Coxeter-Dynkin diagrams and singularities

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There is a deep and only partially understood connection between the classification and structure of singularities and the Coxeter-Dynkin diagrams introduced by H.S.M. Coxeter for classification of reflection-generated groups, and by E.B. Dynkin for classification of semisimple Lie algebras.

One of the main problems of the theory of singularities is classification of singular objects of increasing complexity. The first objects in this theory are *stable* singularities that are not destroyed by small perturbations. The next important class consists of *simple* singularities: only a finite number of non-equivalent singularities appear as their small perturbations.

Consider, for example, classification of critical points of germs of analytic functions $(\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$. Each simple object in this classification appears, after a change of variables in $(\mathbf{C}^3, 0)$, in the following (A, D, E) list [1, 5, 8]:

$A_k,\ k \geq 1$	$D_k,\ k \geq 4$	E_6	E_7	E_8
$x^{k+1} + y^2 + z^2$	$x^{k-1} + xy^2 + z^2$	$x^4 + y^3 + z^2$	$x^3y + y^3 + z^2$	$x^5 + y^3 + z^2$
cyclic	*dihedral*	*tetrahedral*	*octahedral*	*icosahedral*

These are exactly Kleinian singularities [5, 8] associated with finite subgroups of $SL_2(\mathbf{C})$. The algebra of invariants of the natural action of such a group on $\mathbf{C}^2$ was computed by Klein. It is generated by three polynomials x, y, z, with a single relation. These relations appear in the (A, D, E) list above, for cyclic groups and for binary dihedral, tetrahedral, octahedral, icosahedral groups, respectively. Direct connection between finite subgroups of $SL_2(\mathbf{C})$ and (extended) Dynkin diagrams is provided by McKay correspondence (see [8, 9]).

Several constructions link functions from the (A, D, E) list to the corresponding Coxeter-Dynkin diagrams.

The zero set $V_0 = \{f = 0\}$ of each function f in the list has an isolated singularity at 0. A *resolution of the singularity* is an analytic mapping from a nonsingular manifold to V_0 which is one-to-one on $V_0 \setminus 0$. There is a unique *minimal* resolution, such that any other resolution can be mapped through it. The *exceptional divisor* (preimage of 0) of the minimal resolution consists of μ spheres $\mathbf{CP}^1$ (here μ is the index in $A_\mu,\ D_\mu$, or E_μ). In the *dual diagram* of the exceptional divisor, a point corresponds to each sphere, and an edge connects two points when the corresponding spheres intersect. For the functions in the (A, D, E) list, the dual diagrams are the corresponding Coxeter-Dynkin diagrams (A, D, E). This was found by Du Val in 1934 (see [5, 7]).

Let $f(x)$ be an analytic function in $\mathbf{C}^{n+1}$ with an isolated critical point at 0. For a small $\delta > 0$ and a non-zero $\epsilon \ll \delta$, the *Milnor fiber* of f is defined as the intersection of the ball $|x| < \delta$ with the level set $f = \epsilon$. It is a complex n-dimensional manifold with homotopy type of a bouquet of $\mu = \dim_{\mathbf{C}} \mathbf{C}[[x]]/(\partial f/\partial x_i)$ spheres of (real) dimension n. These spheres are called *vanishing cycles*, as they are contracted to the origin as $\epsilon \to 0$. For a function from the (A, D, E) list (with $n = 3$), vanishing cycles can be naturally chosen to form a basis of a root system of the corresponding (A, D, E) type in the two-dimensional homology of the Milnor fiber with respect to the intersection index [4, 6].

A *deformation* of $f(x)$ is an analytic function $F(x, \lambda)$ such that $F(x, 0) = f(x)$. A deformation is *versal* if it is transversal to the orbit through f of the group of changes of variables x. A versal deformation with the minimal (μ-dimensional) parameter space is called *miniversal*. It is defined uniquely up to an isomorphism.

Let G be a simple, simply connected group over $\mathbf{C}$. Let u be a subregular unipotent element of G (i.e., the conjugacy class C of u in G has codimension $r + 2$ where r is rank of G). Let T be a maximal torus of G, W the Weyl group, and $p : G \to T/W$ projection defined as $p(x) =$ set of all elemants of T conjugate to the semisimple part of x. Let V be a transversal to C through u in G. Brieskorn proved that $p : V \to T/W$ is the miniversal deformation of the corresponding Kleinian singularity [8].

The *discriminant* Σ of a miniversal deformation consists of those λ for which 0 is a critical value of $F(x, \lambda)$ (as a function of x). For each $\lambda \notin \Sigma$, the zero set V_λ of $F(x, \lambda)$ is nonsingular, diffeomorphic to the Milnor fiber of f. This allows one to define a representation of the fundamental group π of the complement of Σ in the n-dimensional homology of the Milnor fiber, the *monodromy group* W of f. This group is generated by reflections in vanishing cycles.

For the (A, D, E) list, W is the Weyl group of the root system of the corresponding (A, D, E) type, while π is the corresponding braid group (see [1]). The space of parameters of the versal deformation can be obtained as the space of invariants of the action of W in the μ-dimensional complex space $H_2(V_\lambda, \mathbf{C})$, and Σ corresponds to singular orbits (mirrors) of this action.

Several other classification problems produce the same A, D, E list: Lagrangian and Legendrian singularities that appear in optics as the singularities of caustics and wave fronts [2,4]. These singularities are closely related to the singularities of critical points of the corresponding hamiltonians.

One more classification problem with the same list appears in quiver representations (see [8, 9]). A *quiver* is a set of points, some of them connected by arrows. A representation of a quiver associates to each point a linear space, and to each arrow a linear mapping of the linear spaces corresponding to its beginning and ending points. A quiver is simple if there are only finite number of equivalence classes of its irreducible representations. P. Gabriel proved that connected simple quivers appear exactly when one replaces edges of a Dynkin diagram from the (A, D, E) list by arrows with arbitrary directions.

The above examples represent only a starting point of the presence of Dynkin diagrams in theory of singularities. One way of generalization is to search for Dynkin diagrams of other Lie groups in various classification problems. For example, series B_k and C_k, and exceptional singularities F_4 and G_2 appear in classification of singularities of hypersurfaces over positive-characteristic fields [7], classification of the *boundary singularities* [4], and classification of fractions of analytic functions [3]; non-crystallographic reflection groups H_3 and H_4 appear in obstacle problems in $\mathbf{R}^2$ and $\mathbf{R}^3$ [2,4].

Another way is to study more complicated singularities that are not simple (see [4]). Dynkin diagrams can be defined in this case, although they do not seem to correspond to any Lie algebras. The corresponding Coxeter groups are infinite and their action is not discrete.

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