than a) and b) exist; numerical experiments only show singularities of types a) and b).

Proposition 1. *Function*

$$\Phi_n(b, \lambda) = \int_\gamma y^2(z) dz, \quad Z_{n+1}^{QES} \to \mathbf{C},$$

where y is the eigenfunction corresponding to (b, λ), has infinitely many zeros $(b_k, \lambda_k),\ b_k \to \infty$. When n is even, Φ_n has infinitely many zeros with negative b_k and real λ_k.

Proof. We have

$$\Phi_n(b) = \int_\gamma p_n^2(z) e^{2h(z)} dz.$$

We remind that coefficients of p_n and h are algebraic functions of b. When $n = 0$, we can take $p_0 = 1$, and then

$$\Phi_0(b) = \int_\gamma e^{(2/3)z^3 - 2bz} dz = 2^{2/3} i\pi \mathrm{Ai}(2^{2/3} b),$$

where Ai is the Airy function [1]. Airy function is a real entire function of order 3/2 with infinitely many negative simple zeros.

To generalize this to other values of n, we express Φ_n as a linear combination of Φ_0 and Φ_0' with coefficients depending on b algebraically. Differentiating $\Phi_0(b)$ with respect to b, we obtain

$$\int_\gamma z^k e^{(2/3)z^3 - 2bz} dz = (-2)^{-k} \Phi_0^{(k)}(b),$$

and thus

$$\Phi_n(b) = p_n^2(-D/2)\Phi_0(b),$$

where $D = d/db$. Now all $\Phi_0^{(k)}$ are linear combinations of Φ_0 and Φ_0' with polynomial coefficients because Ai satisfies the differential equation $\mathrm{Ai}''(s) = s\mathrm{Ai}(s)$. So Φ_n is of the form

$$\Phi_n(b) = A_n(b)\Phi_0(b) + B_n(b)\Phi_0'(b), \tag{23}$$

where A_n and B_n are algebraic functions.

We claim that every linear combination ϕ of Φ_0 and Φ_0' with algebraic coefficients has infinitely many zeros. We prove this claim by contradiction. Suppose that such a linear combination

$$\phi = a_0 \Phi_0 + a_1 \Phi_0' \tag{24}$$

has finitely many zeros. Let F be a compact Riemann surface spread on the Riemann sphere on which a_0 and a_1 are meromorphic. Then ϕ is meromorphic on $F \backslash E$, where E is the finite set of points of F lying over ∞. At the points of E, ϕ has isolated essential singularities. As ϕ has finitely many zeros and poles on $F\backslash E$, we conclude that ϕ'/ϕ is meromorphic on $F\backslash E$. The growth estimate $\log|\phi(b)| \leq O(|b|^{3/2})$, $b \to \infty$, implies that the points of E are removable singularities of ϕ'/ϕ. Thus ϕ is the exponent of an Abelian integral. Now consider (24) as a linear differential equation of first order with respect to Φ_0, whose coefficients belong to the minimal field K that contains $\mathbf{C}(b)$, is algebraically closed, and contains a primitive of every element, and the exponent of a primitive of every element. As every first order linear differential equation can be solved by integration we conclude that $\Phi_0 \in K$ which implies that $\mathrm{Ai} \in K$. But this is not so by a well-known classical theorem of Picard and Vessiot, [12, Theorem 6.6]. This proves our claim.

When n is even, according to Theorem 3, we have a real analytic branch $\lambda(b)$ defined for all real b with sufficiently large absolute value. The graph of this branch is a part of $\Gamma_{n,n/2}$. Using this branch we rewrite the equation $\Phi_n(b) = 0$ as

$$\Phi_0'(b)/\Phi_0(b) = A(b),$$

where A is a real branch of an algebraic function on $(-\infty, B)$ with some $B \in \mathbf{R}$. This last equation has infinitely many negative solutions because Φ_0 has infinitely many negative zeros and they are interlaced with zeros of Φ_0'. This completes the proof of the proposition.

Using the asymptotics of the zeros of Airy's function [1] we obtain that the crossing points satisfy $b_k \sim -((3/4)\pi k)^{2/3}$, $k \to \infty$.

8 Asymptotics as $b \to +\infty$

Now we study asymptotics of the eigenvalues λ as $b \to +\infty$ and make conclusions about polynomials Q_{n+1}. Our main result here is the explicit formula (30) for the top quasi-homogeneous part of Q^*_{n+1}.

First we obtain a preliminary estimate of solutions $\lambda(b)$ of equation (16) for large b:

$$\lambda(b) \sim b^2 + O(\sqrt{b}), \quad b \to \infty. \tag{25}$$

To prove this, consider the recurrence (15). For a monomial $a^m b^k$ we define the weight as $m + 2k$. Then (15) implies that

$$j!a_j = a^j + \sum_{m=1}^{[j/2]} c_{m,j} b^m a^{j-2m} + \text{terms of lower weight.}$$

Vanishing of the constant term in (14) gives

$$Q^*_{n+1}(a,b) = aa_n + ba_{n-1} + 2a_{n-2} = 0,$$

so Q^*_{n+1} is a sum of a quasi-homogeneous polynomial in a and b of weight $2(n+1)$ and a polynomial of lower weight. This means that $a = O(\sqrt{b})$ and $\lambda(b) = b^2 - 2a$ satisfies (25).

To obtain more precise asymptotics we use singular perturbation arguments from [10], which we state in Appendix A for the reader's convenience.

Suppose that b is real and $b \to +\infty$. In the equation (1) we set

$$\zeta = \epsilon u - i\epsilon^{-2}, \quad b = \epsilon^{-4}, \quad W(u) = w(\epsilon u - i\epsilon^{-2}).$$

The result is

$$W'' + (\epsilon^6 u^4 - 4i\epsilon^3 u^3 - 4u^2 - 2iJ\epsilon^3 u)W + (2J + \epsilon^2\lambda - \epsilon^{-6})W = 0, \tag{26}$$

or

$$-W'' - \left(u^2(b^{-3/4}u - 2i)^2 - 2iJb^{-3/4}u\right)W = (2J + b^{-1/2}\lambda - b^{3/2})W. \tag{27}$$

When $\epsilon \to 0$, we obtain the limit eigenvalue problem

$$-W'' + 4u^2 W = \mu W, \tag{28}$$

which is a harmonic oscillator with eigenvalues $\mu_k = 2(2k+1),\ k = 0, 1, 2, \ldots$. By a general result from [10] (see Appendix), (27) implies that for each k, there must be a unique eigenvalue $\lambda_k(b)$ which satisfies

$$\lambda_k = b^2 + (\mu_k - 2J + o(1))\sqrt{b}. \tag{29}$$

Moreover, for each compact set K in the λ-plane there exists $b_0 > 0$ such that for $b > b_0$ there are no other eigenvalues $\lambda(b) \in K$, except those satisfying (29).

We conclude from (25) that QES eigenvalues must satisfy (29). That is for each QES eigenvalue λ there exists k such that (29) holds. Now we have to find out what are the values of k for the QES eigenvalues.

To do this, we consider zeros of eigenfunctions. We know that k-th eigenfunction of (28) has $[k/2]$ zeros in the right half-plane, the same number of zeros in the left half-plane, and one zero on $i\mathbf{R}$ if k is odd. (In fact all these last zeros belong to the real line but this is irrelevant for our argument.) So for every $m = 0, 1, \ldots$ there are two eigenfunctions of the harmonic oscillator (with $k = 2m$ and $k = 2m + 1$) which have m zeros in the right half-plane, and one of them ($k = 2m + 1$) has a zero on $i\mathbf{R}$.

Theorem 3 implies that for each given n and for each $m \leq [n/2]$ and b sufficiently large positive, there is exactly one curve $\Gamma_{n,m}$, such that the corresponding eigenfunctions have m zeros in the right half-plane[1]. We refer to [10] for the argument showing that the zeros of eigenfunctions w in the right half-plane do not escape to infinity as $b \to +\infty$. Zeros of w on $i\mathbf{R}$ *do* escape to infinity, except possibly one of them. Thus the branches of QES eigenvalues must be $\lambda_0, \ldots, \lambda_n$ satisfying (29).

Putting $\lambda_k = b^2 - 2a(k)$, and $J = n + 1$ in (29) we obtain

$$a(k) \sim \sqrt{b}(n - 2k), \quad 0 \leq k \leq n.$$

We conclude that the top weight term of the polynomial Q^*_{n+1} is

$$\prod_{k=0}^{n} \left(a - (n-2k)\sqrt{b} \right) = \begin{cases} (a^2 - b)(a^2 - 3b) \ldots (a^2 - nb), & n \text{ is odd}, \\ a(a^2 - 2b) \ldots (a^2 - nb), & n \text{ is even}. \end{cases} \tag{30}$$

This implies that the degree of the discriminant of Q^*_{n+1} is $n(n+1)/2$, and the genus of the QES spectral locus is $n(n-2)/4$ when n is even and $(n-1)^2/4$ when n is odd.

9 Asymptotics as $b \to -\infty$

When $b \to -\infty$, our operator (1) also degenerates to a harmonic oscillator. However none of the QES eigenvalues of (1) tend to the eigenvalues of this harmonic oscillator as $b \to -\infty$. To study this limit, we set $z = \epsilon u,\ b = -\epsilon^{-4}$ and $W(u) = w(\epsilon u)$ in (1). The result is

$$W'' + \left(\epsilon^6 u^4 - 2u^2 + 2iJ\epsilon^3 u + \epsilon^2 \lambda \right) W = 0. \tag{31}$$

As $\epsilon \to 0$, this tends to the harmonic oscillator

$$-W'' + 2u^2 W = \mu W,$$

[1]Remember that we are working here with eigenfunctions $w(\zeta) = y(i\zeta)$, where y is an eigenfunction from Theorem 3.

whose eigenvalues are $\mu_k = \sqrt{2}(2k+1)$, $k = 0, 1, 2\ldots$. So by the results in [10] (see Appendix A), for every k and for $b < -b_k$, there is an eigenvalue $\lambda_k(b)$ which satisfies

$$\lim_{\epsilon \to 0} \epsilon^2 \lambda_k(b) = \sqrt{2}(2k+1),$$

or $\lambda_k(b) \sim \sqrt{-b}$. Comparison with (25) shows that these eigenvalues λ_k cannot come from the QES spectrum.

We thank Per Alexandersson for making Fig. 1, and for help with computations which led us to the discovery of (12), (18), Stefan Boettcher for sending us pictures of $Z_J(\mathbf{R})$ which inspired this work, Evgenii Mukhin and Alexandre Varchenko for useful discussions, and Vladimir Marchenko for his insightful suggestion to look at the Darboux transform.

Appendix A. Singular perturbation of polynomial potentials

Here we state the main singular perturbation result of [10] and verify that the eigenvalue problems (26) and (31) satisfy all conditions that imply continuity of the discrete spectrum at $\epsilon = 0$.

Consider the eigenvalue problem

$$-y'' + P_\epsilon(z,b)y = \lambda y, \quad y(z) \to 0, \quad z \in R_1 \cup R_2. \tag{32}$$

Here z is the independent variable, P is a polynomial in z whose coefficients depend on parameters $\epsilon > 0$ and $b \in \mathbf{C}$, dependence on b is holomorphic, and R_1, R_2 are two rays in the complex plane defined by $R_k = \{te^{i\theta_k} \in \mathbf{C}_z : t > 0\}$, $k = 1, 2$.

Suppose that

$$P_\epsilon(z,b) = \sum_{j=0}^{d} a_j(b,\epsilon) z^j,$$

where $a_d(\epsilon) > 0$ does not depend on b, $P_0(z,b) = a_m(b,0)z^m + \ldots$, where $m < d$, and the dots stand for the terms of smaller degree in z.

Let

$$P_\epsilon^*(z,b) = \sum_{j=m}^{d} a_j(b,\epsilon) z^j.$$

For every polynomial potential $P(z) = a_n z^n + \ldots$ of degree n, the *separation rays* are defined by

$$\{z \in \mathbf{C} : a_n z^{n+2} < 0\}.$$

Turning points are just zeros of the potential P in the complex plane.[2] *Vertical line* at a point z is the line defined by $P(z)dz^2 < 0$. If P depends on parameters, then the separation rays, turning points and the vertical line field depend on the same parameters.

We assume that there exists $\delta > 0$ and $\epsilon_0 > 0$ and a compact $K \subset \mathbf{C}_b$, such that for all $\epsilon \in (0, \epsilon_0)$ and for all $b \in K$ and $k \in \{1, 2\}$ the following conditions are satisfied:

(i) $|\arg z - \theta_k| \geq \delta$ for all turning points $z \in \mathbf{C}\backslash\{0\}$ of P_ϵ^*,

[2]This terminology is somewhat unusual but convenient here. In the standard terminology turning points are zeros of $P - \lambda$.

(ii) For every point $z \in R_k$, the smallest angle between R_k and the vertical line with respect to P_ϵ^* at this point is at least δ.

(iii) R_k are not separation rays, for P_ϵ, $\epsilon > 0$ or P_0.

(iv) All coefficients $a_j(b,\epsilon)$ are bounded from above and $|a_m(b,\epsilon)|$ is bounded from below.

Theorem A. *If the conditions (i)–(iv) are satisfied, then the spectral determinant F_ϵ of the eigenvalue problem (32) converges as $\epsilon \to 0$ to the spectral determinant of (32) with $\epsilon = 0$:*

$$F_\epsilon \to F_0, \quad \epsilon \to 0,$$

uniformly for $(b,\lambda) \in K \times K_1$, for every compact $K_1 \subset \mathbf{C}_z$.

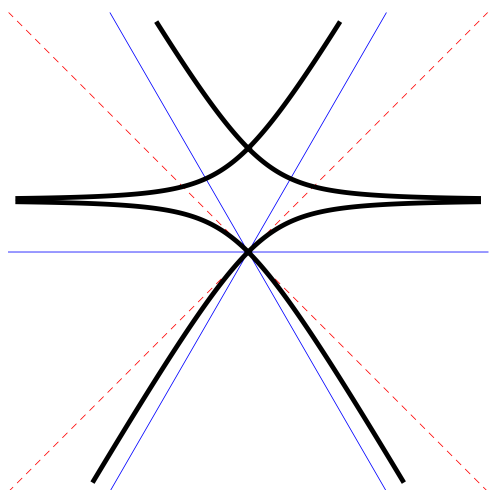

Fig 1. Stokes complex of P_ϵ^*.

Now we verify that the family of potentials in (26) satisfies all conditions (i)-(iv) with $d = 4, m = 2$. We have

$$P_\epsilon^*(z) = -\epsilon^6 z^4 + 4i\epsilon^3 z^3 + 4z^2,$$

$P_0^*(z) = 4z^2$. The turning points are 0 and $2i\epsilon^{-3}$. The separation rays are $\arg z \in \{0, \pi, \pm\pi/3, \pm 2\pi/3\}$ for $P_\epsilon^*, \epsilon > 0$ (shown in thin solid lines in Fig. 1), and $\arg z \in \{\pm\pi/4, \pm 3\pi/4\}$ for P_0 (dashed lines in Fig. 1). The normalization rays are $\arg z \in \{-\pi/2 \pm \pi/3\}$. The bold lines in Fig. 1 represent the Stokes complex, that is the integral curves of the vertical direction field $P_\epsilon^*(z)dz^2 < 0$ that are adjacent to the turning points.

Thus conditions (i),(iii) and (iv) evidently hold. It remains to verity (ii).

To do this we parametrize R_1 as $z = te^{-i\pi/6} : \; t > 0$ and find the direction of the line field $\arg dz$ at z by inserting this parametrization to $\arg(P_\epsilon^*(z)dz^2) = \pi$. We obtain

$$\arg P_\epsilon^*(z) \in (-\pi/2, \pi/3), \quad \pm \arg dz^2 \in (2\pi/3, 4\pi/3),$$

so the angle between dz and R_1 is at least $\pi/6$. Verification for R_2 is similar.

We leave to the reader to verify that conditions of Theorem A are satisfied for (31).

Appendix B. Explicit expressions

We remind that $\lambda = b^2 - 2a$ where $p_n(z) = z^n + az^{n-1} + \ldots$. Since $Q_{n+1}(b, \lambda)$ and $Q_{n+1}^*(b, a)$ are normalized so that they are monic polynomials in λ and a, respectively, we have $Q^*(b, a) = (-1)^{n+1} Q_{n+1}(b, b^2 - 2a)/2^{n+1}$. We use the notation $C^*(b, a) = C(b, b^2 - 2a)$, where C is the constant from (12). Then (18) can be rewritten as

$$C^*(b, a) = \alpha_n^* \frac{\partial}{\partial a} Q_{n+1}^*, \tag{33}$$

where $\alpha_n^* = (-1)^n 2^n \alpha_n = 2^{-n}$.

Here are results of symbolic computations with Maple.

For $n = 1$:

$$p_1(z) = z + a,$$

$$Q_2^*(b, a) = a^2 - b,$$

$$C^*(b, a) = a.$$

For $n = 2$:

$$p_2(z) = z^2 + az + \left(\frac{a^2}{2} - b\right),$$

$$Q_3^*(b, a) = a^3 - 4ab + 2,$$

$$C^*(b, a) = \frac{3}{4}a^2 - b = \frac{1}{4}\frac{\partial Q_2^*}{\partial a}.$$

For $n = 3$:

$$p_2(z) = z^3 + az^2 + \left(\frac{1}{2}a^2 - \frac{3}{2}b\right) z - \frac{7}{6}ab + \frac{1}{6}a^3 + 1,$$

$$Q_4^*(b, a) = a^4 - 10a^2b + 12a + 9b^2,$$
$$C^*(b, a) = \frac{1}{2}a^3 - \frac{5}{2}ab + \frac{3}{2} = \frac{1}{8}\frac{\partial Q_4^*}{\partial a}.$$

For $n = 4$:

$$p_4(z) = z^4 - az^3 + \left(\frac{1}{2}a^2 - 2b\right) z^2 - \left(2 + \frac{1}{6}a^3 - \frac{5}{2}ab\right) z - \frac{2}{3}a^2b + b^2 + \frac{5}{4}a + \frac{1}{24}a^4,$$

$$Q_5^*(b, a) = 42a^2 - 96b - 20a^3b + 64ab^2 + a^5,$$
$$C^*(b, a) = \frac{21}{4}a + \frac{5}{16}a^4 - \frac{15}{4}a^2b + 4b^2 = \frac{1}{16}\frac{\partial Q_5^*}{\partial a}.$$

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