MULTIPLICITY OF A ZERO OF AN ANALYTIC FUNCTION ON A TRAJECTORY OF A VECTOR FIELD

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To Vladimir Igorevich Arnol'd on his 60th birthday

Abstract. The multiplicity μ of a zero of a restriction of an analytic function P in $\mathbb{C}^n$ to a trajectory of a vector field ξ with analytic coefficients is equal to the sum of the Euler characteristics of Milnor fibers associated with a deformation of P. When P is a polynomial of degree p and ξ is a vector field with polynomial coefficients of degree q, this allows one to compute μ in purely algebraic terms, and to give an upper bound for μ in terms of $n,\ p,\ q$, single exponential in n and polynomial in $p,\ q$. This implies a single exponential in n bound on degree of nonholonomy of a system of polynomial vector fields in $\mathbb{C}^n$.

Introduction

Let $P(x)$ be a germ at the origin of an analytic function in $\mathbb{C}^n$, where $x=(x_1,\dots,x_n)$, and let $\xi=\xi_1(x)\partial/\partial x_1+\dots+\xi_n(x)\partial/\partial x_n$ be a germ at the origin of an analytic vector field. Suppose that $\xi(0)\neq 0$, and let γ be a trajectory of ξ through the origin. Suppose that $P|_\gamma\not\equiv 0$, and let $\mu(P|_\gamma)$ be the multiplicity of a zero of $P|_\gamma$ at the origin. Let $\xi P=\xi_1\partial P/\partial x_1+\dots+\xi_n\partial P/\partial x_n$ be derivative of P in the direction of ξ, and let $\xi^k P$ be the kth iteration of this derivative.

We show (Theorem 1) that $\mu(P|_\gamma)$ is a sum of the Euler characteristics of "Milnor fibers" $X_k=\{\tilde P=\xi\tilde P=\dots=\xi^{k-1}\tilde P=0\}$ associated with a deformation $\tilde P$ of P. For a polynomial P of degree p and a vector field ξ with polynomial coefficients of degree q, X_k are (semi-)algebraic sets. This allows one to compute $\mu(P|_\gamma)$ in purely algebraic terms (Theorem 3), and to give an upper bound (Theorem 2) for $\mu(P|_\gamma)$ in terms of $n,\ p,\ q$, single exponential in n and polynomial in p and q. This estimate improves previous results [9, 1] which were double exponential in n.

For a system $\Xi=\{\xi_i\}$ of vector fields in $\mathbb{R}^n$ with polynomial coefficients of degree not exceeding q, this implies a single exponential in n and polynomial in q estimate for the degree of nonholonomy of Ξ, i.e., for the minimal order of brackets of ξ_i necessary to

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generate a subspace of maximal possible dimension at each point of $\mathbb{R}^n$. This improves an estimate in [1], which was double exponential in n.

For $n = 2$, our estimate coincides with the estimate for the multiplicity of a Pfaffian intersection [5, 2]. In case $n = 3$, a similar estimate was obtained in [3].

The main result of this paper can be reformulated as follows. Let $x(t) : \mathbb{C}_{t,0} \to \mathbb{C}^n_{x,0}$ be a germ of an analytic vector-function satisfying a system of nonlinear algebraic differential equations $S_i(x(t), t)dx_i/dt = Q_i(x(t), t)$ where S_i and Q_i are polynomial in (x, t) of degree q, and $S_i(0, 0) \neq 0$. Let $p(t) = P(x(t), t)$ where P is a polynomial in (x, t) of degree p. Suppose that $p(t) \not\equiv 0$. Then the multiplicity of a zero of $p(t)$ at $t = 0$ can be computed in purely algebraic terms, and there is an estimate for this multiplicity in terms of n, p, q, single exponential in n and polynomial in p and q.

1. The main result

Definition 1. A germ $\tilde{P}(x, \epsilon)$ of an analytic function at the origin in $\mathbb{C}^{n+1}$ is called a *deformation* of P if $\tilde{P}(x, 0) = P(x)$. For a fixed ϵ, we write $\tilde{P}^\epsilon(x)$ for the function $\tilde{P}(x, \epsilon)$ considered as a function of x.

For a real positive δ, let B_δ be a closed ball in $\mathbb{C}^n$ of radius δ centered at the origin.

Proposition 1. *Let* $\mathbf{P}(x) = \big(P_1(x), \ldots, P_k(x)\big)$ *be a k-tuple of germs of analytic functions at the origin in* $\mathbb{C}^n$*, and let* $\tilde{\mathbf{P}}(x, \epsilon) = \big(\tilde{P}_1(x, \epsilon), \ldots, \tilde{P}_k(x, \epsilon)\big)$ *be a deformation of* $\mathbf{P}(x)$*. Then, for a small positive δ and for a nonzero $\epsilon \in \mathbb{C}$ much smaller than δ, the homotopy type of the set* $\{\tilde{\mathbf{P}}^\epsilon = 0\} \cap B_\delta$ *does not depend on δ and ϵ, and on the choice of metrics in* $\mathbb{C}^n$*. This set is called the Milnor fiber of* $\tilde{\mathbf{P}}$*.*

Proof. This follows from Lê Dũng Tráng's generalization of Milnor's fibration theorem [7]. One has to consider fibration of an analytic set $X = \{\tilde{\mathbf{P}} = 0\} \cap B_\delta \subset \mathbb{C}^{n+1}$ by nonzero level sets of the function $\epsilon : X \to \mathbb{C}$.

Definition 2. Let $\epsilon : X \to \mathbb{C}$ be an analytic function on an analytic set X, such that $X^\epsilon = X \cap \{\epsilon = \text{const}\}$ is nonsingular for small $\epsilon \neq 0$. Let Z be an analytic subset of X^0, and let $\{Z_\alpha\}$ be a Whitney stratification of Z, where Z_α are nonsingular analytic manifolds and their closures are analytic subsets of X. It is called Thom's A_ϵ stratification, if the following holds:

Let x_ν be a sequence of points in X converging to $x^0 \in Z_\alpha$, such that tangent spaces to $X^{\epsilon(x_\nu)}$ at x_ν have a limit T as $\nu \to \infty$. Then T contains the tangent space to Z_α at x^0.

According to [6], a stratification with this property always exists.

Definition 3. Let $\mathbf{P}(x) = \big(P_1(x), \ldots, P_k(x)\big)$, and let $\tilde{\mathbf{P}}(x, \epsilon)$ be a deformation of $\mathbf{P}(x)$. Suppose that, for small $\epsilon \neq 0$, the Milnor fiber X^ϵ of $\tilde{\mathbf{P}}$ is nonsingular.

Let X be the closure of $\bigcup_{\epsilon \neq 0} X^\epsilon$, and let $Z = X \cap \{\epsilon = 0\}$. Let $\{Z_\alpha\}$ be a Thom's A_ϵ stratification of $Z \setminus 0$.

Let $l(x)$ be an analytic function in $\mathbb{C}^n$ such that the set Γ^ϵ of critical points of $l|_{X^\epsilon}$, for small $\epsilon \neq 0$, is finite. Let ν be the number of these points, counted with their multiplicities, converging to the origin as $\epsilon \to 0$. The closure Γ of $\bigcup_{\epsilon \neq 0} \Gamma^\epsilon$ is called the *polar curve* of $\tilde{\mathbf{P}}$ relative to l, and ν is the *multiplicity* of Γ.

Proposition 2. *Let* $\mathbf{P}(x) = \big(P_1(x), \dots, P_k(x)\big)$, *and let* $\tilde{\mathbf{P}}(x, \epsilon)$ *be a deformation of* $\mathbf{P}(x)$. *Suppose that, for small* $\epsilon \neq 0$, *the Milnor fiber* X^ϵ *of* $\tilde{\mathbf{P}}$ *is nonsingular. Let* X *be the closure of* $\bigcup_{\epsilon\neq 0} X^\epsilon$, *and let* $Z = X \cap \{\epsilon = 0\}$. *Let* $\{Z_\alpha\}$ *be an* A_ϵ *stratification of* $Z \setminus 0$.

Let $l(x)$ *be an analytic function in* $\mathbb{C}^n$ *such that* $\{l(x) = 0\}$ *is transversal to all* Z_α. *Let* Γ *be the polar curve of* $\tilde{\mathbf{P}}$ *relative to* l, *and let* ν *be the multiplicity of* Γ.

Then the Milnor fiber of $\tilde{\mathbf{P}}$ *can be obtained from the Milnor fiber of* $(\tilde{\mathbf{P}}, l)$ *by attaching* ν *cells of dimension* $n - k$.

Proof. This follows from the proof of the "generic hyperplane section" theorem in [7].

Theorem 1. *Let* $P(x)$ *be a germ at* $0 \in \mathbb{C}^n$ *of an analytic function, and let* $\tilde{P}(x, \epsilon)$ *be a deformation of* $P(x)$. *Let* ξ *be a germ at* $0 \in \mathbb{C}^n$ *of an analytic vector field. Suppose that* $\xi(0) \neq 0$, *and let* γ *be a trajectory of* ξ *through* 0. *Suppose that* $P|_\gamma \not\equiv 0$, *and let* μ *be the multiplicity of a zero of* $P|_\gamma$ *at* 0. *Let* X_k *be the Milnor fiber of* $\tilde{\mathbf{P}}_k = (\tilde{P}, \xi\tilde{P}, \dots, \xi^{k-1}\tilde{P})$. *Suppose that* X_k *is a nonsingular* $(n-k)$*-dimensional set, for* $k = 1, \dots, n$, *and let* $\chi(X_k)$ *be the Euler characteristic of* X_k. *Then*

$$\mu = \chi(X_1) + \dots + \chi(X_n). \tag{1}$$

The proof of this theorem will be given in the next section. A. Khovanskii suggested an alternative proof, valid also when the Milnor fibers X_k are singular. In fact, the following holds:

Theorem 1′. *Let* P, $\tilde{P}$, ξ, γ, μ, *and* X_k *be the same as in Theorem 1, without any non-singularity conditions on* X_k. *Then*

$$\mu = \sum_{k=1}^{\mu} \chi(X_k). \tag{2}$$

Proof. (See [4].) Let $y = (y_1, \dots, y_n)$ be a system of coordinates in $\mathbb{C}^n$ where $\xi = \partial/\partial y_1$, let $y' = (y_2, \dots, y_n)$, and let π be projection $\mathbb{C}^n_y \to \mathbb{C}^{n-1}_{y'}$ along y_1-axis. Let us choose a metric in $\mathbb{C}^n$ so that a small ball B^n in $\mathbb{C}^n$ is a product of a small ball B^{n-1} in $\mathbb{C}^{n-1}$ and a small disk D in $\mathbb{C}$, where B^{n-1} and D are chosen so that $\{P = 0\} \cap (B^{n-1} \times \partial D) = \emptyset$. Then each fiber of the projection $\pi : \{P = 0\} \cap B^n \to B^{n-1}$ consists of exactly μ points (counting multiplicities). For small enough ϵ, the same is true for $\tilde{P}^\epsilon$ instead of P.

The set X_k consists of those points y where the multiplicity of a zero of $\tilde{P}^\epsilon$ restricted to $\{y' = \text{const}\}$ is at least k. In particular, $X_k = \emptyset$ for $k > \mu$. For $1 \le k \le \mu$, let $\zeta_k(y') = \chi(X_k \cap \pi^{-1}y')$. Since each set $\pi^{-1}y' \cap X_k$ is finite, its Euler characteristic equals to the number of points in it, not counting multiplicities. Hence

$$\sum_{k=1}^{\mu} \zeta_k(y') \equiv \mu.$$

Fubini theorem for the integral over Euler characteristic [11] implies

$$\int_{B^{n-1}} \zeta_k d\chi = \int_{B^n} \mathbf{1}_{X_k} d\chi = \chi(X_k).$$

Here $\mathbf{1}_{X_k}$ is the characteristic function of the set X_k. At the same time,

$$\int_{B^{n-1}} \left(\sum_{k=1}^{\mu} \zeta_k\right) d\chi = \int_{B^{n-1}} \mu d\chi = \mu\chi(B^{n-1}) = \mu.$$

Theorem $1'$ follows from these two equalities.

Remark. Theorem 1 follows from Theorem $1'$: when X_k, for $k = 1, \dots n$, are nonsingular, we can modify $\tilde{P}$ so that topology of X_k remains unchanged for $k = 1, \dots n$, and $X_k = \emptyset$ for $k > n$.

Lemma 1. *Let $l(x)$ be a germ of an analytic function such that $\xi l(0) \neq 0$. Let δ be a small positive number. For $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, let $P_c(x) = P(x) + c_1 + c_2 l(x) + \cdots + c_n l^{n-1}(x)$. Let $X_{k,c} = \{P_c = \xi P_c = \cdots = \xi^{k-1} P_c = 0\} \cap B^n_\delta$.*

(i) *For a generic c, the set $X_{k,c}$ is nonsingular $(n-k)$-dimensional, for $k = 1, \dots, n$.*

(ii) *For a generic c, the deformation $\tilde{P}(x, \epsilon) = P_{\epsilon c}(x)$ satisfies conditions of Theorem 1.*

Proof. For a small positive δ, we can choose a coordinate system $(y_1, \dots, y_n)$ in B^n_δ so that $y_1 = l(x)$ and trajectories of ξ are defined by $y_2 = \text{const}, \dots, y_n = \text{const}$. This means that, in the new coordinates, $\xi = u(y)\partial/\partial l$ where $u(0) \neq 0$. Accordingly,

$$X_{k,c} = \{y \in B_\delta,\ P_c(y) = \frac{\partial}{\partial l} P_c(y) = \cdots = \frac{\partial^{k-1}}{\partial l^{k-1}} P_c(y) = 0\}.$$

Let $Q(y, c)$ be $P_c(y)$ considered as a function of $2n$ variables y and c. Let

$$Z_k = \cup_x X_{k,c} = \{y \in B_\delta,\ c \in \mathbb{C}^n,\ Q(y, c) = \frac{\partial}{\partial l} Q(y, c) = \cdots = \frac{\partial^{k-1}}{\partial l^{k-1}} Q(y, c) = 0\}.$$

For $k = 1, \dots, n$, the set Z_k is nonsingular $(2n-k)$-dimensional, because differentials of $\partial^i Q(y, c)/\partial l^i$ are independent near $y = 0$:

$$\frac{\partial}{\partial c_j} \frac{\partial^{i-1}}{\partial l^{i-1}} Q(0, c) = (i-1)!\delta_{ij} \quad \text{for } 1 \leq i, j \leq n.$$

Let $\pi : Z_k \to \mathbb{C}^n_c$ be a natural projection. The set $X_{k,c}$ is nonsingular if and only if c is not a critical value of π. Due to Sard's theorem, this holds for a general c. This proves (i).

To prove (ii), note that, for $\tilde{P}(x, \epsilon) = P_{\epsilon c}(x)$, the Milnor fiber of $(\tilde{P}, \xi\tilde{P}, \dots, \xi^{k-1}\tilde{P})$ coincides with $X_{k,\epsilon c}$, for small nonzero ϵ. Consider the set $W_k \subset Z_k$ of critical points of π. For $c \neq 0$, let L_c denote a linear subspace in $\mathbb{C}^n$ generated by c. For a generic c, the intersection of W_k with $\pi^{-1}(L_c \setminus 0)$ is zero-dimensional (or empty). Otherwise, this intersection would be at least one-dimensional, and $\pi(W_k)$ would be n-dimensional, in contradiction to Sard's theorem. This implies that, for a generic c and small enough ϵ, the set $X_{k,\epsilon c}$ is nonsingular.

2. Proof of Theorem 1

Let us choose a coordinate system $y=(y_1,\dots,y_n)$ in a neighborhood of the origin in $\mathbb{C}^n$ so that $\xi=\partial/\partial y_1$ in this coordinate system. In particular, trajectory γ of ξ through the origin becomes y_1-axis, and μ equals to the multiplicity of a zero of $P(y_1,0,\dots,0)$ at the origin. Let $\tilde{P}(y,\epsilon)$ be a deformation of P satisfying conditions of Theorem 1, i.e., the Milnor fiber X_k of $\tilde{\mathbf{P}}_k=(\tilde{P},\xi\tilde{P},\dots,\xi^{k-1}\tilde{P})$ is nonsingular $(n-k)$-dimensional, for $k=1,\dots,n$.

We proceed by induction on n. For $n=1$ the statement is obvious. Suppose that it holds for $n-1$. We want to apply it to the subspace $\{y_n=0\}$ of $\mathbb{C}^n$. Let $\tilde{P}'=\tilde{P}|_{y_n=0}$, and let $\tilde{\mathbf{P}}'_k=(\tilde{P}',\xi\tilde{P}',\dots,\xi^{k-1}\tilde{P}')$, for $k=1,\dots,n-1$.

First of all, to satisfy conditions of Theorem 1, the Milnor fiber X'_k of $\tilde{\mathbf{P}}'_k$ should be nonsingular. Singularities of X'_k coincide with zero critical values of y_n restricted to X_k. Consider these critical values as functions of ϵ. For large enough N, none of these critical values equals ϵ^N identically. Let us replace $\tilde{P}(y,\epsilon)$ by $\tilde{P}(y_1,\dots,y_{n-1},y_n-\epsilon^N,\epsilon)$. If N is large enough, this does not change topology of X_k, and makes X'_k nonsingular.

Due to inductive hypothesis,

$$\mu=\chi(X'_1)+\cdots+\chi(X'_{n-1}). \tag{3}$$

Next, we want to apply Proposition 2 to $l=y_n$. Let us show that, for a generic choice of y_n, conditions of Proposition 2 are satisfied.

Let X be the closure of $\{\tilde{\mathbf{P}}_k(y,\epsilon)=0,\ \epsilon\neq 0\}$, and let $X_0=X\cap\{\epsilon=0\}$. Let $\{Z_\alpha\}$ be a Thom's A_ϵ stratification of $X_0\backslash 0$. As $P|_\gamma\not\equiv 0$, none of Z_α contains y_1-axis. Hence a generic linear hyperplane H containing y_1-axis is transversal to all Z_α. To satisfy conditions of Proposition 2, we can choose y_n so that $H=\{y_n=0\}$.

Due to Proposition 2, X_k can be obtained from X'_k by attaching ν_k cells of dimension $n-k$, where ν_k is the number of critical points of $y_n|_{X_k}$ counted with their multiplicities. In particular,

$$\chi(X'_k)=\chi(X_k)-(-1)^{n-k}\nu_k. \tag{4}$$

The critical points of $y_n|_{X_k}$ are defined by linear dependence at the points of X_k of the following 1-forms:

$$d(\tilde{P}^\epsilon),\ d(\xi\tilde{P}^\epsilon),\ \dots\ d(\xi^{k-1}\tilde{P}^\epsilon),\ dy_n.$$

In other words, rank of the following $k\times(n-1)$-matrix A_k should be less than k:

$$A_k=\begin{pmatrix} \frac{\partial}{\partial y_1}\tilde{P}^\epsilon & \frac{\partial}{\partial y_2}\tilde{P}^\epsilon & \dots & \frac{\partial}{\partial y_{n-1}}\tilde{P}^\epsilon \\ \frac{\partial}{\partial y_1}\xi\tilde{P}^\epsilon & \frac{\partial}{\partial y_2}\xi\tilde{P}^\epsilon & \dots & \frac{\partial}{\partial y_{n-1}}\xi\tilde{P}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial y_1}\xi^{k-1}\tilde{P}^\epsilon & \frac{\partial}{\partial y_2}\xi^{k-1}\tilde{P}^\epsilon & \dots & \frac{\partial}{\partial y_{n-1}}\xi^{k-1}\tilde{P}^\epsilon \end{pmatrix}.$$

Taking into account that $\xi = \partial/\partial y_1$, we find that, at the points of X_k, all the entries in the first column of the matrix A_k are zero, except for the last entry which is $\xi^k \tilde{P}^\epsilon$:

$$A_k = \begin{pmatrix} 0 & \frac{\partial}{\partial y_2}\tilde{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}}\tilde{P}^\epsilon \\ 0 & \frac{\partial}{\partial y_2}\xi\tilde{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}}\xi\tilde{P}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial}{\partial y_2}\xi^{k-2}\tilde{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}}\xi^{k-2}\tilde{P}^\epsilon \\ \xi^k\tilde{P}^\epsilon & \frac{\partial}{\partial y_2}\xi^{k-1}\tilde{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}}\xi^{k-1}\tilde{P}^\epsilon \end{pmatrix}.$$

Let B_k be the matrix A_k with the first column removed, and let C_k be the matrix A_k with the first column and the last row removed. For $k = 1, \ldots, n-2$, we have

$$B_k = C_{k+1} = \begin{pmatrix} \frac{\partial}{\partial y_2}\tilde{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}}\tilde{P}^\epsilon \\ \frac{\partial}{\partial y_2}\xi\tilde{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}}\xi\tilde{P}^\epsilon \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial y_2}\xi^{k-1}\tilde{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}}\xi^{k-1}\tilde{P}^\epsilon \end{pmatrix}.$$

For $k = 1, \ldots, n-1$, rank of A_k is less than k if either $\xi^k\tilde{P}^\epsilon = 0$ and rank of B_k is less than k, or if $\xi^k\tilde{P}^\epsilon \neq 0$ and rank of C_k is less than $k-1$. Geometrically, first of these two conditions defines those points of X_{k+1} where X_k is not transversal to $(n-2)$-dimensional space $L = (y_1 = \text{const},\ y_n = \text{const})$, and the second condition defines those points in X_k where X_{k-1} is not transversal to L. (For $k = 1$, the second condition is empty.)

For a generic choice of coordinates y_1 and y_n, L is a generic $(n-2)$-dimensional linear subspace in $\mathbb{C}^n$. From Thom's transversality theorem, the set of points where X_{k-1} is not transversal to L is one-dimensional and does not intersect X_{k+1}, which has codimension two in X_{k-1}.

This means that, for generic coordinates y, the set of critical points of $y_n|_{X_k}$ is a union of two disjoint sets: $X_k \cap \{\xi^k\tilde{P}^\epsilon = 0\} \cap \{\operatorname{rank} B_k < k\}$ and $X_k \cap \{\operatorname{rank} C_k < k-1\}$. Hence $\nu_k = \nu'_k + \nu''_k$, where ν'_k and ν''_k are the numbers of critical points of $y_n|_{X_k}$ in these two sets, counted with their multiplicities.

Taking into account that $B_k = C_{k+1}$ and $X_k \cap \{\xi^k\tilde{P}^\epsilon = 0\} = X_{k+1}$, we have $\nu'_k = \nu''_{k+1}$, for $k = 1, \ldots, n-2$. For $k = 1$, we have $\nu_1 = \nu'_1$. For $k = n-1$, we have $\nu'_{n-1} = \chi(X_n)$, the number of points in the set X_n.

Replacing ν_k in (4) by $\nu'_k + \nu''_k$ and substituting (4) into (3), we see that all the values ν'_k and ν''_k cancel out, except ν'_{n-1}, and (3) implies (1).

3. Algebraic case

Theorem 2. *Let P be a polynomial in $\mathbb{C}^n$ of degree not exceeding $p \geq n-1$, and let ξ be a vector field with polynomial coefficients of degree not exceeding $q \geq 1$. Suppose that $\xi(0) \neq 0$, and let γ be a trajectory of ξ through 0.*

(i) Let $P|_\gamma \not\equiv 0$, and let μ be the multiplicity of a zero of $P|_\gamma$ at 0. Then μ is less than

$$2^{2n-1}\sum_{k=1}^{n}[p + (k-1)(q-1)]^{2n}. \tag{5}$$

(ii) Let $P|_\gamma \equiv 0$, and let P_ν be any sequence of polynomials of degree not exceeding p converging to P as $\nu \to \infty$. Then the number of isolated zeros of $P_\nu|_\gamma$ converging to the origin as $\nu \to \infty$ is less than (5).

Proof. (i) From Lemma 1, there exists a deformation $\tilde{P}$ of P satisfying conditions of Theorem 1, such that P^ϵ is a polynomial of degree not exceeding p. Hence degree of $\xi^i \tilde{P}^\epsilon$ does not exceed $p + i(q-1)$. Thus the Milnor fiber X_k of $(\tilde{P}, \xi\tilde{P}, \dots, \xi^{k-1}\tilde{P})$ is defined by polynomial equations of degree not exceeding $d = p + (k-1)(q-1)$. From [8], the sum of Betti numbers of X_k does not exceed $d(2d-1)^{2n-1}$, which is less than $2^{2n-1}d^{2n}$. The estimate (5) follows now from Theorem 1.

(ii) The statement follows from (i) and the results of [12]. An alternative argument was suggested by Khovanskii. Let $\mathcal{L}$ denote the linear space of all polynomials of degree not exceeding p modulo polynomials identically vanishing on γ. Let P_ν be a sequence of polynomials P_ν converging to P such that M zeros of $P_\nu|_\gamma$ converge to the origin as $\nu \to \infty$. These polynomials define a sequence of points Q_ν in $\mathcal{L}$. Note that the zeros of $P_\nu|_\gamma$ depend only on Q_ν, and do not change when we multiply Q_ν by a constant. If we define any norm in $\mathcal{L}$, we obtain a sequence of points $Q_\nu/|Q_\nu|$ in $\mathcal{L}$ that has a non-zero limit point Q_0. Let P_0 be a polynomial of degree not exceeding p such that its image in $\mathcal{L}$ is Q_0. Obviously, $P_0|_\gamma$ has a zero of the multiplicity M at 0. Hence M is less than (5).

We want to show that, for a polynomial P and a vector field ξ with polynomial coefficients, the value of μ in (1) can be computed in purely algebraic terms. First, we need another expression for μ, valid also for non-algebraic P and ξ.

Theorem 3. *Let $P(x)$ be a germ at $0 \in \mathbb{C}^n$ of an analytic function, and let ξ be a germ at $0 \in \mathbb{C}^n$ of an analytic vector field. Suppose that $\xi(0) \neq 0$, and let γ be a trajectory of ξ through 0. Suppose that $P|_\gamma \not\equiv 0$, and let μ be the multiplicity of a zero of P_γ at 0. Let $\tilde{P}(x,\epsilon)$ be a deformation of $P(x)$ satisfying conditions of Theorem 1, and let $\tilde{P}^\epsilon$ be $\tilde{P}(x,\epsilon)$ considered as a function of x, for a fixed nonzero ϵ. Let $l_1(x), \dots, l_{n-1}(x)$ be generic linear forms in $\mathbb{C}^n$. For a small positive δ and a small nonzero $\epsilon \ll \delta$, let*

$$X_{i,k} = \{x \in B^n_\delta,\ \tilde{P}^\epsilon(x) = \xi\tilde{P}^\epsilon(x) = \dots = \xi^{k-1}\tilde{P}^\epsilon(x) = l_1(x) = \dots = l_{n-k-i}(x) = 0\},$$

for $k = 1, \dots, n$ and $i = 0, \dots, n-k$. Let $\nu_{0,k}$ be the number of points in $X_{0,k}$ converging to the origin as $\epsilon \to 0$. For $i = 1, \dots, n-k$, let $\nu_{i,k}$ be the multiplicity of the polar curve of $(\tilde{P}, \xi\tilde{P}, \dots, \xi^{k-1}\tilde{P}, l_1, \dots, l_{n-k-i})$ relative to $l_{n-k-i+1}$, i.e., the number of critical points of $l_{n-k-i+1}|_{X_{i,k}}$ converging to the origin as $\epsilon \to 0$. Then

$$\mu = \sum_{k=1}^{n} \sum_{i=0}^{n-k} (-1)^i \nu_{i,k}. \tag{6}$$

Proof. Let X_k be the Milnor fiber of the deformation $\tilde{\mathbf{P}}_k = (\tilde{P}, \xi\tilde{P}, \dots, \xi^{k-1}\tilde{P})$. Then $X_{i,k}$ is the intersection of X_k with a generic linear $(k+i)$-dimensional subspace $L^{k+i} = \{l_1 = \dots = l_{n-k-i} = 0\}$. In particular, $X_{n-k,k} = X_k$. We suppose X_k to be nonsingular

$(n-k)$-dimensional, hence $X_{i,k}$ is a nonsingular i-dimensional set, and, for a generic linear form $l_{n-k-i+1}$, all critical points of $l_{n-k-i+1}|_{X_{i,k}}$ are non-degenerate.

In particular, $X_{0,k}$ is zero-dimensional, and $\chi(X_{0,k}) = \nu_{0,k}$. From Proposition 2, for $k = 1, \ldots, n$ and $i = 1, \ldots, k$, we have

$$\chi(X_{i,k}) - \chi(X_{i-1,k}) = (-1)^i \nu_{i,k}.$$

Hence

$$\chi(X_k) = \sum_{i=0}^{n-k} (-1)^i \nu_{i,k}.$$

From Theorem 1, $\mu = \sum_{k=1}^{n} \chi(X_k) = \sum_{k=1}^{n} \sum_{i=0}^{n-k} (-1)^i \nu_{i,k}$.

Corollary. *For a polynomial P in $\mathbb{C}^n$ of degree not exceeding p, and for a vector field ξ in $\mathbb{C}^n$ with polynomial coefficients of degree not exceeding q, the value of μ in (6) can be computed as the number of solutions of a finite system of algebraic equations and inequalities. The number of equations and inequalities in this system, and their degrees, can be estimated in terms of n, p, and q.*

Proof. For polynomial P and ξ, the sets $X_{i,k}$ in Theorem 3 are semi-algebraic, and each number $\nu_{i,k}$ in (6) is defined as the number of solutions of a system of algebraic equations and inequalities, with an estimate for the number of equations and inequalities and for their degrees in terms of n, p, and q.

4. Degree of nonholonomy

Definition 4. Let $\Xi = \{\xi_i\}$ be a system of vector fields in $\mathbb{C}^n$ or $\mathbb{R}^n$. Let L_x be a vector space spanned by the values of ξ_i, and of their brackets of all orders, at a point x. Here ξ_i themselves are considered as brackets of order one, $[\xi_i, \xi_j]$ as brackets of order two, $[\xi_i, [\xi_j, \xi_k]]$ as brackets of order three, and so on. *Degree of nonholonomy* of Ξ at x is the minimal number N_x such that the values at x of ξ_i, and of their brackets of order not exceeding N_x, generate L_x.

Theorem 4. *Let $\Xi = \{\xi_i\}$ be a system of vector fields in $\mathbb{C}^n$ or $\mathbb{R}^n$ with polynomial coefficients of degree not exceeding $\beta \geq 1$. Let d be dimension of the vector space L_x spanned by the values at x of ξ_i and their brackets of all orders. Then degree of nonholonomy of Ξ at x is less than*

$$2^{d-2}\left(1 + 2^{2n(d-2)-2}\beta^{2n}\sum_{k=1}^{n}(k+3)^{2n}\right) \quad \text{for } d > 2, \tag{7}$$

$$1 + 2^{2n-1}\beta^{2n}\sum_{k=1}^{n}(k+1)^{2n}, \quad \text{for } d = 2. \tag{8}$$

Proof. According to Proposition 1 of [1], there exist vector fields $\chi_0, \chi_1, \ldots, \chi_{d-1}$ with polynomial coefficients, such that

(i) χ_0 and χ_1 are some of ξ_i, and $\chi_0(x) \neq 0$;
(ii) for $j > 1$, χ_j is either one of ξ_i or a linear combination of brackets $[\chi_\mu, f\chi_\nu]$ where $\mu, \nu < j$ and f is a linear function;
(iii) for a generic $c = (c_1, \dots, c_{d-2})$, $\chi_0 \wedge \dots \wedge \chi_{d-1}$ does not vanish identically at the points of a trajectory γ of $\chi_c = \chi_0 + c_1\chi_1 + \dots + c_{d-2}\chi_{d-2}$ through x.

In particular, each χ_j is a vector field with polynomial coefficients of degree not exceeding $\max(1, 2^{j-1})\beta$. Let $Q = \chi_0 \wedge \dots \wedge \chi_{d-1}$. We have

$$Q = \sum_{i_1,\dots,i_d} Q_{i_1\dots i_d} \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_d}},$$

where $Q_{i_1\dots i_d}$ are polynomials of degree not exceeding $p = 2^{d-1}\beta$. Due to (iii), some of these polynomials do not vanish identically on a trajectory γ through x of a vector field χ_c with polynomial coefficients of degree not exceeding $q = \max(1, 2^{d-3})\beta$. Due to Theorem 2, the multiplicity μ of a zero of such a polynomial restricted to γ is less than

$$2^{2n-1}\sum_{k=1}^{n}[p + (k-1)(q-1)]^{2n} < 2^{2n-1}\beta^{2n}\sum_{k=1}^{n}[2^{d-1} + (k-1)\max(1, 2^{d-3})]^{2n}. \tag{9}$$

Each derivation of Q along χ_c decreases this multiplicity by 1. Hence the result of μ consecutive derivations of Q along χ_c does not vanish at x. From (ii), χ_j are linear combinations, with polynomial coefficients, of brackets of ξ_i of order not exceeding 2^{d-2}, and χ_c is a combination of brackets of ξ_i of order not exceeding $\max(1, 2^{d-3})$. Taking into account a formula for a derivation along χ_c:

$$\partial_{\chi_c}(\chi_0 \wedge \dots \wedge \chi_{d-1}) = \sum_{i=0}^{d-1} \chi_0 \wedge \dots \wedge [\chi_c, \chi_i] \wedge \dots \wedge \chi_{d-1},$$

we see that the result of μ derivations of Q along χ_c is a linear combination, with polynomial coefficients, of wedge-products of vector fields which are brackets of ξ_i of order not exceeding

$$2^{d-2} + \max(1, 2^{d-3})\mu. \tag{10}$$

From (9), this order is less than (7) for $d > 2$, and (8) for $d = 2$. Since the result of μ derivations of Q along χ_c does not vanish at x, there exist d brackets of ξ_i of order not exceeding (10) which are linearly independent at x, hence generate L_x.

5. Noetherian functions

Definition 5. (Khovanskii, unpublished; see [10].) A *Noetherian chain* of order m and degree α is a system $f(x) = (f_1(x), \dots, f_m(x))$ of germs of analytic functions at the origin $\mathbf{0}$ of a complex or real n-dimensional space, satisfying

$$\frac{\partial f_i}{\partial x_j} = g_{ij}(x, f(x)), \text{ for } i = 1, \dots, m \text{ and } j = 1, \dots, n, \tag{11}$$

where g_{ij} are polynomials in x and f of degree not exceeding $\alpha \geq 1$. A function $\phi(x) = P(x, f(x))$, where P is a polynomial in x and f of degree not exceeding p, is called a *Noetherian function* of degree p, with the Noetherian chain f.

The following two theorems can be reduced to Theorems 2 and 4 by adding m new variables corresponding to m functions of a Noetherian chain (see [1]).

Theorem 5. *Let $f = (f_1, \dots, f_m)$ be a Noetherian chain of order m and degree α, and let $\xi = \sum_j \phi_j(x)\partial/\partial x_j$ be a vector field with the coefficients ϕ_j Noetherian of degree q, with the Noetherian chain f. Let ψ be a Noetherian function of degree p, with the Noetherian chain f. Suppose that $\xi(0) \neq 0$ and that ψ does not vanish identically on the trajectory γ of ξ through 0. Then the multiplicity of the zero of $\psi|_\gamma$ at 0 is less than*

$$2^{2(n+m)-1} \sum_{k=1}^{n+m} [p + (k-1)(q + \alpha - 1))]^{2(n+m)}. \tag{12}$$

Theorem 6. *Let $f = (f_1, \dots, f_m)$ be a Noetherian chain in $\mathbb{C}^n$ or $\mathbb{R}^n$ of order m and degree $\alpha \geq 1$. Let $\Xi = \{\xi_i\}$ be a set of vector fields with Noetherian coefficients:*

$$\xi_i = \sum_j Q_{ij}(x, f(x)) \frac{\partial}{\partial x_j}$$

with Q_{ij} polynomial in x and f of degree not exceeding $\beta \geq 1$. Let d be dimension of the vector space spanned by the values of the vector fields ξ_i and their brackets of all orders at a point x. Then degree of nonholonomy of Ξ at x is less than

$$2^{d-2}\left(1 + 2^{2(n+m)(d-2)-2}(\alpha+\beta)^{2(n+m)} \sum_{k=1}^{n+m} (k+3)^{2(n+m)}\right) \quad \text{for } d > 2, \tag{13}$$

$$1 + 2^{2(n+m)-1}(\alpha+\beta)^{2(n+m)} \sum_{k=1}^{n+m} (k+1)^{2(n+m)}, \quad \text{for } d = 2. \tag{14}$$

Remark. The "integration over Euler characteristic" arguments allow one to obtain an effective estimate on the multiplicity of an isolated intersection defined by Noetherian functions of degree p in n variables, with a Noetherian chain of order m and degree α, in terms of n, m, α, and p. The proof is given in a joint paper of A. Khovanskii and the author [4].

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