BETTI NUMBERS OF SEMIALGEBRAIC AND SUB-PFAFFIAN SETS

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Abstract. Let X be a subset in $[-1,1]^{n_0} \subset \mathbb{R}^{n_0}$ defined by a formula

$$X = \{\mathbf{x}_0 |\, Q_1\mathbf{x}_1 Q_2\mathbf{x}_2 \cdots Q_\nu\mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\nu) \in X_\nu)\},$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and X_ν be either an open or a closed set in $[-1,1]^{n_0+\cdots+n_\nu}$ being a difference between a finite CW-complex and its subcomplex. We express an upper bound on each Betti number of X via a sum of Betti numbers of some sets defined by quantifier-free formulae involving X_ν.

In important particular cases of semialgebraic and semi-Pfaffian sets defined by quantifier-free formulae with polynomials and Pfaffian functions respectively, upper bounds on Betti numbers of X_ν are well known. Our results allow to extend the bounds to sets defined with quantifiers, in particular to sub-Pfaffian sets.

Introduction

Well-known results of Petrovskii, Oleinik [12, 11], Milnor [10], and Thom [14] provide an upper bound for the sum of Betti numbers of a semialgebraic set defined by a Boolean combination of polynomial equations and inequalities. A refinement of these results can be found in [1]. For semi-Pfaffian sets the analogous bounds were obtained by Khovanskii [8] (see also [17]). In this paper we describe a reduction of estimating Betti numbers of sets defined by formulae with quantifiers to a similar problem for sets defined by a quantifier-free formulae.

More precisely, let X be a subset in $[-1,1]^{n_0} \subset \mathbb{R}^{n_0}$ defined by a formula

$$X = \{\mathbf{x}_0 |\, Q_1\mathbf{x}_1 Q_2\mathbf{x}_2 \cdots Q_\nu\mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\nu) \in X_\nu)\}, \tag{0.1}$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and X_ν be either an open or a closed set in $[-1,1]^{n_0+\cdots+n_\nu}$ being a difference between a finite CW-complex and one of its subcomplexes. For instance, if $\nu = 1$ and $Q_1 = \exists$, then X is the projection of X_ν.

We express an upper bound on each Betti number of X via a sum of Betti numbers of some sets defined by quantifier-free formulae involving X_ν. In conjunction with Petrovskii-Oleinik-Thom-Milnor's result this implies a new upper bound for semialgebraic sets defined by formulae with quantifiers, which is significantly better than a bound following from the cylindrical cell decomposition approach. In conjunction with Khovanskii's result our method produces an analogous upper bound for restricted sub-Pfaffian sets defined by formulae with quantifiers. Apparently in this case no general upper bounds were previously known.

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Throughout the paper each topological space is assumed to be a difference between a finite CW-complex and one of its subcomplexes.

Example 0.1. The closure X of the interior of a compact set $Y \subset [-1,1]^n$ is homotopy equivalent to

$$X_{\varepsilon,\delta} = \{\mathbf{x} |\ \exists \mathbf{y}(\|\mathbf{x}-\mathbf{y}\| \le \delta)\ \forall \mathbf{z}(\|\mathbf{y}-\mathbf{z}\| < \varepsilon)\ (\mathbf{z} \in Y)\}$$

for small enough $\delta, \varepsilon > 0$ such that $\delta \gg \varepsilon$. Representing $X_{\varepsilon,\delta}$ in the form (0.1), we conclude that X is homotopy equivalent to $X_{\varepsilon,\delta} = \{\mathbf{x} |\ \exists \mathbf{y}\ \forall \mathbf{z}\ X_2\}$, where

$$X_2 = \{(\mathbf{x},\mathbf{y},\mathbf{z}) |\ (\|\mathbf{x}-\mathbf{y}\| \le \delta \wedge (\|\mathbf{y}-\mathbf{z}\| \ge \varepsilon \vee \mathbf{z} \in Y))\}$$

is a closed set in $[-1,1]^{3n}$. Our results allow to bound from above Betti numbers of X in terms of Betti numbers of X_2.

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1. A spectral sequence associated with a surjective map

Definition 1.1. A continuous map $f:\ X \to Y$ is *locally split* if for any $y \in Y$ there is an open neighbourhood U of y and a section $s:\ U \to X$ of f (i.e., s is continuous and $fs = \mathrm{Id}$). In particular, a projection of an open set in $\mathbb{R}^n$ on a subspace of $\mathbb{R}^n$ is always locally split.

Theorem 1.2. *Let $f:\ X \to Y$ be a surjective cellular map. Assume that f is either closed or locally split. Then for any Abelian group G, there exists a spectral sequence $E^r_{p,q}$ converging to $H_*(Y,G)$ with*

$$E^1_{p,q} = H_q(W_p, G) \tag{1.1}$$

where

$$W_p = \underbrace{X \times_Y \cdots \times_Y X}_{p+1 \text{ times}} \tag{1.2}$$

In particular,

$$\dim H_k(Y,G) \le \sum_{p+q=k} \dim H_q(W_p, G), \tag{1.3}$$

for all k.

For a locally split map f, this theorem can be derived from [3], Corollary 1.3. We present below a proof for a closed map f.

Remark 1.3. In the sequel we use Theorem 1.2 only for projections of either closed or open sets in $\mathbb{R}^n$. If f is a projection of an open set, then (1.3) easily follows from the analogous result for closed maps which will be proved below, without references to [3]. Indeed, for an open set Z define its *shrinking* $S(Z)$ as the closed set $Z \setminus N(\partial Z)$ where N denotes an open neighbourhood. For a small enough $N(\partial Z)$, the set Z is homotopy equivalent to $S(Z)$ (recall that Z is a difference between a finite CW–complex and a subcomplex). Let X be open and $S(X)$ be its shrinking with a sufficiently small $N(\partial X)$. It induces shrinkings $S(Y) = f(S(X))$ and $S(W_p) = S(X) \times_{S(Y)} \cdots \times_{S(Y)} S(X)$ which are homotopy equivalent to Y and W_p respectively. The statement for open sets X and Y follows from the statement for closed sets applied to $f:\ S(X) \to S(Y)$.

Definition 1.4. For a sequence $(P_0, \dots, P_p)$ of topological spaces, their *join* $P_0 * \cdots * P_p$ can be defined as follows. Let $\Delta^p = \{s_0 \geq 0, \dots, s_p \geq 0,\ s_0 + \cdots + s_p = 1\}$ be the standard p-simplex. Then $P_0 * \cdots * P_p$ is the quotient space of $P_0 \times \cdots \times P_p \times \Delta^p$ over the following relation:

(1.4)
$$(x_0, \dots, x_p, s) \sim (x'_0, \dots, x'_p, s) \text{ if } s = (s_0, \dots, s_p) \text{ and } x_i = x'_i \text{ whenever } s_i \neq 0.$$

Given a continuous surjective map $f_i : P_i \to Y$ for each $i = 0, \dots, p$, the *fibered join* $P_0 *_Y \cdots *_Y P_p$ is defined as the quotient space of $P_0 \times_Y \cdots \times_Y P_p \times \Delta^p$ over the relation (1.4).

Definition 1.5. For a space Z, 1-*st suspension* of Z is defined as the suspension (see [9]) of $(Z \sqcup \{point\})$. For an integer $p > 0$, the p-th iteration of this operation will be called *p-th suspension of Z*.

Lemma 1.6. *Let $f_i : P_i \to Y$, $i = 0, \dots, p$, be continuous surjective maps and $P = P_0 *_Y \cdots *_Y P_p$ their fibered join. There is a natural map $F : P \to Y$ induced by the maps $f_0, \dots, f_p$. For a point $y \in Y$ the fiber $F^{-1}y$ coincides with the join $f_0^{-1}y * \cdots * f_p^{-1}y$ of the fibers of f_i.*

There is a natural map $\pi : P \to \Delta^p$. The fiber of π over an interior point of Δ^p is $P_0 \times_Y \cdots \times_Y P_p$. For each $i = 0, \dots, p$, there is a natural embedding

$$\phi_i : P(i) = P_0 *_Y \cdots *_Y P_{i-1} *_Y P_{i+1} *_Y \cdots *_Y P_p \to P. \tag{1.5}$$

Its image coincides with $\pi^{-1}(\{s_i = 0\})$, and the space $P/\left(\bigcup_i \phi_i(P(i))\right)$ is homotopy equivalent to the p-th suspension of $P_0 \times_Y \cdots \times_Y P_p$.

Proof. Straightforward. □

Definition 1.7. Let $f : X \to Y$ be a surjective continuous map. Its *join space* $J^f(X)$ is the quotient space of the disjoint union of spaces

$$J^f_p(X) = \underbrace{X *_Y \cdots *_Y X}_{p+1 \text{ times}}, \quad p = 0, 1, \dots, \tag{1.6}$$

identifying $J^f_{p-1}(X)$ with each of its images $\phi_i(J^f_{p-1}(X))$ in $J^f_p(X)$ for $i = 0, \dots, p$, where ϕ_i is defined in (1.5). When Y is a point, we write $J_p(X)$ instead of $J^f_p(X)$ and $J(X)$ instead of $J^f(X)$.

Lemma 1.8. *Let $\phi : J_p(X) \to J(X)$ be the natural map induced by the maps ϕ_i. Then $\phi(J_{p-1}(X))$ is contractible in $\phi(J_p(X))$.*

Proof. Let x be a point in X. For $t \in [0, 1]$, the maps

$$g_t(x, x_1, \dots, x_p, s) \mapsto (x, x_1, \dots, x_p, (1 - t + ts_0, ts_1, \dots, ts_p))$$

define a contraction of $\phi_0(J_{p-1}(X))$ to the point $x \in X$ where X is identified with its embedding in $J_p(X)$ as $\pi^{-1}(1, 0, \dots, 0)$. It is easy to see that the maps g_t are compatible with the equivalence relations in Definition 1.7 and define a contraction of $\phi(J_{p-1}(X))$ to a point in $\phi(J_p(X))$. □

Lemma 1.9. *The join space $J(X)$ is homologically trivial.*

Proof. Any cycle in $J(X)$ belongs to $\phi(J_p(X))$ for some p, while according to Lemma 1.8 $\phi(J_p(X))$ is contractible in $J(X)$. Hence the cycle is homologous to 0. □

Proof of Theorem 1.2. Let f be closed. Let $F : J^f(X) \to Y$ be the natural map induced by f. Then F is also closed. Its fiber $F^{-1}y$ over a point $y \in Y$ coincides with the join space $J(f^{-1}y)$ which is homologically trivial according to Lemma 1.9. It follows that $\tilde{\bar{H}}^*(J(f^{-1}y)) \cong 0$, where $\bar{H}^*$ is the Alexander cohomology ([13], p. 308), since $\bar{H}^*(Z)) \cong H^*(Z)$ for any locally contractible space Z ([13], p. 340), in particular for a difference between CW-complex and a subcomplex.

Vietoris-Begle theorem ([13], p. 344) applied to $F : J^f(X) \to Y$, implies

$$\bar{H}^*(J^f(X), G) \cong \bar{H}^*(Y, G)$$

and therefore

$$H_*(J^f(X), G) \cong H_*(Y, G).$$

By Lemma 1.6, the space $J^f_p(X)/\left(\bigcup_{q<p} J^f_q(X)\right)$ is homotopy equivalent to the p-th suspension of W_p. Theorem 1.2 follows now from the spectral sequence associated with filtration of $J^f(X)$ by the spaces $J^f_p(X)$. □

Remark 1.10. For a map f with 0-dimensional fibers, a similar spectral sequence, "image computing spectral sequence" was applied to problems in theory of singularities and topology by Vassiliev [15], Goryunov-Mond [6], Goryunov [5], Houston [7], and others.

Remark 1.11. Let $X, Y \subset \mathbb{R}^n$ and a surjective cellular map f satisfies the following property. For any convergent sequence in Y there is an infinite subsequence which is an f-image of a convergent sequence in X. This condition includes both the closed and the locally split cases and may be more convenient for applications. For $f : X \to Y$ Theorem 1.2 is also true. A proof will appear elsewhere.

2. Alexander's duality and Mayer-Vietoris inequality

Let

$$I^n_i := \bigcap_{1 \le j \le n} \{-i \le x_j \le i\} \subset \mathbb{R}^n.$$

Define the "thick boundary" $\partial I^n_i := I^n_{i+1} \setminus I^n_i$. The following lemma is a version of Alexander's duality theorem.

Lemma 2.1. (Alexander's duality) *If $X \subset I^n_i$ is an open set in I^n_i, then for any $q \in \mathbb{Z}$, $q \le n-1$,*

$$H_q(I^n_i \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup \partial I^n_i, \mathbb{R}). \tag{2.1}$$

If $X \subset I^n_i$ is a closed set in I^n_i, then for any $q \in \mathbb{Z}$, $q \le n-1$,

$$H_q(I^n_i \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup closure(\partial I^n_i), \mathbb{R}). \tag{2.2}$$

Proof. For definiteness let X be closed. Compactifying $\mathbb{R}^n$ at infinity as $\mathbb{R}^n \cup \infty \simeq S^n$, we have, by Alexander's duality [9],

$$\tilde{H}_q(S^n \setminus (X \cup closure(\partial I^n_i), \mathbb{R}) \cong \tilde{H}_{n-q-1}((X \cup closure(\partial I^n_i), \mathbb{R}).$$

The first group is isomorphic to $H_q(I^n_i \setminus X, \mathbb{R})$ when $q > 0$, and to $\tilde{H}_0(I^n_i \setminus X, \mathbb{R}) + \mathbb{R} \cong H_0(I^n_i \setminus X, \mathbb{R})$ when $q = 0$. Combining these two cases, we obtain (2.2). □

Lemma 2.2. (Mayer-Vietoris inequality) *Let $X_1, \dots, X_m \subset I_1^n$ be all open or all closed in I_1^n. Then*

$$\mathrm{b}_i\Big(\bigcup_{1 \le j \le n} X_j\Big) \le \sum_{J \subset \{1,\dots,n\}} \mathrm{b}_{i-|J|+1}\Big(\bigcap_{j \in J} X_j\Big)$$

and

$$\mathrm{b}_i\Big(\bigcap_{1 \le j \le n} X_j\Big) \le \sum_{J \subset \{1,\dots,n\}} \mathrm{b}_{i+|J|-1}\Big(\bigcup_{j \in J} X_j\Big),$$

where b_i is the ith Betti number.

Proof. A well-known corollary to Mayer-Vietoris sequence [9]. □

3. Thom-Milnor's and Khovanskii's bounds

Necessary definitions regarding semi-Pfaffian and sub-Pfaffian sets can be found in [8], [4]. In this paper we consider only *restricted* sub-Pfaffian sets.

To apply our results to semialgebraic sets and to restricted sub-Pfaffian sets, defined by formulae with quantifiers, we need the following known upper bounds on Betti numbers for sets defined by quantifier-free formulae.

Let $X = \{\varphi\} \subset \mathbb{R}^n$ be a semialgebraic set, where φ is a Boolean combination with no negations of s atomic formulae of the kind $f \ge 0$, f being polynomials in n variables, $\deg(f) < d$. We will refer to the sequence (n, s, d) as *format* of φ. It follows from [14, 10, 1] that the sum of Betti numbers of X is

$$\mathrm{b}(X) \le O(sd)^n. \tag{3.1}$$

Now let $X = \{\varphi\} \subset \mathbb{R}^n$ be a semi-Pfaffian set, where φ is a Boolean combination with no negations of s atomic formulae of the kind $f \ge 0$, f being Pfaffian functions in an open domain $G \subset \mathbb{R}^n$ of order r, degree (α, β), having a common Pfaffian chain. We will refer to the sequence (n, s, r, α, β) as *format* of φ. It follows from [8, 17] that the sum of Betti numbers of X is

$$\mathrm{b}(X) \le s^n 2^{r(r-1)/2} O(n\beta + \min\{n, r\}\alpha)^{n+r}. \tag{3.2}$$

Let $X \subset \mathbb{R}^{n_0}$ be a semialgebraic set defined by a formula

$$Q_1 \mathbf{x}_1 Q_2 \mathbf{x}_2 \cdots Q_\nu \mathbf{x}_\nu F(\mathbf{x}_0, \mathbf{x}_1, \dots \mathbf{x}_\nu), \tag{3.3}$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \ne Q_{i+1}$, $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n_i}) \in \mathbb{R}^{n_i}$, and F is a quantifier-free Boolean formula with no negations having s atoms of the kind $f \ge 0$ of degrees less than d. The *cylindrical algebraic decomposition* technique from [2, 16] allows to bound from above the number of cells in a representation of X as a CW-complex. In particular,

$$\mathrm{b}(X) \le (sd)^{2^{O(n)}}. \tag{3.4}$$

4. Basic notation

Let $X = \widetilde{X_0} \subset I_1^{n_0}$ be a set defined by a formula (0.1). For example, X could be a sub-Pfaffian or a semialgebraic set defined by (3.3), where F is a quantifier-free Boolean formula with no negations. For definiteness assume that $Q_1 = \exists$ and X is open in $I_1^{n_0}$.

Define

$$X_i := \{(\mathbf{x}_0, \dots, \mathbf{x}_i) |\, Q_{i+1}\mathbf{x}_{i+1} Q_{i+2}\mathbf{x}_{i+2} \cdots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu) \in X_\nu)\}$$

for odd i and

$$X_i := I_1^{n_0+\cdots n_i} \setminus \{(\mathbf{x}_0, \dots , \mathbf{x}_i)|\, Q_{i+1}\mathbf{x}_{i+1}Q_{i+2}\mathbf{x}_{i+2} \cdots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots , \mathbf{x}_\nu) \in X_\nu)\}$$

for even i. Then $\pi_i(X_i) = \widetilde{X_{i-1}}$, where $\pi_i : \mathbb{R}^{n_0+\cdots+n_i} \to \mathbb{R}^{n_0+\cdots+n_{i-1}}$.

Let

$$I_i^m := \bigcap_{1\le j\le m} \{-i \le x_j \le i\}.$$

For a set $I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \cdots \times I_1^{m_1}$ define $\partial(I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \cdots \times I_1^{m_1})$ as

$$(I_{i+1}^{m_i} \times I_i^{m_{i-1}} \times \cdots \times I_2^{m_1}) \setminus (I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \cdots \times I_1^{m_1})$$

for even i and as the closure of this difference for odd i.

We can assume without loss of generality that all functions defining $X_\nu \subset I_1^{n_0+\cdots+n_\nu}$ can be extended to $I_\nu^{n_0+\cdots+n_\nu}$.

Define

$$B_i^i := \partial(I_{i-1}^{n_0+(p_1+1)n_1} \times I_{i-2}^{(p_2+1)n_2} \times \cdots \times I_1^{(p_{i-1}+1)n_{i-1}}) \times I_1^{n_i}.$$

For any j, $i < j \le \nu$ define $B_j^i := \widetilde{B_{j-1}^i} \times I_1^{n_j}$

Let $t_i = n_0 + n_1(p_1 + 1) + \cdots + n_i(p_i + 1)$.

Definition 4.1.

1. Let $Y \subset I_v^{n_0+(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \cdots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\cdots+n_i}$, where $1 \le l \le i$, $v \ge i$, and let $J \subset \{(j_l, \dots , j_i)|\, 1 \le j_k \le p_k + 1,\ l \le k \le i\}$. Then define $\prod_{i,J}^{l} Y$ as an intersection of sets

$$\{(\mathbf{x}_0, \mathbf{x}_1^{(1)}, \dots , \mathbf{x}_1^{(p_1+1)}, \dots , \mathbf{x}_i^{(1)}, \dots , \mathbf{x}_i^{(p_i+1)})|\mathbf{x}_0 \in I_v^{n_0}, \mathbf{x}_k^{(m)} \in I_{v-k+1}^{n_k} (1 \le k \le l-1),$$

$$\mathbf{x}_k^{(m)} \in I_1^{n_k}\ (l \le k \le i),\ (\mathbf{x}_0, \mathbf{x}_1^{(1)}, \dots , \mathbf{x}_{l-1}^{(p_{l-1}+1)}, \mathbf{x}_l^{(j_l)}, \dots , \mathbf{x}_i^{(j_i)}) \in Y\}$$

   over all $(j_l, \dots , j_i) \in J$.

2. Let $Y \subset I_v^{n_0+(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \cdots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\cdots+n_i+n_{i+1}}$. Define $\prod_{i,J}^{l,i+1} Y$ as an intersection of sets

$$\{(\mathbf{x}_0, \mathbf{x}_1^{(1)}, \dots , \mathbf{x}_1^{(p_1+1)}, \dots , \mathbf{x}_i^{(1)}, \dots , \mathbf{x}_i^{(p_i+1)}, \mathbf{x}_{i+1})|$$

$$\mathbf{x}_0 \in I_v^{n_0},\ \mathbf{x}_k^{(m)} \in I_{v-k+1}^{n_k}\ (1 \le k \le l-1),\ \mathbf{x}_k^{(m)} \in I_1^{n_k}\ (l \le k \le i),\ \mathbf{x}_{i+1} \in I_1^{n_{i+1}},$$

$$(\mathbf{x}_0, \mathbf{x}_1^{(1)}, \dots , \mathbf{x}_{l-1}^{(p_{l-1}+1)}, \mathbf{x}_l^{(j_l)}, \dots , \mathbf{x}_i^{(j_i)}, \mathbf{x}_{i+1}) \in Y\}$$

   over all $(j_l, \dots , j_i) \in J$.

3. If $l = i$ and $J = \{j|\, 1 \le j \le p_i + 1\}$ we use the notation $\prod_i^i Y$ for $\prod_{i,J}^i Y$.

Lemma 4.2. *Let* $Y \subset I_v^{n_0+(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \cdots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\cdots+n_i+n_{i+1}}$. *Then for any* $J \subset \{j|1 \le j \le p_{i+1}+1\}$, $J' \subset \{(j_l, \dots , j_i)|1 \le j_k \le p_k+1, l \le k \le i\}$ *we have*

$$\prod_{i+1,J}^{i+1} \prod_{i,J'}^{l,i+1} Y = \prod_{i+1,J'\times J}^{l} Y.$$

Proof. Straightforward. □

Definition 4.3. Let Y, l, i, J be as in Definition 4.1. Define $\coprod_{i,J}^{l} Y$ and $\coprod_{i,J}^{l,i+1} Y$ similar to $\prod_{i,J}^{l} Y$ and $\prod_{i,J}^{l,i+1} Y$ respectively, replacing in Definition 4.1 "intersection" by "union".

Lemma 4.4. (De Morgan law)

$$\bigsqcup\nolimits_{i,J}^{l} Y = \left(\prod\nolimits_{i,J}^{l} \widetilde{Y}\right)^{\sim};$$

$$\bigsqcup\nolimits_{i,J}^{l,i+1} Y = \left(\prod\nolimits_{i,J}^{l,i+1} \widetilde{Y}\right)^{\sim}.$$

Proof. Straightforward. □

Definition 4.5. Define

$$\pi_i : \mathbb{R}^{n_0+\cdots+n_i} \to \mathbb{R}^{n_0+\cdots+n_{i-1}},$$

and for $j < i$,

$$\pi_{i,j} : \mathbb{R}^{t_j+n_{j+1}+\cdots+n_i} \to \mathbb{R}^{t_j+n_{j+1}+\cdots+n_{i-1}}.$$

Lemma 4.6. *Let* $Y \subset I_v^{n_0+(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \cdots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\cdots+n_i+n_{i+1}}$ *Then*

$$\bigsqcup\nolimits_{i,J}^{l} \pi_{i+1,l-1}(Y) = \pi_{i+1,i}\left(\bigsqcup\nolimits_{i,J}^{l,i+1} Y\right).$$

Proof. Straightforward. □

5. Case of a single quantifier block

According to Theorem 1.2,

$$\mathrm{b}_{q_0}(X) = \mathrm{b}_{q_0}(\widetilde{X_0}) \le \sum_{p_1+q_1=q_0} \mathrm{b}_{q_1}\left(\prod\nolimits_{1,J_1^1}^{1} X_1\right), \tag{5.1}$$

where $J_1^1 = \{1, \dots, p_1+1\}$.

Let $\nu = 1$, then (3.3) turns into $\exists \mathbf{x}_1 F(\mathbf{x}_0, \mathbf{x}_1)$, where $X_1 = \{F(\mathbf{x}_0, \mathbf{x}_1)\}$ and $F(\mathbf{x}_0, \mathbf{x}_1)$ is a Boolean combination with no negations of s atomic formulae of the kind $f > 0$.

5.1. **Polynomial case.** Suppose that X_1 is semialgebraic, with f's being polynomials of degrees $\deg(f) < d$. For any $k \le \dim(X)$, we bound the Betti number $\mathrm{b}_k(X)$ from above in the following way. Observe that $\prod_{1,J_1^1}^{1} X_1$ is an open set definable by a Boolean combination with no negations of $(p_1+1)s$ atomic formulae of the kind $g > 0$, $\deg(g) < d$ in $t_1 = n_0 + (p_1+1)n_1$ variables.

According to (3.1), for any $q_1 \le \dim(X)$,

$$\mathrm{b}_{q_1}\left(\prod\nolimits_{1,J_1^1}^{1} X_1\right) \le O(p_1 s d)^{n_0+(p_1+1)n_1}.$$

Then due to (5.1), for any $k \le \dim(X) \le n_0$,

$$\mathrm{b}_k(X) \le \sum_{p_1+q_1=k} O(p_1 s d)^{n_0+(p_1+1)n_1} \le (ksd)^{O(n_0+kn_1)}.$$

5.2. **Pfaffian case.** Suppose that X_1 is sub-Pfaffian, with f's being Pfaffian functions in an open domain $G \subset \mathbb{R}^n$ of order r, degree (α, β), having a common Pfaffian chain. Since quantifier elimination is generally impossible for Pfaffian functions, we will only use the approach based on (5.1). More precisely, observe that $\prod^1_{1,J^1_1} X_1$ is an open set definable by a Boolean combination with no negations of $(p+1)s$ atomic formulae of the kind $g > 0$, where g are Pfaffian functions in an open domain $G \subset \mathbb{R}^{n_0+(p+1)n_1}$ of degrees (α, β), order $(p+1)r$ in $n_0 + (p+1)n_1$ variables, having a common Pfaffian chain. According to (3.2), for any $q_1 \le \dim(X)$,

$$\mathrm{b}_{q_1}\Big(\prod\nolimits^1_{1,J^1_1} X_1\Big) \le ((p_1+1)s)^{n_0+(p_1+1)n_1} 2^{(p_1+1)r((p_1+1)r-1)/2}\cdot$$

$$\cdot O((n_0+p_1n_1)\beta + \min\{p_1r, n_0+p_1n_1\}\alpha)^{n_0+(p_1+1)(n_1+r)}.$$

Then due to (5.1), for any $k \le \dim(X) \le n_0$,

$$\mathrm{b}_k(X) \le \sum_{p_1+q_1=k} \mathrm{b}_{q_1}\Big(\prod\nolimits^1_{1,J^1_1} X_1\Big) \le$$

$$k((k+1)s)^{n_0+(k+1)n_1} 2^{(k+1)r((k+1)r-1)/2}\cdot$$

$$\cdot O((n_0+kn_1)\beta + \min\{kr, n_0+kn_1\}\alpha)^{n_0+(k+1)(n_1+r)}.$$

Let $d > \alpha + \beta$. Relaxing the obtained bound, we get

$$\mathrm{b}_k(X) \le (ks)^{O(n_0+kn_1)} 2^{(O(kr))^2} ((n_0+kn_1)d)^{O(n_0+kn_1+kr)}.$$

6. Case of two or three quantifier blocks

In this section we obtain a generalization of (5.1) to the case of three blocks of quantifiers, as a preparation for cumbersome general formulae in the next section.

Recall that

$$\pi_i : \mathbb{R}^{n_0+\cdots+n_i} \to \mathbb{R}^{n_0+\cdots+n_{i-1}},$$

for $j < i$,

$$\pi_{i,j} : \mathbb{R}^{t_j+n_{j+1}+\cdots+n_i} \to \mathbb{R}^{t_j+n_{j+1}+\cdots+n_{i-1}}.$$

Let $\nu = 3$, then the original formula becomes $\exists \mathbf{x}_1 \forall \mathbf{x}_2 \exists \mathbf{x}_3 ((\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in X_3)$. Thereby,

$$X_1 = \{\forall \mathbf{x}_2 \exists \mathbf{x}_3 ((\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in X_3)\},\ \widetilde{X_2} = \{\exists \mathbf{x}_3 ((\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in X_3)\},$$

$X = \widetilde{X_0}$ is open in $I_1^{n_0}$.

According to Theorem 1.2,

$$\mathrm{b}_{q_0}(\widetilde{X_0}) \le \sum_{p_1+q_1=q_0} \mathrm{b}_{q_1}\Big(\prod\nolimits^1_{1,J^1_1} X_1\Big).$$

Applying in succession Lemma 4.4 (De Morgan law), Lemma 2.1 (Alexander's duality), definitions of π_2 and $\pi_{2,1}$, and Lemma 4.6 we get

$$\mathrm{b}_{q_1}\Big(\prod\nolimits^1_{1,J^1_1} X_1\Big) = \mathrm{b}_{q_1}\Big(\widetilde{\Big(\bigsqcup\nolimits^1_{1,J^1_1} \widetilde{X_1}\Big)}\Big) \le$$

$$\le \mathrm{b}_{t_1-q_1-1}\Big(\bigsqcup\nolimits^1_{1,J^1_1} \widetilde{X_1} \cup \partial I_1^{t_1}\Big) = \mathrm{b}_{t_1-q_1-1}\Big(\bigsqcup\nolimits^1_{1,J^1_1} \pi_2(X_2) \cup \pi_{2,1}(\partial I_1^{t_1} \times I_1^{n_2})\Big) =$$

$$= \mathrm{b}_{t_1-q_1-1}\Big(\pi_{2,1}\Big(\bigsqcup\nolimits^{1,2}_{1,J^1_1} X_2 \cup \partial I_1^{t_1} \times I_1^{n_2}\Big)\Big).$$

Due to Theorem 1.2, the last expression does not exceed

$$\sum_{p_2+q_2=t_1-q_1-1} \mathrm{b}_{q_2}\Big(\prod{}^2_2\Big(\bigsqcup{}^{1,2}_{1,J^1_1} X_2 \cup \partial I^{t_1}_1 \times I^{n_2}_1\Big)\Big) =$$

$$= \sum_{p_2+q_2=t_1-q_1-1} \mathrm{b}_{q_2}\Big(\prod{}^2_2\Big(\bigsqcup{}^{1,2}_{1,J^1_1} X_2 \cup B^2_2\Big)\Big).$$

Due to Lemma 4.4 (De Morgan law) and Lemma 2.1 (Alexander's duality),

$$\mathrm{b}_{q_2}\Big(\prod{}^2_2\Big(\bigsqcup{}^{1,2}_{1,J^1_1} X_2 \cup B^2_2\Big)\Big) = \mathrm{b}_{q_2}\Big(\widetilde{\Big(\bigsqcup{}^2_2\Big(\prod{}^{1,2}_{1,J^1_1} \widetilde{X_2} \cap \widetilde{B^2_2}\Big)\Big)}\Big) =$$

$$= \mathrm{b}_{t_2-q_2-1}\Big(\bigsqcup{}^2_2\Big(\prod{}^{1,2}_{1,J^1_1} \widetilde{X_2} \cap \widetilde{B^2_2}\Big) \cup \partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1)\Big).$$

By Lemma 2.2 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1\le k_2\le p_2+1} \quad \sum_{\hat{J}^2_2\subset\{1,\dots,p_2+1\},\ |\hat{J}^2_2|=k_2}$$

$$\mathrm{b}_{t_2-q_2-k_2}\Big(\prod{}^2_{2,\hat{J}^2_2}\Big(\prod{}^{1,2}_{1,J^1_1} \widetilde{X_2} \cap \widetilde{B^2_2}\Big) \cup \partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1)\Big).$$

(We estimate a Betti number of the union of cylindrical sets from the definition of the symbol $\bigsqcup{}^2_2$ by a sum of Betti numbers of intersections of various combinations of these sets.)

By Lemma 4.2,

$$\mathrm{b}_{t_2-q_2-k_2}\Big(\prod{}^2_{2,\hat{J}^2_2}\Big(\prod{}^{1,2}_{1,J^1_1} \widetilde{X_2} \cap \widetilde{B^2_2}\Big) \cup \partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1)\Big) =$$

$$= \mathrm{b}_{t_2-q_2-k_2}\Big(\prod{}^1_{2,J^1_1\times\hat{J}^2_2} \widetilde{X_2} \cap \prod{}^2_{2,J^2_1\times\hat{J}^2_2} \widetilde{B^2_2} \cup \partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1)\Big),$$

with $J^2_1 = \{1\}$. By Lemma 2.2 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1\le s_2\le q_2+k_2+1} \quad \sum_{J^1_2\subset J^1_1\times\hat{J}^2_2,\ J^2_2\subset J^2_1\times\hat{J}^2_2,\ |J^1_2|+|J^2_2|=s_2}$$

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup{}^1_{2,J^1_2} \widetilde{X_2} \cup \bigsqcup{}^2_{2,J^2_2} \widetilde{B^2_2} \cup \partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1)\Big),$$

taking into the account that

$$\dim\Big(\bigsqcup{}^1_{2,J^1_2} \widetilde{X_2} \cup \bigsqcup{}^2_{2,J^2_2} \widetilde{B^2_2} \cup \partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1)\Big) \le t_2$$

and therefore

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup{}^1_{2,J^1_2} \widetilde{X_2} \cup \bigsqcup{}^2_{2,J^2_2} \widetilde{B^2_2} \cup \partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1)\Big) = 0$$

for $s_2 > q_2 + k_2 + 1$.

We have

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup{}^1_{2,J^1_2} \widetilde{X_2} \cup \bigsqcup{}^2_{2,J^2_2} \widetilde{B^2_2} \cup \partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1)\Big) =$$

$$= \mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup{}^1_{2,J^1_2} \pi_3(X_3) \cup \bigsqcup{}^2_{2,J^2_2} \pi_{3,1}(B^2_3) \cup \pi_{3,2}(\partial(I^{t_1}_2 \times I^{n_2(p_2+1)}_1) \times I^{n_3}_1)\Big) =$$

$$= \mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\pi_{3,2}\Big(\bigsqcup{}^{1,3}_{2,J^1_2} X_3 \cup \bigsqcup{}^{2,3}_{2,J^2_2} B^2_3 \cup B^3_3\Big)\Big).$$

Due to Theorem 1.2 the last expression does not exceed

$$\sum_{p_3+q_3=t_2-q_2-k_2+s_2-1} \mathrm{b}_{q_3}\Big(\prod{}^3_3\Big(\bigsqcup{}^{1,3}_{2,J^1_2} X_3 \cup \bigsqcup{}^{2,3}_{2,J^2_2} B^2_3 \cup B^3_3\Big)\Big).$$

In case of a sub-Pfaffian or a semialgebraic X it is now possible to estimate

$$\mathrm{b}_{q_3}\Big(\prod{}_3^3\Big(\bigsqcup{}_{2,J_2^1}^{1,3} X_3 \cup \bigsqcup{}_{2,J_2^2}^{2,3} B_3^2 \cup B_3^3\Big)\Big)$$

via the format of X_3.

7. Arbitrary number of quantifiers

Theorem 7.1. *For any i the Betti number $\mathrm{b}_{q_0}(X)$ does not exceed*

$$\sum_{p_1+q_1=q_0} \sum_{p_2+q_2=t_1-q_1-1} \sum_{1\le k_2\le p_2+1} \sum_{\hat J_2^2\subset\{1,\dots,p_2+1\},\ |\hat J_2^2|=k_2}$$
$$\sum_{1\le s_2\le q_2+k_2+1} \sum_{J_2^1\subset J_1^1\times \hat J_2^2,\ J_2^2\subset J_1^2\times\hat J_2^2,\ |J_2^1|+|J_2^2|=s_2} \sum_{p_3+q_3=t_2-k_2+s_2-1} \cdots$$
$$\cdots \sum_{1\le k_{i-1}\le p_{i-1}+1} \sum_{\hat J_{i-1}^{i-1}\subset\{1,\dots,p_{i-1}+1\},\ |\hat J_{i-1}^{i-1}|=k_{i-1}} \sum_{1\le s_{i-1}\le q_{i-1}+k_{i-1}+1}$$
$$\sum_{J_{i-1}^1\subset J_{i-2}^1\times\hat J_{i-1}^{i-1},\dots,\ J_{i-1}^{i-1}\subset J_{i-2}^{i-1}\times\hat J_{i-1}^{i-1},\ |J_{i-1}^1|+\dots+|J_{i-1}^{i-1}|=s_{i-1}} \sum_{p_i+q_i=t_{i-1}-q_{i-1}-k_{i-1}+s_{i-1}-1}$$
$$\mathrm{b}_{q_i}\Big(\prod{}_i^i\Big(\bigsqcup{}_{i-1,J_{i-1}^1}^{1,i} X_i \cup \bigcup_{2\le r\le i-1} \bigsqcup{}_{i-1,J_{i-1}^r}^{r,i} B_i^r \cup B_i^i\Big)\Big). \tag{7.1}$$

Proof. Induction on i. Suppose (7.1) is true. Due to Lemma 4.4 (De Morgan law) and Lemma 2.1 (Alexander's duality),

$$\mathrm{b}_{q_i}\Big(\prod{}_i^i\Big(\bigsqcup{}_{i-1,J_{i-1}^1}^{1,i} X_i \cup \bigcup_{2\le r\le i-1} \bigsqcup{}_{i-1,J_{i-1}^r}^{r,i} B_i^r \cup B_i^i\Big)\Big) =$$
$$= \mathrm{b}_{q_i}\Big(\Big(\bigsqcup{}_i^i\Big(\prod{}_{i-1,J_{i-1}^1}^{1,i} \widetilde{X_i} \cap \bigcap_{2\le r\le i-1} \prod{}_{i-1,J_{i-1}^r}^{r,i} \widetilde{B_i^r} \cap \widetilde{B_i^i}\Big)\Big)^{\sim}\Big) \le$$
$$\le \mathrm{b}_{t_i-q_i-1}\Big(\bigsqcup{}_i^i\Big(\prod{}_{i-1,J_{i-1}^1}^{1,i} \widetilde{X_i} \cap \bigcap_{2\le r\le i-1} \prod{}_{i-1,J_{i-1}^r}^{r,i} \widetilde{B_i^r} \cap \widetilde{B_i^i}\Big) \cup$$
$$\cup\, \partial(I_i^{n_0+(p_1+1)n_1}\times\cdots\times I_1^{(p_i+1)n_i})\Big).$$

By Lemma 2.2 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1\le k_i\le p_i+1} \sum_{\hat J_i^i\subset\{1,\dots,p_i+1\},\ |\hat J_i^i|=k_i}$$
$$\mathrm{b}_{t_i-q_i-k_i}\Big(\prod{}_{i,\hat J_i^i}^i\Big(\prod{}_{i-1,J_{i-1}^1}^{1,i} \widetilde{X_i} \cap \bigcap_{2\le r\le i-1} \prod{}_{i-1,J_{i-1}^r}^{r,i} \widetilde{B_i^r} \cap \widetilde{B_i^i}\Big) \cup$$
$$\cup\, \partial(I_i^{n_0+(p_1+1)n_1}\times\cdots\times I_1^{(p_i+1)n_i})\Big),$$

where, by Lemma 4.2,

$$\mathrm{b}_{t_i-q_i-k_i}\Big(\prod{}_{i,\hat J_i^i}^i\Big(\prod{}_{i-1,J_{i-1}^1}^{1,i} \widetilde{X_i} \cap \bigcap_{2\le r\le i-1} \prod{}_{i-1,J_{i-1}^r}^{r,i} \widetilde{B_i^r} \cap \widetilde{B_i^i}\Big) \cup$$
$$\cup\, \partial(I_i^{n_0+(p_1+1)n_1}\times\cdots\times I_1^{(p_i+1)n_i})\Big) =$$
$$= \mathrm{b}_{t_i-q_i-k_i}\Big(\prod{}_{i,J_{i-1}^1\times\hat J_i^i}^1 \widetilde{X_i} \cap \bigcap_{2\le r\le i} \prod{}_{i,J_{i-1}^r\times\hat J_i^i}^r \widetilde{B_i^r} \cup$$

$$\cup\, \partial(I_i^{n_0+(p_1+1)n_1} \times \cdots \times I_1^{(p_i+1)n_i})\Big),$$

where $J_{i-1}^i = \{1\}$. By Lemma 2.2 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1 \le s_i \le q_i+k_i+1} \quad \sum_{J_i^1 \subset J_{i-1}^1 \times \hat{J}_i^i,\ \dots,\ J_i^i \subset J_{i-1}^i \times \hat{J}_i^i,\ |J_i^1|+\cdots+|J_i^i|=s_i}$$

$$\mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits_{i,J_i^1}^1 \widetilde{X_i} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits_{i,J_i^r}^r \widetilde{B_i^r} \cup\ \partial(I_i^{n_0+(p_1+1)n_1} \times \cdots \times I_1^{(p_i+1)n_i})\Big).$$

We have

$$\mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits_{i,J_i^1}^1 \widetilde{X_i} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits_{i,J_i^r}^r \widetilde{B_i^r} \cup\ \partial(I_i^{n_0+(p_1+1)n_1} \times \cdots \times I_1^{(p_i+1)n_i})\Big) =$$

$$= \mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits_{i,J_i^1}^1 \pi_{i+1}(X_{i+1}) \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits_{i,J_i^r}^r \pi_{i+1,r-1}(B_{i+1}^r) \cup$$

$$\cup\, \pi_{i+1,i}(\partial(I_i^{n_0+(p_1+1)n_1} \times \cdots \times I_1^{(p_i+1)n_i}) \times I_1^{n_{i+1}})\Big) =$$

$$= \mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\pi_{i+1,i}\Big(\bigsqcup\nolimits_{i,J_i^1}^{1,i+1} X_{i+1} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits_{i,J_i^r}^{r,i+1} B_{i+1}^r \cup B_{i+1}^{i+1}\Big)\Big).$$

Due to Theorem 1.2 the last expression does not exceed

$$\sum_{p_{i+1}+q_{i+1}=t_i-q_i-k_i+s_i-1}$$

$$\mathrm{b}_{q_{i+1}}\Big(\prod\nolimits_{i+1}^{i+1}\Big(\bigsqcup\nolimits_{i,J_i^1}^{1,i+1} X_{i+1} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits_{i,J_i^r}^{r,i+1} B_{i+1}^r \cup B_{i+1}^{i+1}\Big)\Big).$$

□

8. Upper bounds for sub-Pfaffian sets

We first estimate from above the number of additive terms in (7.1). These terms can be partitioned into $i-1$ groups of the kind

$$\sum_{1\le k_j\le p_j+1} \quad \sum_{\hat{J}_j^j \subset \{1,\dots,p_j+1\},\ |\hat{J}_j^j|=k_j}$$

$$\sum_{1 \le s_j \le q_j+k_j+1} \quad \sum_{J_j^1 \subset J_{j-1}^1 \times \hat{J}_j^j,\dots,\ J_j^j \subset J_{j-1}^j \times \hat{J}_j^j,\ |J_j^1|+\cdots+|J_j^j|=s_j}$$

$$\sum_{p_{j+1}+q_{j+1}=t_j-q_j-k_j+s_j-1},$$

where $1 \le j \le i-1$.

The number of terms in

$$\sum_{1\le k_j\le p_j+1} \quad \sum_{\hat{J}_j^j \subset \{1,\dots,p_j+1\},\ |\hat{J}_j^j|=k_j}$$

is 2^{p_j+1}. The number of terms in

$$\sum_{1 \le s_j \le q_j+k_j+1} \quad \sum_{J_j^1 \subset J_{j-1}^1 \times \hat{J}_j^j,\dots,\ J_j^j \subset J_{j-1}^j \times \hat{J}_j^j,\ |J_j^1|+\cdots+|J_j^j|=s_j}$$

does not exceed $2^{j(q_j+k_j+1)}$. The number of terms in

$$\sum_{p_{j+1}+q_{j+1}=t_j-q_j-k_j+s_j-1}$$

does not exceed t_j+1.

It follows that the total number of terms in jth group does not exceed

$$2^{p_j+1+j(q_j+k_j+1)}(t_j+1) \le 2^{O(jt_{j-1})}.$$

Since $t_j = n_0+n_1(p_1+1)+\cdots+n_j(p_j+1)$, $p_l \le t_{l-1}$, and therefore $t_j \le 2^j n_0 n_1 \cdots n_j$, the number of terms in jth group does not exceed $2^{O(j2^j n_0 n_1 \cdots n_{j-1})}$. It follows that the total number of terms in (7.1) does not exceed $2^{O(i^2 2^i n_0 n_1 \cdots n_{i-2})}$.

We now find an upper bound for

$$\mathrm{b}_{q_\nu}\Big(\prod\nolimits_\nu^\nu \Big(\bigsqcup\nolimits^{1,\nu}_{\nu-1,J^1_{\nu-1}} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup\nolimits^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \cup B^\nu_\nu\Big)\Big).$$

Assume that $X_\nu = \{F(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_\nu)\}$, where F is a quantifier-free Boolean formula with no negations having s atoms of the kind $f>0$, f's are polynomials or Pfaffian functions of degrees less than d or (α,β) respectively.

The set $\bigsqcup^{1,\nu}_{\nu-1,J^1_{\nu-1}} X_\nu \subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having $|J^1_{\nu-1}|s \le s_{\nu-1}s \le (2t_{\nu-2}+1)s$ atoms of degrees less than d (for polynomials) or less than (α,β) (for Pfaffian functions) and at most $2t_{\nu-1}+2n_\nu$ linear atoms (defining $I_1^{t_{\nu-1}+n_\nu}$).

For any $2\le r\le \nu$ the set $B^r_r \subset \mathbb{R}^{t_{r-1}+n_r}$ is defined by a Boolean formula with no negations having $4t_{r-1}+2n_r$ linear atomic inequalities. Therefore, all sets of the kind B^r_j for $j\ge r$ are defined by Boolean formulae with no negations having $4t_{r-1}+2(n_r+\cdots+n_j)$ linear inequalities. In particular, the set $B^r_\nu \subset \mathbb{R}^{t_{r-1}+n_r+\cdots+n_\nu}$ is defined by $4t_{r-1}+2(n_r+\cdots+n_\nu)$ linear atomic inequalities.

For any $2\le r\le \nu-1$ the set $\bigsqcup^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having at most

$$\begin{aligned}(4t_{r-1}+2(n_r+\cdots+n_\nu))|J^r_{\nu-1}|+2t_{\nu-1}+2n_\nu \le\\ \le (4t_{r-1}+2(n_r+\cdots+n_\nu))s_{\nu-1}+2t_{\nu-1}+2n_\nu \le\\ \le (4t_{r-1}+2(n_r+\cdots+n_\nu))(2t_{\nu-2}+1)+2t_{\nu-1}+2n_\nu\end{aligned}$$

linear atoms.

It follows that the set $\bigcup_{2\le r\le \nu-1} \bigsqcup^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having at most

$$((4t_{\nu-1}+2(n_2+\cdots+n_\nu))(2t_{\nu-2}+1)+2t_{\nu-1}+2n_\nu)(\nu-2)$$

linear atoms.

The set

$$\prod\nolimits_\nu^\nu \Big(\bigsqcup\nolimits^{1,\nu}_{\nu-1,J^1_{\nu-1}} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup\nolimits^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \cup B^\nu_\nu\Big) \subset \mathbb{R}^{t_\nu}$$

is defined by a Boolean formula with no negations having at most

$$((2t_{\nu-2}+1)s+2t_{\nu-1}+2n_\nu+((4t_{\nu-1}+2(n_2+\cdots+n_\nu))(2t_{\nu-2}+1)+2t_{\nu-1}+2n_\nu)(\nu-2))\cdot$$
$$\cdot(t_{\nu-1}+1) \le s t_{\nu-1}^{O(1)}$$

atoms of degrees less than d for polynomials or less than (α,β) for Pfaffian functions.

8.1. **Polynomial case.** Let functions f in formula F be polynomials of degrees $\deg(f) < d$. Then, according to (3.1),

$$\mathrm{b}_{q_\nu}\Big(\prod{}_\nu^\nu\Big(\bigsqcup{}^{1,\nu}_{\nu-1,J^1_{\nu-1}} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup{}^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \cup B^\nu_\nu\Big)\Big) \le$$

$$\le O(ds)^{t_\nu} t_{\nu-1}^{O(t_\nu)} \le (2^\nu dsn_0n_1\cdots n_{\nu-1})^{O(2^\nu n_0n_1\cdots n_\nu)}.$$

Using (7.1) in case $i=\nu$, we get

$$\mathrm{b}_{q_0}(X) \le (2^{\nu^2} dsn_0n_1\cdots n_{\nu-1})^{O(2^\nu n_0n_1\cdots n_\nu)}$$

(compare with (3.4)).

8.2. **Pfaffian case.** Let f be Pfaffian functions in an open domain $G \subset \mathbb{R}^n$ of order r, degree (α,β), having a common Pfaffian chain. Then, according to (3.2),

$$\mathrm{b}_{q_\nu}\Big(\prod{}_\nu^\nu\Big(\bigsqcup{}^{1,\nu}_{\nu-1,J^1_{\nu-1}} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup{}^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \cup B^\nu_\nu\Big)\Big) \le$$

$$\le 2^{r(r-1)/2}(st_{\nu-1})^{O(t_\nu)} O(t_\nu\beta + \min\{t_\nu, r\}\alpha)^{t_\nu+r} \le$$

$$\le 2^{r(r-1)/2} s^{O(2^\nu n_0n_1\cdots n_\nu)} (2^\nu n_0n_1\cdots n_\nu(\alpha+\beta))^{O(2^\nu n_0n_1\cdots n_\nu + r)}.$$

Using (7.1) in case $i=\nu$, we get

$$\mathrm{b}_{q_0}(X) \le$$

$$\le 2^{r(r-1)/2 + O(\nu r + \nu^2 2^\nu n_0n_1\cdots n_\nu)} s^{O(2^\nu n_0n_1\cdots n_\nu)} (n_0n_1\cdots n_\nu(\alpha+\beta))^{O(2^\nu n_0n_1\cdots n_\nu + r)} \le$$

$$\le 2^{r(r-1)/2}(2^{\nu^2} sn_0n_1\cdots n_\nu(\alpha+\beta))^{O(2^\nu n_0n_1\cdots n_\nu + r)}.$$

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