Complements of Subanalytic Sets and Existential Formulas for Analytic Functions

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ABSTRACT: We show that the complement of a subanalytic set defined by real analytic functions from any subalgebra closed under differentiation is a subanalytic set defined by the functions from the same subalgebra. This result has an equivalent formulation in logic: Consider an expression built from functions as above using equalities and inequalities as well as existential and universal quantifiers. Such an expression is equivalent to an existential expression involving functions from the same class, provided that the variables approach neither infinity nor the boundary of the domain.

1. Introduction. The class of subanalytic sets was first considered by Gabrielov [8, 9] as a natural extension of the Tarski-Seidenberg [22, 24] theorem on elimination of quantifiers from semialgebraic expressions and of Łojasiewicz's [19, 20] theory of semianalytic sets. It was studied by Hironaka [16], where the term "subanalytic" was introduced, and by other authors [1, 2, 4-6, 13-15, 21, 23, 25].

Subanalytic sets are defined as images of relatively proper real analytic maps of semi-analytic sets. The main theorem of [8] asserts that the complement of any subanalytic set is subanalytic. This geometric property is equivalent to "quantifier simplification" i.e., the possibility to replace an expression involving real analytic equalities and inequalities, with existential and universal quantifiers, by an equivalent existential expression, provided that the variables are bounded and never approach the boundary of the domain of definition of the functions.

In [4, 6] this property is treated, within the framework of mathematical logic, as model completeness and o-minimality of the extension of the field of real numbers by "restricted" real analytic functions. The term "restricted" replaces the compactness conditions. It means that the functions are defined and analytic in the closed unit cube $[0,1]^n$ in $\mathbf{R}^n$, and extended by zero outside the cube. The result in [4] is even stronger: any expression involving equalities and inequalities with restricted analytic functions is equivalent to an expression without quantifiers when we allow "bounded division" of restricted analytic functions, i.e., for any two functions P and Q, consider also P/Q restricted to $\{P < Q\}$.

The natural question of whether quantifier simplification is possible for special classes of analytic functions was treated in [26] for Pfaffian functions.

In this paper, we show that, for a subanalytic set defined by real analytic functions from a set Φ, its complement (within the unit cube) is again a subanalytic set, defined by functions from the algebra A generated by the functions from Φ, their partial derivatives, constants 0 and 1, and coordinate functions. In terms of mathematical logic, this means that any formula with functions from A and with universal and existential quantifiers, is equivalent to an existential formula with the functions from A.

The proof is purely geometric. It is based on the finiteness properties of semianalytic sets.

As already shown in [8], quantifier simplification immediately implies various geometric properties of subanalytic sets (for example, Łojasiewicz inequality and the existence of Whitney stratifications) in the same way that the Tarski-Seidenberg theorem implies these properties for semialgebraic sets.

Some of our results are new also for the semianalytic sets defined by functions from a subalgebra closed under differentiation. Namely, the closure and the frontier of such a set (inside the domain of definition of the functions) can be defined as semianalytic sets within the same subalgebra. Also, there exists a stratification of such a set, with the strata defined by functions from the same subalgebra.

2. Application to Pfaffian functions.

When the functions in the original expression are Pfaffian (see [17]), the corresponding classes of semianalytic and subanalytic sets will be called semi-Pfaffian and sub-Pfaffian, respectively. Our method implies that the complement of a sub-Pfaffian set is sub-Pfaffian, with the defining functions from the algebra generated over polynomials by the Pfaffian functions from the original expression. The constructiveness of our techniques, combined with effective bounds on finiteness properties for Pfaffian functions given in [17] and [10], allows one to effectively estimate complexity of the existential expression, in terms of complexity of the original expression and of the Pfaffian functions involved. For semi- and sub-Pfaffian sets, complexity of closure, frontier, and stratification can be effectively estimated. The detailed estimates will appear in a separate paper. One step in this direction, an estimate for the complexity of a weak stratification of a semi-Pfaffian set, has appeared in [11].

One important class of Pfaffian functions are the sparse polynomials, or fewnomials [17]. Outside coordinate hyperplanes, they can be considered as Pfaffian functions of complexity depending only on the number of non-zero monomials, independent of their

degrees. Effective estimates on the complexity of quantifier simplification for Pfaffian expressions yield the corresponding estimates for sparse polynomials outside coordinate hyperplanes.

For Pfaffian functions in an unbounded domain or at the boundary of their domain of definition (e.g., for sparse polynomials at the coordinate hyperplanes) most of the finiteness properties are still true, and our method can be extended to this case, except for Lemma 1 below (describing the frontier of a semianalytic set) which involves Taylor expansions. Thus the problem of construction of an equivalent existential formula in this case is equivalent to a positive answer to the following question:

Consider $\mathbf{R}^n$ as a subset of $\mathbf{RP}^n$, with $\mathbf{RP}^{n-1}$ identified with the points at infinity. For a semi-Pfaffian set $X \subset \mathbf{R}^n$, its frontier at infinity is defined as $\overline{X} \cap \mathbf{RP}^{n-1}$. Does the frontier at infinity of a k-dimensional semi-Pfaffian set belong to a "sub-Pfaffian" subset of dimension smaller than k? (Here "sub-Pfaffian" is to be understood as a projection of a semi-Pfaffian set with no compactness conditions.)

At present, the only known results in this direction are Wilkie's theorem [26] (see also [7]) on model completeness for expressions with exponential polynomials in an unbounded domain, and the theorem of Cano, Moussu and Lion [3] on the frontier of a Pfaffian hypersurface.

3. Definitions and the main results.

Definition 1. Let $\Phi = \{\phi_j\}$ be a set of real analytic functions ϕ_j defined and analytic in a neighborhood of the closed unit cube $\mathbf{I}^{n_j} \subset \mathbf{R}^{n_j}$ (where $\mathbf{I} = [0,1]$). For every $n \geq 0$, we define $A_n = A_n(\Phi)$ as the minimal set of functions with the following properties:

1. The constants 0 and 1 and a coordinate function x_1 on $\mathbf{R}$ belong to A_1.
2. $\phi_j \in A_{n_j}$, for each j.
3. If $\phi, \phi' \in A_n$ then $\phi \pm \phi' \in A_n$ and $\phi \cdot \phi' \in A_n$.
4. If $\phi(x_1, \ldots, x_n) \in A_n$ then $\phi(x_{i(1)}, \ldots, x_{i(n)}) \in A_{n+m}$, for any mapping $i : \{1, \ldots, n\} \to \{1, \ldots, m+n\}$.
5. If $\phi(x) \in A_n$ then $\partial \phi(x)/\partial x_\nu \in A_n$, for $\nu = 1, \ldots, n$.

Definition 2. A subset $X \subset \mathbf{I}^n$ is called Φ-*semianalytic* if it is a finite union of sets of the form

$$\{x : \ f_i(x) = 0, \text{ for } i = 1, \ldots, I; \quad g_j(x) > 0, \text{ for } j = 1, \ldots, J\} \tag{1}$$

where the f_i and g_j are analytic functions from $A_n(\Phi)$. A subset $Y \subset \mathbf{I}^n$ is called Φ-*subanalytic* if it is an image of the projection to $\mathbf{R}^n$ of a Φ-semianalytic subset $X \subset \mathbf{I}^{m+n}$.

Remark. Subanalytic sets are usually defined as images of semianalytic sets under relatively proper analytic maps. However, the definition given above is essentially equivalent to this formally more general definition.

Definition 3. For a set $X \subset \mathbf{I}^n$, let $\overline{X}$ denote its closure, $\tilde{X} = \mathbf{I}^n \setminus X$ its complement in $\mathbf{I}^n$, and $\partial X = \overline{X} \setminus X$ its frontier. A semianalytic set $X \subset \mathbf{R}^n$ is non-singular of dimension k at a point $x_0 \in X$ if there exist real analytic functions $h_1(x), \ldots, h_{n-k}(x)$ defined in an open set $U \ni x_0$ such that $dh_1 \wedge \ldots \wedge dh_{n-k} \neq 0$ at x_0 and $X \cap U = \{x \in U : h_1(x) = \ldots = h_{n-k}(x) = 0\}$. A semianalytic set (1) is *effectively non-singular* if the functions $h_1, \ldots, h_{n-k}$ can be chosen from the set f_i. The dimension dim X of a set X is defined as the maximum of its dimensions at non-singular points.

To simplify the notation, let f, g denote the set of functions f_i and g_j, respectively. We write "$f = 0$" instead of "$f_i = 0$, for all i," etc. We also define $|f(x)| = \max_i |f_i(x)|$ and $g_{\min}(x) = \min_j g_j(x)$.

Theorem 1. *Let Y be a Φ-subanalytic subset of $\mathbf{I}^n$. Then $\tilde{Y} = \mathbf{I}^n \setminus Y$ is Φ-subanalytic.*

Corollary. *Every expression constructed from equalities and inequalities between functions from the set Φ, arithmetic operations $+$, $-$, and $\times$, logical operations $\wedge$ and $\vee$, existential and universal quantifiers, is equivalent to an existential expression with functions from $A_N(\Phi)$, for some N, provided that only values in $[0, 1]$ are allowed for all variables.*

This is an easy consequence of Theorem 1.

The proof of Theorem 1 will be given in section 5, based on several properties of Φ-semianalytic sets derived in section 4. As a by-product of the induction procedure in the proof of Theorem 1, we obtain the following additional property of Φ-subanalytic sets:

Theorem 2. *Each Φ-subanalytic set $Y \subset \mathbf{I}^n$ can be represented as a finite disjoint union of Φ-subanalytic subsets Y_i so that each Y_i is a cell homeomorphic to an open k_i-dimensional cube. More precisely, for each Y_i there exists a linear transformation with integer coefficients to a coordinate system $y_1, \ldots, y_{k_i}, y_{k_i+1}, \ldots, y_n$ such that*

$$Y_i = \{a_j(y_1, \ldots, y_{j-1}) < y_j < b_j(y_1, \ldots, y_{j-1}), \text{ for } 1 \leq j \leq k_i;$$

$$y_j = c_j(y_1, \ldots, y_{k_i}), \text{ for } j = k_i + 1, \ldots, n\}. \tag{2}$$

Here a_j and b_j, for $1 \leq j \leq k_i$, are analytic functions of $y_1, \ldots, y_{j-1}$ with Φ-subanalytic graphs, defined in an open domain

$$\Omega_j = \{a_\nu(y_1, \ldots, y_{\nu-1}) < y_\nu < b_\nu(y_1, \ldots, y_{\nu-1}), \text{ for } 1 \leq \nu \leq j\},$$

and $a_j < b_j$ everywhere in Ω_j. For for $k_i \leq j \leq n$, the functions c_j are analytic in Ω_{k_i}, with Φ-subanalytic graphs.

4. Preliminary lemmas.

Lemma 1.   Let $X = \{x : f(x) = 0,\ g(x) > 0\}$ where the functions f_i and g_j are analytic in $\mathbf{I}^n$. Then $\overline{X}$ and ∂X are finite unions of semianalytic sets, each defined by a finite system of equalities and inequalities between polynomials in x, the functions f_i, g_j, and their partial derivatives.

Proof.   Let us show first that there exist $c > 0$ and $\kappa > 0$ such that the closure of the set

$$\{x' \in X,\ g(x') \geq c|x' - x|^\kappa\} \tag{3}$$

contains x if and only if $x \in \overline{X}$. This is obviously true when $x \in X$ or $x \notin \overline{X}$. To prove this for $x \in \partial X$, we consider a set

$$A = \{(\epsilon, \delta) : \exists\, x,\ D(x, \epsilon/4) \neq \emptyset,\ \max_{x' \in D(x,\epsilon)} g_{\min}(x') = \delta\}.$$

Here $D(x, \epsilon) = \{x' \in X,\ |x' - x| \leq \epsilon\}$ is a ball in X of radius ϵ, centered at x. Due to [8], A is a subanalytic subset in $\mathbf{R}^2$ (actually, a semianalytic subset, as every subanalytic subset in $\mathbf{R}^2$ is semianalytic according to [20]). We claim that $\overline{A}$ does not contain any points $(\epsilon, 0)$ with $\epsilon > 0$. Otherwise there exist sequences $\{x_\nu\}$ in $\mathbf{I}^n$ and $\epsilon_\nu \to \epsilon > 0$ such that $D(x_\nu, \epsilon_\nu/4) \neq \emptyset$, and

$$\max_{x' \in D(x_\nu, \epsilon_\nu)} g_{\min}(x') \to 0 \text{ as } \nu \to \infty.$$

The sequence $\{x_\nu\}$ has an accumulation point $x_0 \in \mathbf{I}^n$. Due to the properties of the sequence $\{x_\nu\}$, we have $D(x_0, \epsilon/2) \neq \emptyset$ and $g_{\min}(x') \leq 0$ on $D(x_0, \epsilon/2)$, which is impossible.

Due to the "curve lemma" [20], there exist $c_0 > 0$ and $\kappa > 0$ such that

$$\delta \geq c_0 \epsilon^\kappa + o(\epsilon^\kappa) \text{ when } (\epsilon, \delta) \in A,\ \text{for small } \epsilon > 0. \tag{4}$$

We want to show that, for any $c \leq c_0$ and $x \in \partial X$, the closure of the set (3) contains x. Let $\{x_\nu\}$ be a sequence in X converging to x. Let $\epsilon_\nu = |x - x_\nu|$ and $\delta_\nu = \max_{x' \in D(x,\epsilon_\nu)} g_{\min}(x')$, so that $(\epsilon_\nu, \delta_\nu) \in A$. Take $x'_\nu \in D(x, \epsilon_\nu)$ with $g_{\min}(x'_\nu) = \delta_\nu$, so $g_{\min}(x'_\nu) \geq c_0(\epsilon_\nu)^\kappa \geq c|x - x'_\nu|^\kappa$, for large ν. Hence x'_ν belongs to the set (3), for large ν, and $x'_\nu \to x$ as $\nu \to \infty$.

We can suppose that κ in (3) is integer. Due to (4) we can replace $g(x')$ in (3) by its Taylor expansion $\check{g}_x(x')$ of the order κ at x, and the closure of (3) still contains x if and only if $x \in \overline{X}$. This can be reformulated as follows: Let

$$s_x = \{x' : \check{g}_x(x') > 0,\ \check{g}_x(x') \geq c|x' - x|^\kappa\}.$$

Then the closure of the set

$$\{x' \in s_x, \ f(x') = 0\} \tag{5}$$

contains x if and only if $x \in \overline{X}$.

We want to show that there exist $c' > 0$ and $\lambda > 0$ such that the closure of the set

$$\{x' \in s_x : |f(x')| \leq c'|x' - x|^\lambda\} \tag{6}$$

contains x if and only if $x \in \overline{X}$. This is obviously true when $x \in \overline{X}$, or $f(x) \neq 0$, or $g_{\min}(x) < 0$.

To prove this for $x \notin \overline{X}$, $f(x) = g_{\min}(x) = 0$, we consider a subanalytic set

$$B = \{(\epsilon, \delta) : \exists \, x \in \mathbf{I}^n, \ \mathrm{dist}(x, X) \geq \epsilon, \ \max_{0 \leq \zeta \leq \epsilon} \ \min_{x' \in s_x, |x - x'| = \zeta} |f(x')| = \delta\}.$$

We claim that $\overline{B}$ does not contain any points $(\epsilon, 0)$ with $\epsilon > 0$. Otherwise there exists a sequence x_ν in $\mathbf{I}^n$ with $\mathrm{dist}(x_\nu, X) \geq \epsilon_\nu \to \epsilon$ and

$$\max_{0 \leq \zeta \leq \epsilon_\nu} \ \min_{x' \in s_{x_\nu}, |x_\nu - x'| = \zeta} |f(x')| \to 0 \quad \text{as} \quad \nu \to \infty. \tag{7}$$

Let x_0 be an accumulation point of the sequence $\{x_\nu\}$. We have $\mathrm{dist}(x_0, X) \geq \epsilon$, hence $x_0 \notin \overline{X}$. Due to (7), for every $\zeta \leq \epsilon$, there exists a point $x' \in s_{x_0}$ with $\mathrm{dist}(x', x_0) = \zeta$ and $f(x') = 0$. Hence the closure of the set (5) contains x_0, in contradiction to the definition of c and κ.

Due to the "curve lemma" we have $\delta \geq c_1 \epsilon^\lambda + o(\epsilon^\lambda)$, with $c_1 > 0$ and $\lambda > 0$, when $(\epsilon, \delta) \in B$, $\epsilon \searrow 0$. Let us choose $c' > c_1$. Let $x \notin \overline{X}$ and $f(x) = g_{\min}(x) = 0$. Then, for small enough positive ϵ,

$$\max_{0 \leq \zeta \leq \epsilon} \ \min_{x' \in s_x, |x - x'| = \zeta} |f(x')| \geq c'\epsilon^\lambda\}.$$

This implies that the closure of the set (6) does not contain x. We can suppose that λ in (6) is integer.

Finally, we can replace $f(x')$ in (6) by its Taylor expansion $\check{f}_x(x')$ of order λ at x, and the closure of the set

$$S_x = \{x' \in s_x : |\check{f}_x(x')| \leq c'|x' - x|^\lambda\}$$

still contains x if and only if $x \in \overline{X}$.

For a fixed x, the set S_x is semialgebraic in x'. According to the Tarski-Seidenberg theorem, the condition that the closure of S_x contains x is equivalent to a semialgebraic

condition on coefficients of polynomials defining S_x, which are partial derivatives at x of f_i, of order not greater than λ, and of g_j, of order not greater than κ.

This proves Lemma 1 for $\overline{X}$. The statement for ∂X readily follows.

Lemma 2. *Let X be a semianalytic subset of $\mathbf{I}^n$. Then X is a finite disjoint union of non-singular manifolds X_α such that each X_α is defined by a system of equalities and inequalities*

$$f_{\alpha\mu}(x) = 0, \quad g_{\alpha\nu}(x) > 0 \tag{8}$$

where the functions $f_{\alpha\mu}$ belong to the algebra generated by f_i and their partial derivatives, the functions $g_{\alpha\nu}$ belong to the algebra generated by f_i, g_j and the partial derivatives of f_i, and each set X_α with the presentation (8) is effectively non-singular.

Proof. We can suppose that X belongs to the interior of $\mathbf{I}^n$. If this is not so, we can deal with the part of X at the boundary of $\mathbf{I}^n$ separately, as it is a union of subsets of cubes of smaller dimensions. Also, we can suppose that X is defined by equalities and inequalities as in (4), since any semianalytic set can be represented as a disjoint union of sets (4). The case $\dim X = n$ is trivial. Suppose that $\dim X < n$. Then there exists a function $h \not\equiv 0$ in the set $\{f_i\}$. Let ν be the minimal order of h at the points of X i.e., the minimal, over $x \in X$, power of non-zero terms in the Taylor expansion of h at x. Then there exists a partial derivative

$$s_1 = \frac{\partial^\nu h}{\partial x_{i_1} \cdots \partial x_{i_\nu}}$$

of h of the order ν such that $Y_1 = \{x \in X : s_1(x) \neq 0\}$ is a non-empty subset of X. Let

$$h_1 = \frac{\partial^{\nu-1} h}{\partial x_{i_2} \cdots \partial x_{i_\nu}}.$$

Then $h_1 = 0$ and $\partial h_1 / \partial x_{i_1} = s_1 \neq 0$ at every point of Y_1. Let $X' = \{x \in X : s_1(x) = 0\}$ and $U_1 = \{x : h_1(x) = 0, \ s_1(x) \neq 0, \ g(x) > 0\}$. The set X' is a proper subset of X (possibly empty) and the set U_1 is non-singular of dimension $n - 1$.

If X' is not empty, we proceed with X' in the same way as with X i.e., define the partial derivative of h that does not vanish identically on X', and so on.

With Y_1, there are two possibilities. If Y_1 is an open subset of U_1, we can set $X_1 = Y_1$, adding h_1 to the functions f_i and s_1 to the functions g_j (strictly speaking, Y_1 in this case represents two effectively non-singular subsets of X, corresponding to $s_1 > 0$ and $s_1 < 0$).

Otherwise there exists a function h in the set $\{f_i\}$ which does not vanish identically on U_1 in the neighborhood of Y_1. Let ν be the minimal order of $h|_{U_1}$ at the points of Y_1.

For $j \neq i_1$, the set of differential operators (derivatives along U_1)

$$\partial_j = \frac{\partial h_1}{\partial x_j} \frac{\partial}{\partial x_{i_1}} - \frac{\partial h_1}{\partial x_{i_1}} \frac{\partial}{\partial x_j}$$

generate the tangent space to U_1 at every point. Hence there exists a partial derivative $s_2 = \partial_{j_1} \cdots \partial_{j_\nu} h$ such that $Y_2 = \{x \in Y_1 : s_2(x) \neq 0\}$ is non-empty. At the same time, $h_2 = \partial_{j_2} \cdots \partial_{j_\nu} h$ vanishes on U_1 in the neighborhood of every point of Y_2.

Let $i_2 = j_1$, $X'' = \{x \in Y_1 : s_2(x) = 0\}$, and $U_2 = \{x \in U_1 : h_2(x) = 0,\ s_2(x) \neq 0\}$. Then X'' is a proper subset of Y_1, $h_2|_{U_2} \equiv 0$, and $\partial_{i_2} h_2 = s_2 \neq 0$ at every point of U_2. In particular, U_2 is non-singular of dimension $n - 2$.

We proceed with X'' in the same way as with Y_1, i.e., define the partial derivative of h along U_1 that does not vanish identically on X'', and so on.

For $Y_2 = X \cap U_2$, there are two possibilities: either it is an open subset of U_2, or there exists a function h in the set $\{f_i\}$ which does not vanish identically on U_2 in the neighborhood of Y_2. In the first case, Y_2 is effectively non-singular of dimension $n - 2$, with h_1 and h_2 added to the list of the functions f_i and s_1 and s_2 added to the list of the functions g_j. In the second case, we consider derivatives along U_2:

$$\partial_j = \begin{vmatrix} \frac{\partial h_1}{\partial x_{i_2}} & \frac{\partial h_1}{\partial x_j} \\ \frac{\partial h_2}{\partial x_{i_2}} & \frac{\partial h_2}{\partial x_j} \end{vmatrix} \frac{\partial}{\partial x_{i_1}} - \begin{vmatrix} \frac{\partial h_1}{\partial x_{i_1}} & \frac{\partial h_1}{\partial x_j} \\ \frac{\partial h_2}{\partial x_{i_1}} & \frac{\partial h_2}{\partial x_j} \end{vmatrix} \frac{\partial}{\partial x_{i_2}} + \begin{vmatrix} \frac{\partial h_1}{\partial x_{i_1}} & \frac{\partial h_1}{\partial x_{i_2}} \\ \frac{\partial h_2}{\partial x_{i_1}} & \frac{\partial h_2}{\partial x_{i_2}} \end{vmatrix} \frac{\partial}{\partial x_j},$$

for $j \neq i_1,\ i_2$. We define s_3 as a partial derivative of h along U_2 (i.e., a composition of the operators ∂_j) of the minimal order which does not vanish identically on Y_2, choose h_3 such that $s_3 = \partial_{i_3} h_3$, set $U_3 = \{x \in U_2 : h_3(x) = 0,\ s_3(x) \neq 0\}$, and so on.

We repeat this procedure until we finally represent X as a union of effectively non-singular subsets satisfying conditions of Lemma 2. Note that the procedure terminates after a finite number of steps, as at every step we either stop at an effectively non-singular (possibly empty) subset Y, or pass to a non-singular set U of smaller dimension, or add an additional equation $s = 0$ defining a proper subset of a semianalytic set. The latter operation cannot continue indefinitely because of the descending chain condition on zerosets of analytic functions on I^n.

Lemma 3. *Let X be a Φ-semianalytic set in $\mathbf{R}^{m+n}$, and let $Y = \pi X \subset \mathbf{I}^n$, $d = \dim Y$. Then there exist finitely many Φ-semianalytic subsets X'_ν and a Φ-subanalytic subset V of X such that $Y = \bigcup_\nu \pi X'_\nu \bigcup \pi V$ and*

a) X'_ν is effectively non-singular, $\dim X'_\nu = d$, and $\pi : X'_\nu \to Y$ has rank d at every point of X'_ν, for each ν;

$b)$ $\dim(\pi V) < d$;

$c)$ $X'_\mu \cap X'_\nu = \emptyset$, *for* $\mu \neq \nu$.

Proof. We proceed by induction on $k = \dim X$. According to Lemma 2, we can consider the case when X is non-singular set defined by equalities and inequalities as in (4), and $\mathrm{rk}(\partial f/\partial x) = m+n-k$ at every point of X. Let $z = (x_1, \ldots, x_m)$ and $y = (x_{m+1}, \ldots, x_{m+n})$ be coordinates in $\mathbf{R}^m$ and $\mathbf{R}^n$, and let

$$V = X \cap \{\mathrm{rk}(\partial f/\partial z) < m - k + d\}$$

be the critical set of $\pi|_X$. Then $\dim(\pi V) < d$. Preimage $\pi^{-1}y$ of every $y \in Y \setminus \pi V$ is a smooth manifold of dimension $k - d$. We can suppose that its projection to a $(k-d)$-dimensional subspace in $\mathbf{R}^m$ is a diffeomorphism. Otherwise we can deal with a critical set of this projection separately. For $k > d$, this implies that every connected component of $\pi^{-1}y$ has a nonempty frontier. Let $g(x) = g_1(x) \cdots g_J(x)$. Then $g > 0$ at every point of X and $g \equiv 0$ at the frontier of X. Hence g is not constant on every connected component of $\pi^{-1}y$ for every $y \in Y \setminus \pi V$. Let

$$W = X \cap \{\mathrm{rk}(\partial(f, g)/\partial z) = m - k + d\}.$$

Intersection of W with $\pi^{-1}y$, for every $y \in Y \setminus \pi V$, coincides with the critical set of g on $\pi^{-1}y$. Hence intersection of W with every connected component C of $\pi^{-1}y$ is a non-empty proper subset of C. In particular, $\pi W = Y$ and $\dim W < k$, q.e.d.

5. Proof of Theorems 1 and 2.

Let X be a Φ-semianalytic subset in $\mathbf{I}^{m+n}$ and $Y = \pi X \subset \mathbf{I}^n$. Let $x = (x_1, \ldots, x_{m+n})$ and $y = (x_{m+1}, \ldots, x_{m+n})$ be coordinates in $\mathbf{R}^{m+n}$ and $\mathbf{R}^n$, respectively.

We proceed by induction on $d = \dim Y$. As an inductive hypothesis, we suppose that, for any Φ-subanalytic set $Y \subset \mathbf{I}^n$ of dimension $k < d$,

(A_k) the complement $\tilde{Y}$ of Y is Φ-subanalytic,

(B_k) the set Y is a finite union of Φ-subanalytic subsets of type (2),

(C_k) for $n = k + 1$, there exists a closed Φ-subanalytic set $Y' \supset Y$ such that $\dim Y' \leq k$ and each connected component of $\tilde{Y}'$ is a Φ-subanalytic set of type (2).

Let us show first that (A_0)-(C_0) are valid. If $k = 0$ then Y is a finite set, and its connected components are isolated points. Let N be the number of points in Y. For $n = 1$, each point in Y is a Φ-subanalytic set of the form

$$\{y \in Y : \exists y_1 \in Y, \ldots, \exists y_{N-1} \in Y, \ y_1 < \ldots < y_{\nu-1} < y < y_\nu < \ldots < y_{N-1}\}. \qquad (9)$$

Here $\nu = 1, \ldots, N$. For $n > 1$, connected components of Y are defined by (9) if we replace "$<$" by a lexicographic order in $\mathbf{R}^n$. This proves (B$_0$).

If $Y \subset \mathbf{R}^1$, connected components of $\tilde{Y}$ (intervals) are Φ-subanalytic sets of the form

$$\{y \in \mathbf{I} : \exists \, y_1 \in Y, \ldots, \exists \, y_N \in Y, \ y_1 < \ldots < y_{\nu-1} < y < y_\nu < \ldots < y_N\}. \qquad (10)$$

Here $\nu = 1, \ldots, N+1$. This proves (C$_0$).

For $n > 1$, the set $\tilde{Y}$ is a union of Φ-subanalytic sets defined by (10) if we replace "$<$" by a lexicographic order in $\mathbf{R}^n$. This proves (A$_0$).

Now we want to prove (A$_d$)-(C$_d$) when $d > 0$. The idea is to consider Y as a d-parametric family of 0-dimensional sets, and to apply the analogues of the formulas (9) and (10) to this family. As these formulas involve the number N of points in Y, we have to ensure that the number of points in the sets of our d-parametric family does not depend on the parameters. To achieve this, we identify a Φ-subanalytic set $Z \subset \mathbf{I}^d$, of dimension smaller than d, containing all possible values of the parameters where the number of points can change. Due to the inductive hypothesis, there exists a closed Φ-subanalytic subset $Z' \supset Z$, of dimension smaller than d, such that each connected component of $\tilde{Z}'$ is a d-dimensional Φ-subanalytic subset of type (2). The number of points remains constant over any connected component of $\tilde{Z}'$, and the analogues of (9) and (10) can be applied, proving (B) and (C) and representing part of $\tilde{Y}$ over $\tilde{Z}'$ as a Φ-subanalytic set. Finally, the part of Y that lies over Z' has dimension smaller than d itself, and the inductive hypothesis can be applied to it to complete the proof of (A).

According to Lemma 3, we can consider the case when $X = \bigcup_\nu X'_\nu \bigcup V$, a disjoint union, where V is a Φ-semianalytic set with $\dim(\pi V) < d$, the sets X'_ν are effectively non-singular d-dimensional Φ-semianalytic sets, and projection of X'_ν into $\mathbf{R}^n$ has rank d at every point of X'_ν, for each ν.

Consider first the case $d = n$. Let $Z = \bigcup_\nu \pi \partial X'_\nu \bigcup \pi V$. We have $\dim Z < d$. Due to the inductive hypothesis, we can find a closed Φ-subanalytic subset $Z' \supset Z$ of $\mathbf{R}^d$, with $\dim Z' < d$, such that all connected components of $\tilde{Z}'$ are Φ-subanalytic sets of type (2). By the definition of Z, each connected component of $\tilde{Z}'$ is either a subset of Y or of $\tilde{Y}$. In particular, $\tilde{Z}' \cap \tilde{Y}$ is a Φ-subanalytic set, and $\tilde{Z}' \cap Y$ is a finite disjoint union of Φ-subanalytic subsets of type (2).

We have $\dim(Z' \cap Y) < d$, hence $\widetilde{Z' \cap Y}$ is Φ-subanalytic, by the inductive hypothesis, and $Z' \cap Y$ is a finite disjoint union of Φ-subanalytic subsets of type (2). Thus $\tilde{Y} = (\tilde{Z}' \cap \tilde{Y}) \cup (Z' \cap (\widetilde{Z' \cap Y}))$ is Φ-subanalytic i.e., (A$_d$) is valid, and $Y = (\tilde{Z}' \cap Y) \cup (Z' \cap Y)$

is a finite disjoint union of Φ-subanalytic subsets of type (2) i.e., (B_d) is valid. (C_d) is not applicable in the case $d = n$.

Let now $d < n$. According to the Koopman–Brown theorem [18, Theorem 5.I], for a generic direction l in $\mathbf{R}^n$, projection ρ of $\overline{Y}$ along l to a $(n-1)$-dimensional subspace satisfies the following property:

(*) *For each $y \in \overline{Y}$, the set $\rho^{-1}(\rho y)$ is finite.*

Applying the Koopman-Brown theorem inductively, we can show that, for a generic $(n-d)$-dimensional linear subspace L in $\mathbf{R}^n$, projection ρ of Y along L to a d-dimensional subspace $\mathbf{R}^d$ of $\mathbf{R}^n$ satisfies (*).

As condition (*) is open in L, we can choose L to be defined by linear equations with integer coefficients in $\mathbf{R}^n$. Subspace $\mathbf{R}^d$ can be chosen as a coordinate subspace of $\mathbf{R}^n$.

Let $S = \cup S_\nu$ be the union of the critical sets S_ν of projection $\rho\pi : X'_\nu \to \mathbf{R}^d$. Then S is a Φ-semianalytic set. (Since each X'_ν is effectively non-singular, S_ν is defined by zeroes of certain minors of the Jacobian matrix of functions defining X'_ν, in a properly chosen system of linear coordinates.) It follows from (*) that $\dim \rho\pi S < d$. Otherwise, the set of critical values of $\rho\pi$ would be d-dimensional, in contradiction to the Sard's theorem.

Let $Z = \rho\pi(\bigcup_\nu \partial X'_\nu \bigcup V \bigcup S)$. We have $\dim Z < d$.

By the inductive hypothesis, there exists a closed Φ-subanalytic set $Z' \supset Z$ such that $\dim Z' < d$, the set $\tilde{Z}'$ is Φ-subanalytic, and each connected component of $\tilde{Z}'$ is a d-dimensional Φ-subanalytic set of type (2).

By the definition of Z and implicit functions theorem, the set $X' = X \cap \pi^{-1}\rho^{-1}\tilde{Z}'$ is a finite union of graphs of analytic functions over each connected component of $\tilde{Z}'$. The same is true for $Y' = Y \cap \pi^{-1}\rho^{-1}\tilde{Z}'$.

By the condition c) of Lemma 3, the graphs for X' are all disjoint. However, after projection π to $\mathbf{R}^n$, some of the different graphs for Y' can intersect along a subset W of dimension smaller than d. To show that W is a Φ-subanalytic set, we consider a Φ-subanalytic set

$$W' = \{y, \epsilon : y \in Y, \ \rho y \in \tilde{Z}', \ \epsilon \in \mathbf{R}^1, \ \exists y' \in Y, \ y' \neq y, \ \rho y' = \rho y, \ , \ |y' - y| < \epsilon\}.$$

Then $W = \overline{W'} \cap \{\epsilon = 0\}$ is Φ-subanalytic due to Lemma 1.

Adding ρW to Z and replacing Z' correspondingly, we can guarantee that the number N of points in Y over each connected component C of $\tilde{Z}'$ remains constant.

In this case, connected components C_μ of $Y \cap \rho^{-1}C$ are defined by the following existential expression, similar to (9):

$$C_\mu = \{y \in Y : \rho(y) \in C, \ \exists y_1 \in Y, \ldots, \exists y_{N-1} \in Y,$$

$$\rho' y_1 < \ldots < \rho' y_{\mu-1} < \rho' y < \rho' y_\mu < \ldots < \rho' y_{N-1}, \; \rho(y_1) = \ldots = \rho(y_{N-1}) = \rho(y)\}.$$

Here $\mu = 1, \ldots, N$ and "$<$" is the order in the 1-dimensional space L. Obviously, C_μ are of type (2) as long as C is of type (2).

As $\dim Y \cap \rho^{-1} Z' < d$, this, together with the inductive hypothesis, proves (B_d).

For $n = d + 1$, connected components C_μ of $\tilde{Y} \cap \rho^{-1} C$ are defined by the following existential expression, similar to (10):

$$C_\mu = \{y : \rho(y) \in C, \; \exists \, y_1 \in Y, \ldots, \exists \, y_N \in Y,$$

$$y_1 < \ldots < y_{\mu-1} < y < y_\mu < \ldots < y_N, \; \rho(y_1) = \ldots = \rho(y_N) = \rho(y)\}. \tag{11}$$

Here $\mu = 1, \ldots, N + 1$.

Again, C_μ are of type (2) as long as C is of type (2). Setting $Y' = Y \cup \rho^{-1} Z'$, we prove (C_d).

For $d < n - 1$, the set $\tilde{Y} \cap \rho^{-1} C$ is the union of C_μ if "$<$" is defined as a lexicographic order. As $Y_0 = Y \cap \rho^{-1} Z'$ has dimension smaller than d, the set $\tilde{Y}_0$ is Φ-subanalytic due to inductive hypothesis. Hence $\tilde{Y}$, which is the union of $\tilde{Y}_0 \cap \rho^{-1} Z'$ and the sets $\tilde{Y} \cap \rho^{-1} C$ over all connected components C of $\tilde{Z}'$, is Φ-subanalytic. This proves (A_d).

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