Irreducibility of some spectral determinants

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Abstract

This is a complement to our paper arXiv:0802.1461. We study irreducibility of spectral determinants of some one-parametric eigenvalue problems in dimension one with polynomial potentials.

We consider eigenvalue problems of the type

$$-y'' + P(\alpha, z)y = \lambda y,$$

with the boundary condition that $y(z) \to 0$ along appropriately chosen paths in the complex plane (for example, at $\pm\infty$ on the real line). The potential P is a polynomial in the independent variable z, depending on a complex parameter α. The problems we consider have discrete spectra for all complex α. The dependence of the eigenvalues on α is described by the equation

$$F(\alpha, \lambda) = 0,$$

where F is an entire function of two variables which is called the spectral determinant. We study irreducibility of the spectral determinant for certain eigenvalue problems that occur in quantum mechanics.

Problems of this type were considered for the first time in [11, 12] for the case of Mathieu equation with the boundary conditions on a finite interval. We refer to [17, 6, 7, 8] for further development.

Our research was stimulated by the paper of Bender and Wu who discovered in [3] that the spectral determinant of the even quartic oscillator

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has exactly two irreducible components. In our paper [9] we gave a complete rigorous proof of this fact. It was mentioned in [9] that our method applies to several other one-parametric families of eigenvalue problems, and the purpose of the present paper is to give some details of these applications.

We begin with a sketch of the main result of [9] referring to that paper for complete details.

Even quartic oscillator

1. in [9] we considered the eigenvalue problem for the even quartic oscillator:

$$-y'' + (z^4 + \alpha z^2)y = \lambda y, \tag{1}$$

$$y(\pm\infty) = 0 \quad \text{on the real line,} \tag{2}$$

where α is a complex parameter. For real α the problem is self-adjoint. The spectrum is discrete, infinite and simple. The set

$$Z = \{(\alpha, \lambda) \in \mathbf{C}^2 : \lambda \text{ is an eigenvalue of (1), (2)}\}$$

is an analytic subset of $\mathbf{C}^2$.

Theorem 1 *The set Z consists of two irreducible components: one for even eigenfunctions, another for odd ones. These irreducible components are also connected components. Moreover, the set Z is non-singular.*

2. Parametrization of the set Z.

Let G be the set of all odd meromorphic functions f such that $f(z) \to 0$ as $z \to \pm\infty$ on the real line, and the Schwarzian

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is a polynomial of degree 4 with leading coefficient -2:

$$-\frac{1}{2}S_f(z) = z^4 + \alpha z^2 - \lambda.$$

We have a map $\Phi : G \to \mathbf{C}^2,\ f \mapsto (\alpha, \lambda)$.

Proposition 1 *Φ maps G to Z surjectively.*

Remark 1 *If we define the equivalence relation by $f_1 \sim f_2$ if $f_1 = cf_2$ then $\tilde{\Phi} : G/\!\sim\, \to Z$ is a biholomorphic parametrization.*

3. Sketch of the proof. We have $f = y/y_1$ where y is an eigenfunction (it is always even or odd), and y_1 a linearly independent solution of (1) of *opposite parity*. In the opposite direction: $y_1 = 1/\sqrt{f'}$, $y = fy_1$. Now we have the equivalencies:

$$\begin{array}{cc} & f(z) \to 0, \quad |z| \to \infty \quad \text{on the real axis} \\ \Longleftrightarrow & y \text{ is subdominant in } S_0, S_3 \\ \Longleftrightarrow & y \text{ is an eigenfunction} \\ \Longleftrightarrow & \lambda \text{ is an eigenvalue.} \end{array}$$

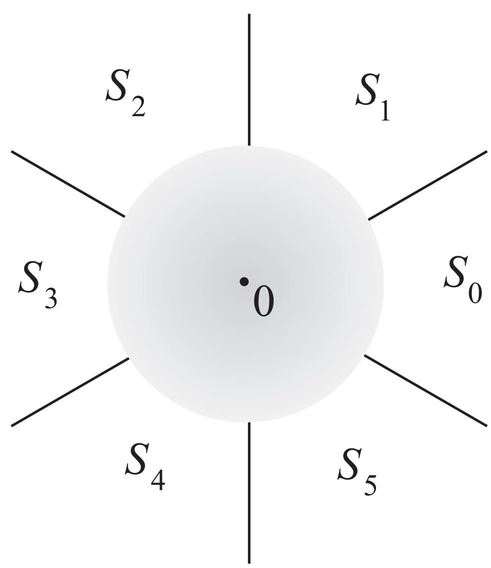


Fig 1. Stokes sectors

4. Deformation of functions in G.

Every function $f \in G$ has no critical points: $f' = (y'y_1 - yy_1')/y_1^2$ so $f'(z) \neq 0$, and the poles are simple. Now f has 6 *asymptotic tracts* corresponding to the Stokes sectors: $f(z) \to w_j$ in S_j. Thus

$$f : \mathbf{C}\backslash f^{-1}(\{w_0, \dots, w_5\}) \to \overline{\mathbf{C}}\backslash\{w_0, \dots, w_5\}$$

is an unramified covering.

Proposition 2 *Let $f_0 \in G$, and let $w_j(t)$ be a centrally symmetric deformation of the asymptotic values such that*

$$w_j(0) \neq w_k(0) \Longrightarrow w_j(t) \neq w_k(t), \quad t \in [0, 1].$$

Then there exists a deformation $f_t \in G$ for $t \in [0, 1]$ such that $f_t(z) \to w_j(t)$ in S_j.

5. Sketch of the proof. Let $\psi_t : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be odd diffeomorphisms, $\psi_t(w_j(0)) = w_j(t)$. Then there exist odd diffeomorphisms $\phi_t : \mathbf{C} \to \mathbf{C}$ such that $g_t = \psi_t \circ f_0 \circ \phi_t$ are meromorphic functions in $\mathbf{C}$ [14]. These functions have no critical points and 6 asymptotic tracts with asymptotic values $w_j(t)$.

For every meromorphic function with no critical points and q asymptotic tracts, the Schwarzian S_f is a polynomial of degree $q-2$ [13]. Putting $f_t(z) = g(c_t z)$ we make $-(1/2)S_f$ monic; it is even because f is odd.

6. We may assume that

$$(w_0, w_1, w_2, w_3, w_4, w_5) = (0, i, 1, 0, -i, -1).$$

How to describe all $f \in G$ with such asymptotic values?

Let $\Psi_f = f^{-1}(\Psi_0)$, where Ψ_0 is the following cell decomposition of the Riemann sphere:

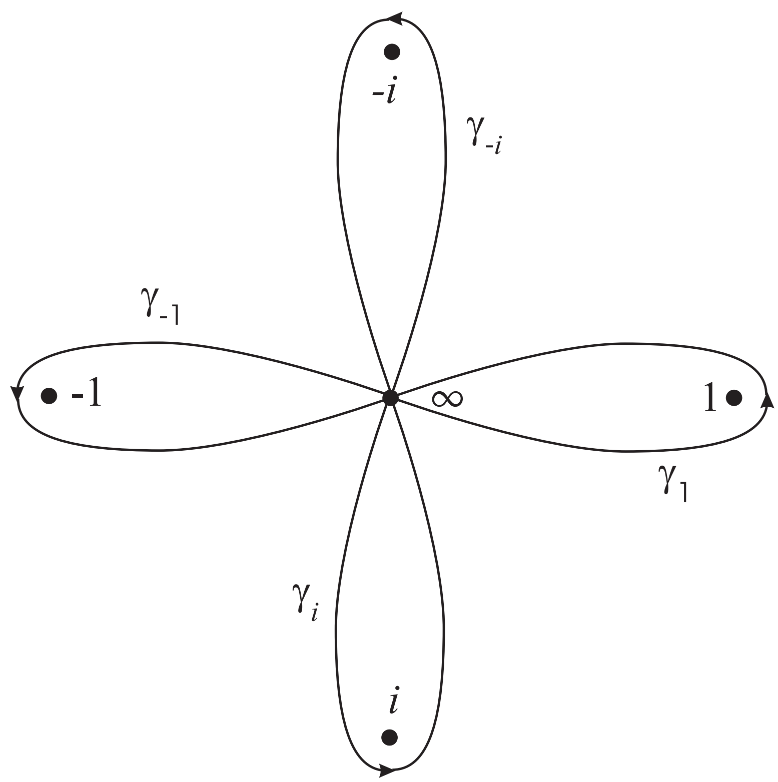


Fig 2. Cell decomposition Ψ_0.

The next figure shows how the preimage of Ψ_0 may look.

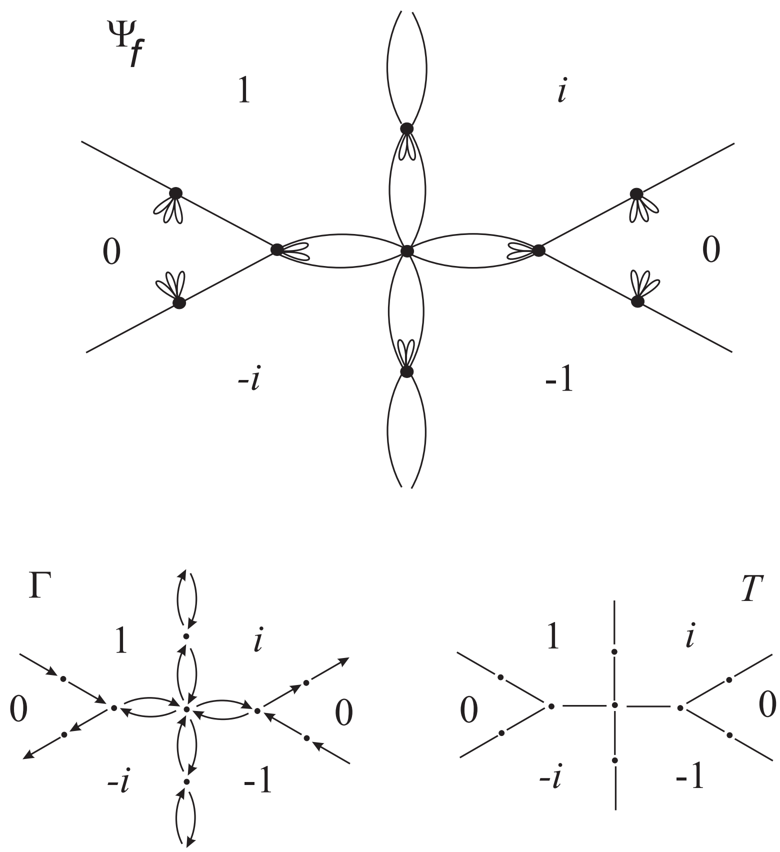


Fig 3. Cell decomposition Ψ_f.

Faces of Ψ_f are labeled by asymptotic values. Removing loops and replacing multiple edges of Ψ_f by simple edges, we obtain a *tree* T. This tree has 6 faces and faces labeled 0 cannot have a common boundary edge. It is possible to classify all such trees.

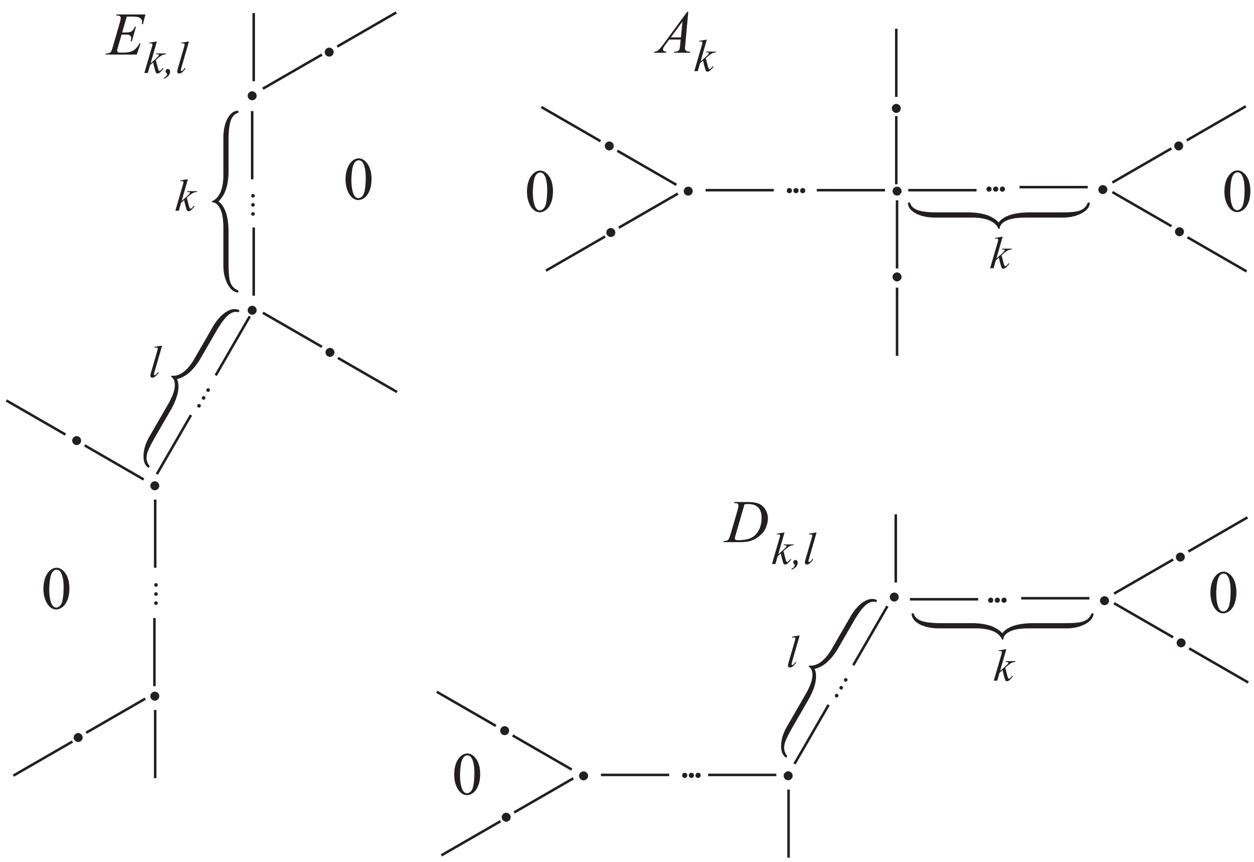
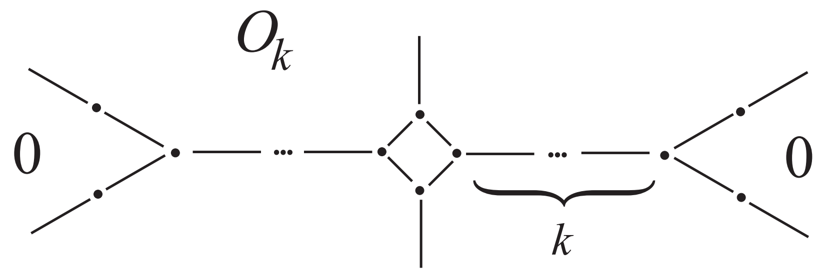


Fig 4. Classification of trees: $A_k : k \geq 0$, $D_{k,l} : k \geq 0, l \geq 1$,
$E_{k,l} : k \geq 1, l \geq 0$.



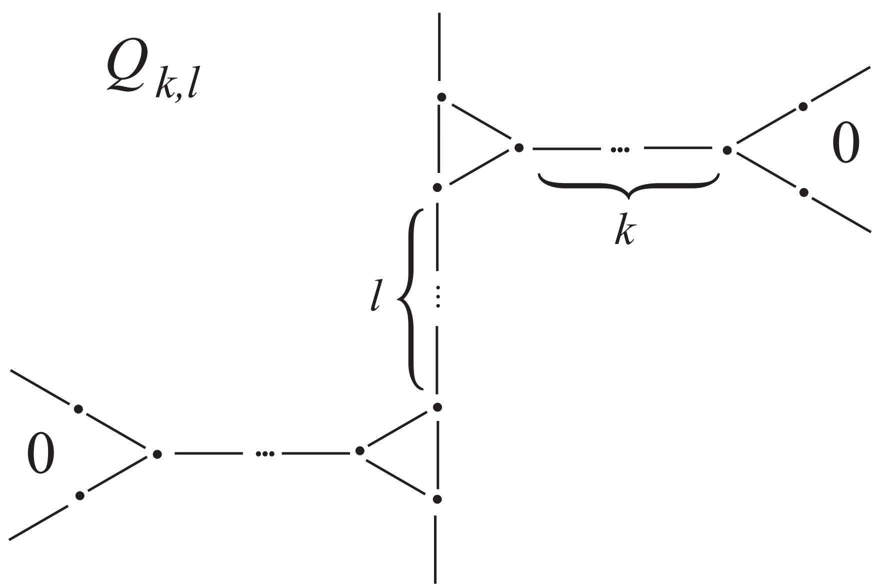


Fig 5. Additional patterns that occur if the cyclic order of the loops of Φ_0 is opposite to the cyclic order of asymptotic tracts in $\mathbf{C}$.

Propoisition 2 implies that we can continuously deform the asymptotic values of a function in G. We consider such deformations that the configuration of 5 asymptotic values describes a closed loop in the space of 5-point configurations symmetric with respect to the origin, and the asymptotic values i and $-i$ are interchanged. Then we compute the action of these deformations on our cell decompositions of the plane, and conclude that there are exactly two orbits.

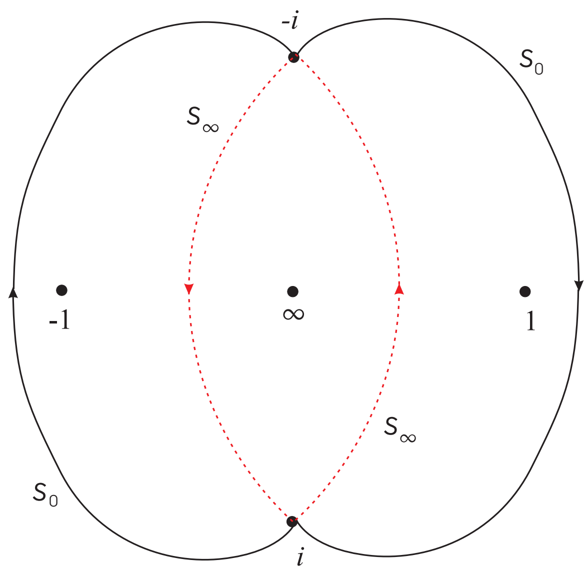


Fig 6. Paths s_0 and s_∞ used in the deformation of Ψ_0.

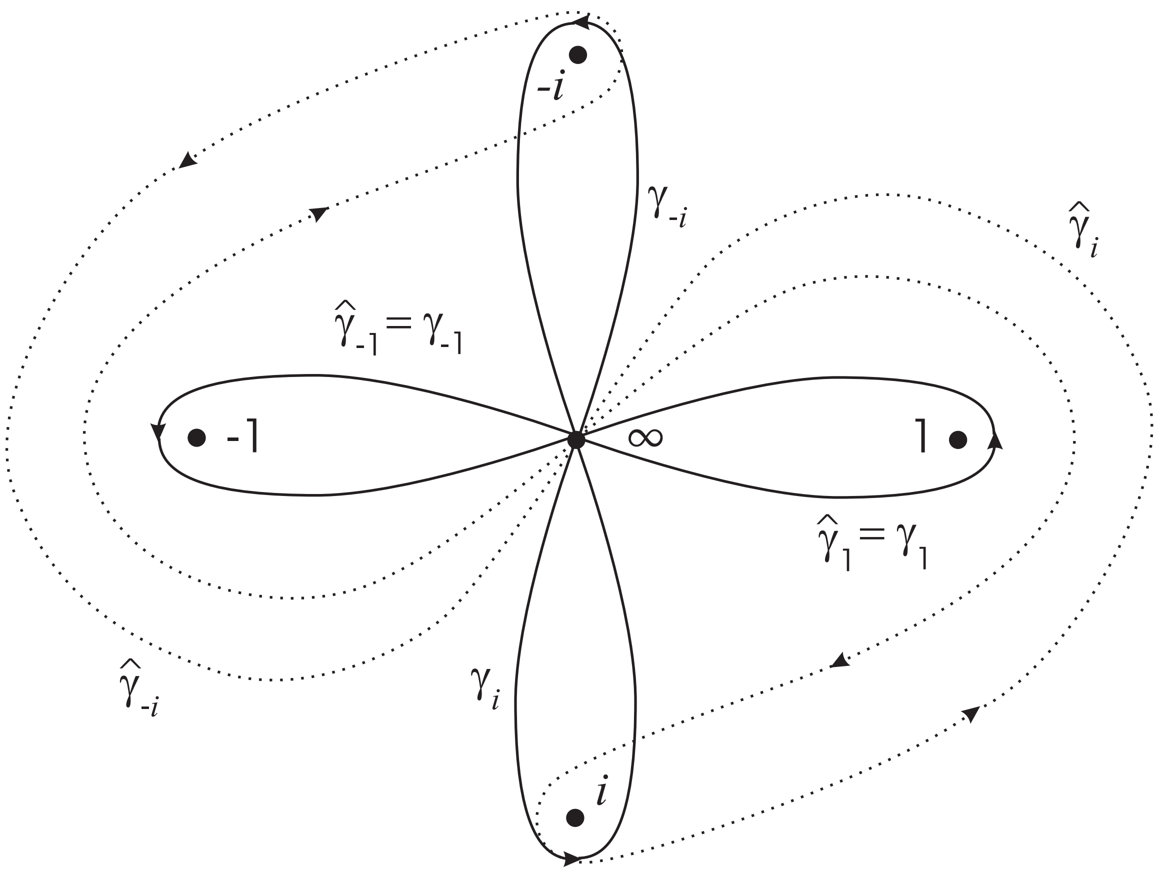


Fig 7. Action of s_0 on Ψ_0. The new loops are expressed in terms of the old ones by the formulas

$$\hat{\gamma}_i = (\gamma_1)^{-1}\gamma_i\gamma_1, \quad \hat{\gamma}_1 = \gamma_1,$$

$$\hat{\gamma}_{-i} = (\gamma_{-1})^{-1}\gamma_{-i}\gamma_{-1}, \quad \hat{\gamma}_{-1} = \gamma_{-1}.$$

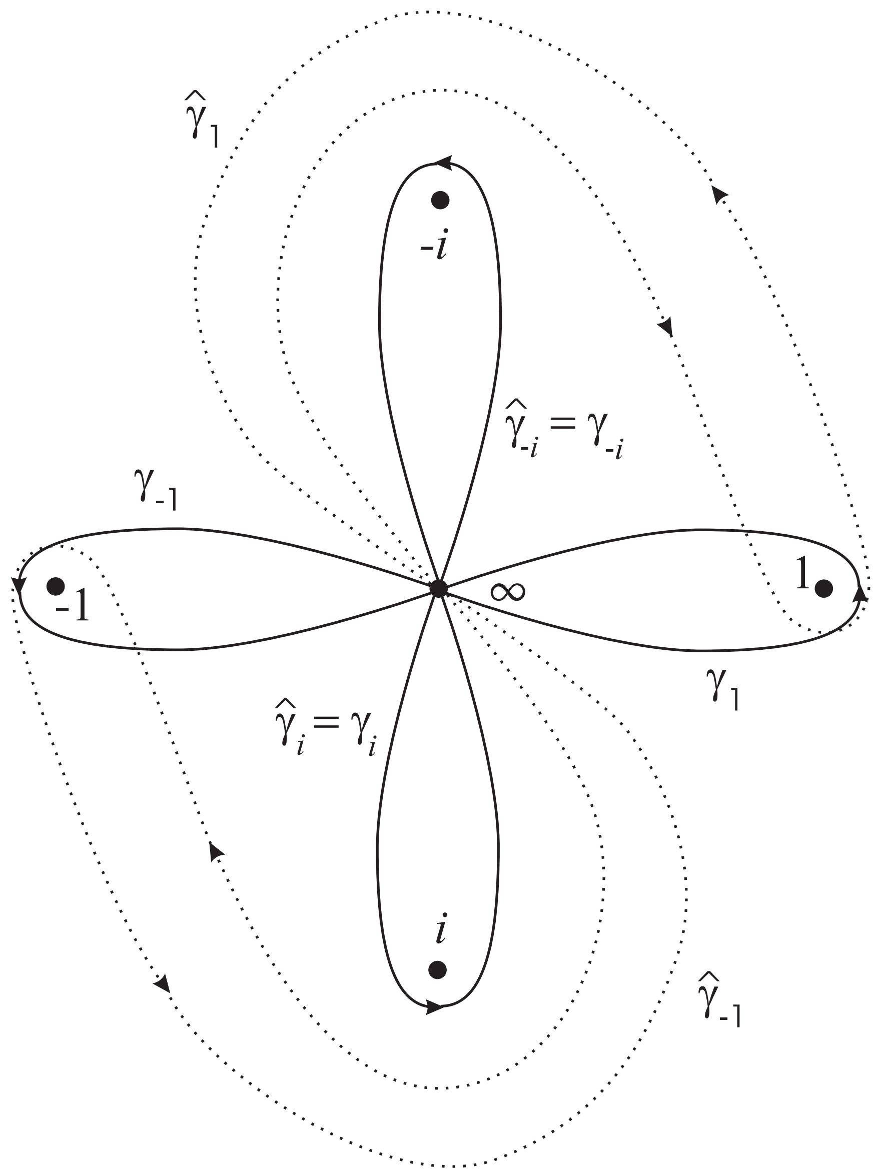


Fig 8. Action of s_∞ on Ψ_0. The new loops are expressed by the formulas

$$\hat{\gamma}_i = \gamma_i, \quad \hat{\gamma}_1 = (\gamma_{-i})^{-1}\gamma_1\gamma_{-i},$$

$$\hat{\gamma}_{-i} = \gamma_{-i}, \quad \hat{\gamma}_{-1} = (\gamma_i)^{-1}\gamma_{-1}\gamma_i.$$

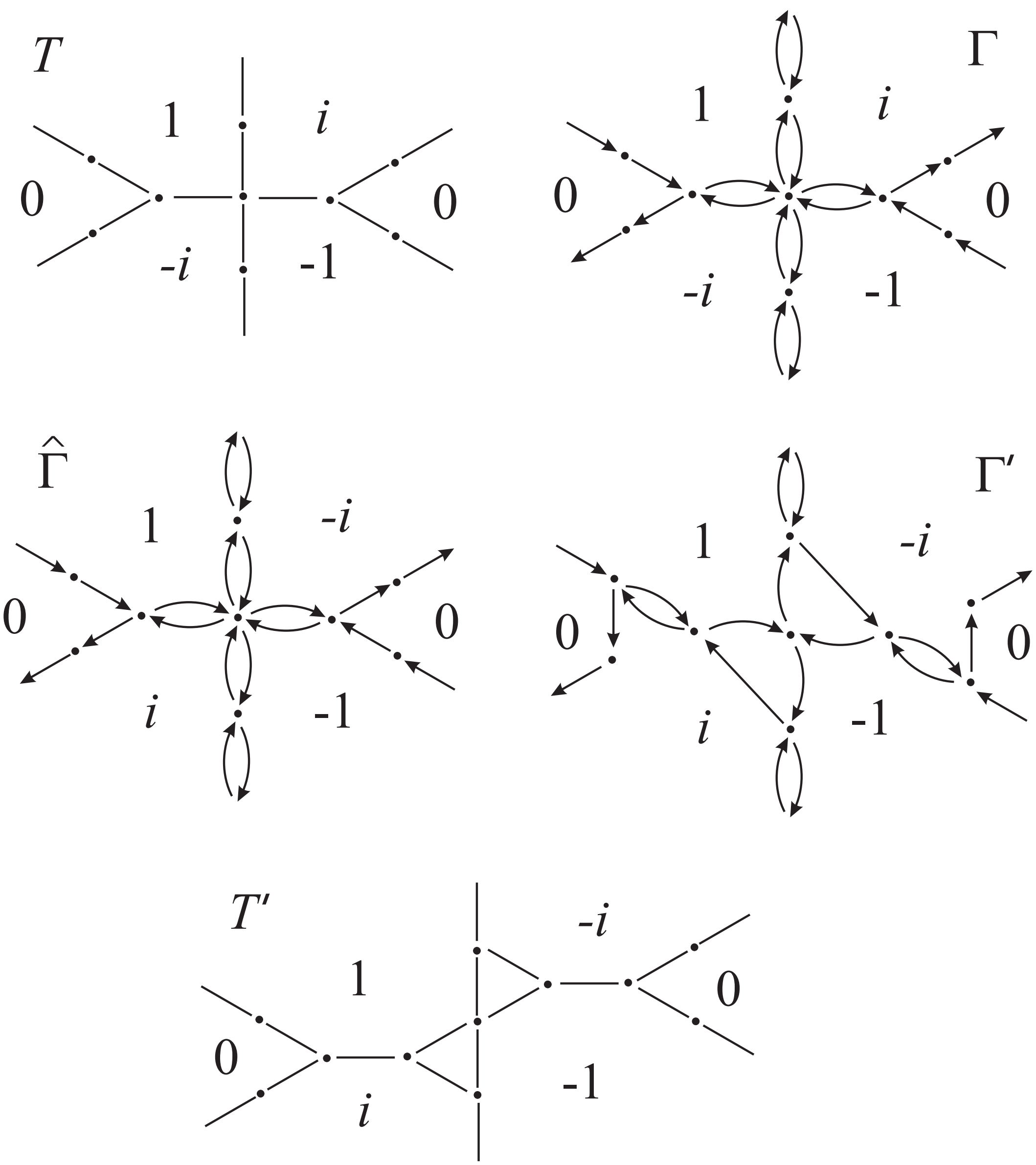


Fig 9. Example. Transformation of A_1 to $Q_{1,0}$ by the action of s_0.

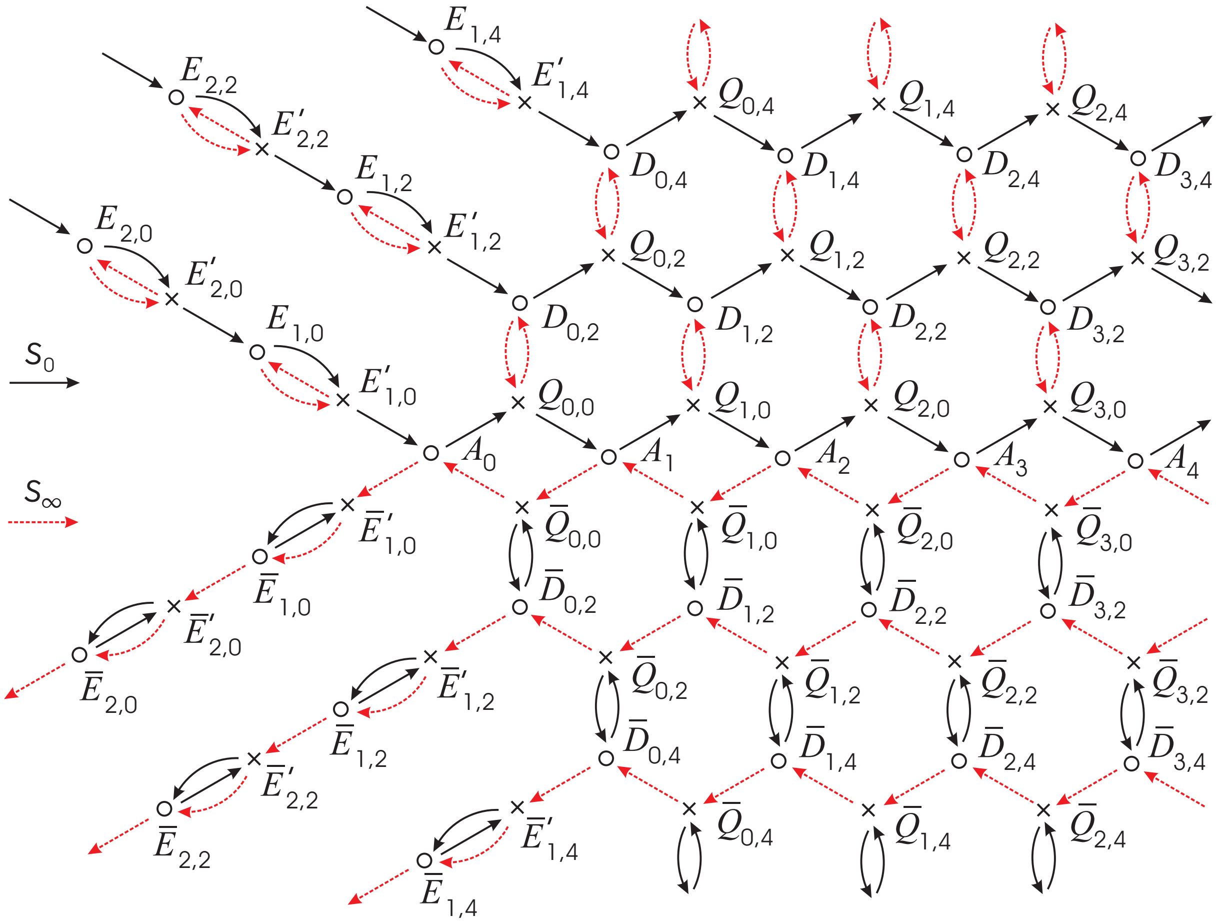


Fig 10. Monodromy action on the even part of G.

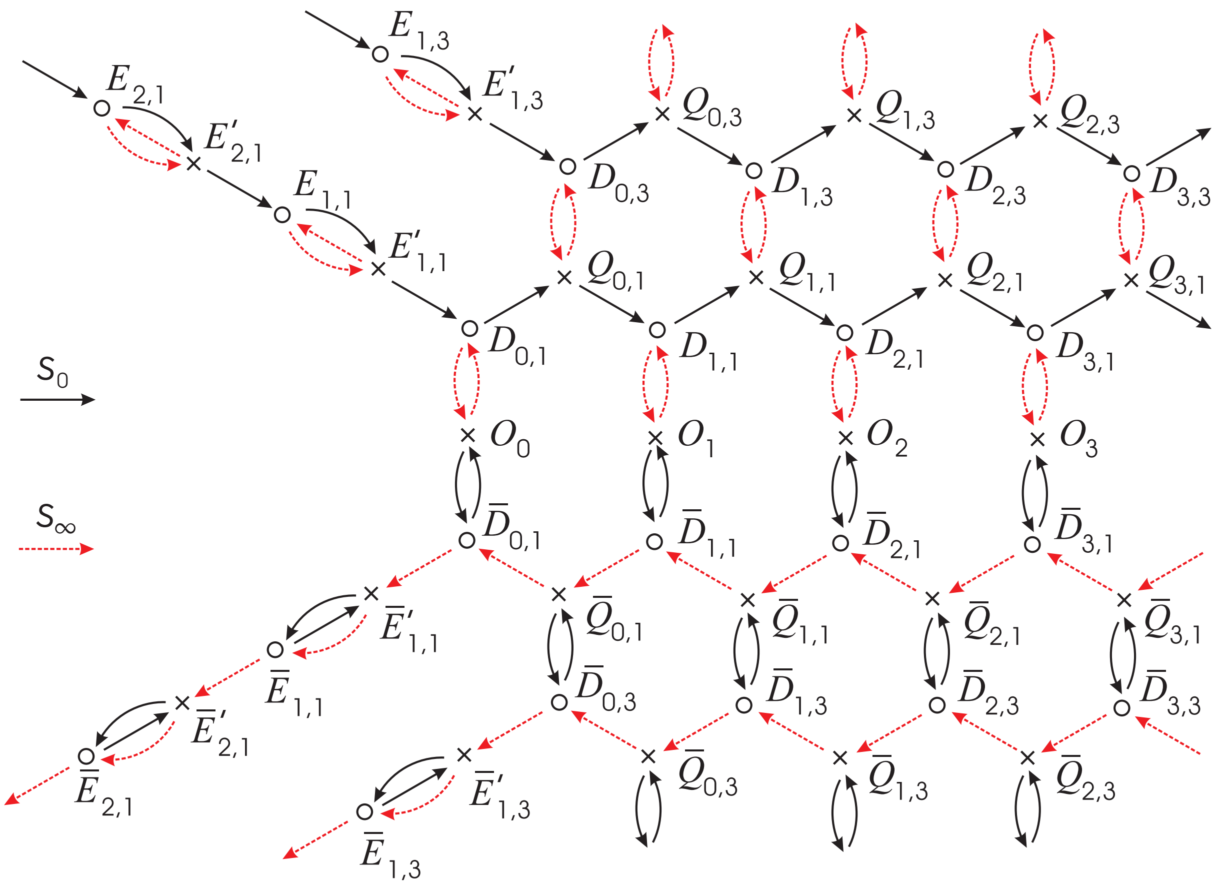


Fig 11. Monodromy action on the odd part of G.

This proof can be generalized to several other eigenvalue problems with polynomial potentials depending on one parameter.

Meromorphic functions in the plane of finite order, without critical points are called *Nevanlinna functions*.

This class of functions coincides with solutions of Schwarz differential equations

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = P,$$

where P is a polynomial. The order of growth of f equals $(\deg P)/2 + 1$, and f has exactly $(\deg P) + 2$ asymptotic tracts.

The general solution of the Schwarz differential equation is a ratio of two linearly independent solutions of the linear differential equation

$$y'' + \frac{1}{2}Py = 0.$$

A Nevanlinna function is determined, up to an affine change of the independent variable, by its asymptotic values and by certain combinatorial information—topology of the cell decomposition of the plane obtained as the preimage of a cell decomposition of the Riemann sphere each of whose faces contains one asymptotic value. Two cell decompositions of the plane are equivalent if they can be obtained one from another by a homeomorphism of the plane preserving orientation.

Our proofs in all cases follow the same pattern. We parametrize the zero locus of the spectral determinant by a class of Nevanlinna functions. To study the irreducible components of this class, we move their asymptotic values to a convenient position. Then we describe all Nevanlinna functions of our class with these asymptotic values by some cell decompositions of the plane. All possible cell decompositions arising in a given problem can be classified by using some related trees. Then we study the monodromy action on the set of these trees.

Quasi-Exactly Solvable sextics (Turbiner-Ushveridze [15])

$$-y'' + (z^6 + 2\alpha z^4 + \{\alpha^2 - (4m + 2p + 3)\} z^2)\, y = \lambda y \tag{3}$$

$$y(\pm\infty) = 0 \text{ on the real line, } m \geq 0,\ p \in \{0,\, 1\}. \tag{4}$$

The problem is self-adjoint for real α. There are $m + 1$ "elementary" eigenfunctions

$$Q(z)\, \exp(-z^4/4 - \alpha z^2/2),$$

where Q is a polynomial of degree $2m + p$.

Consider the set $Z_{m,p}$ of pairs (α, λ) such that λ is an eigenvalue corresponding to an elementary eigenfunction of (3), (4).

Theorem 2 *For each m and p the zet $Z_{m,p}$ is irreducible.*

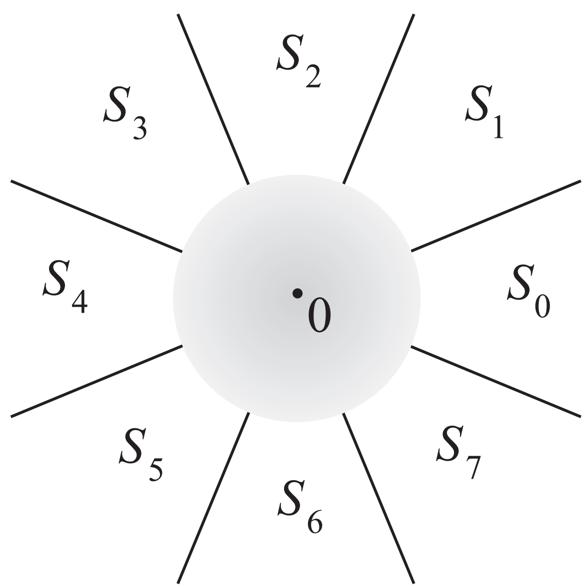


Fig S1. Stokes sectors for the sextic potential

Elementary eigenfunctions are distinguished by the propery that asymptotic values in S_0, S_2, S_4, S_6 are equal to zero. Thus we have 5 asymptotic values. Proposition 2 permits to move them to the points $(0, i, 0, 1, 0, -i, 0, -1)$, and we use the cell decomposition of the Riemann sphere shown in Fig. 2. Applying the reduction procedure illustrated in Fig. 3, we obtain centrally symmetric trees with 8 ends, in which even-numbered faces are labeled by 0, and none of such two faces have a common boundary edge. Next we obtain a classification of such trees.

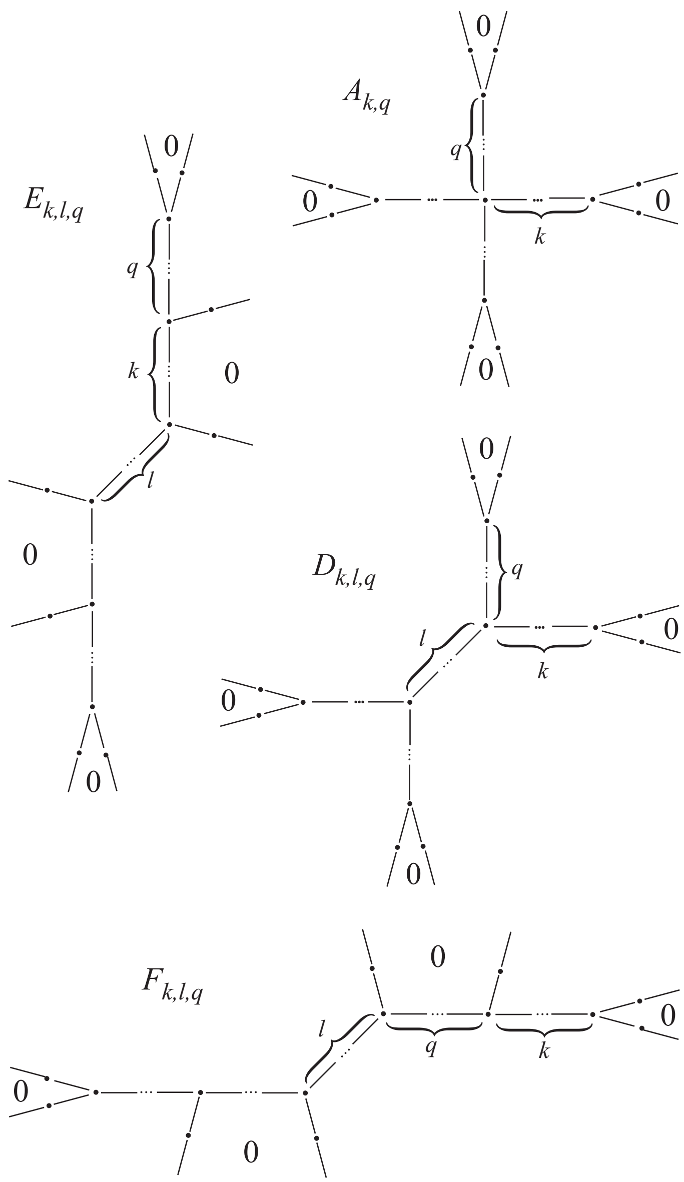


Fig S2. Classification of QES sextic trees

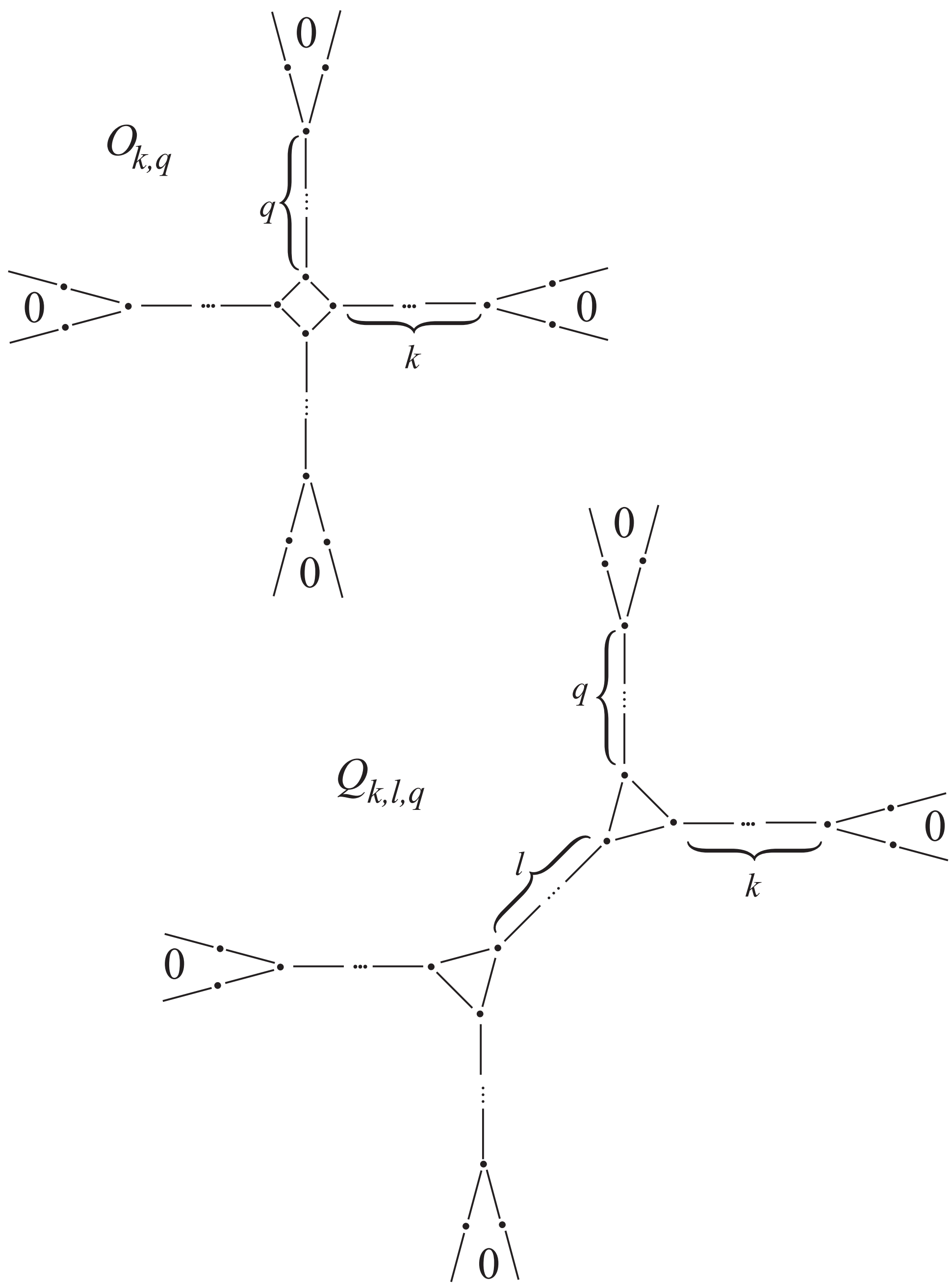


Fig S3. Non-tree QES sextic graphs

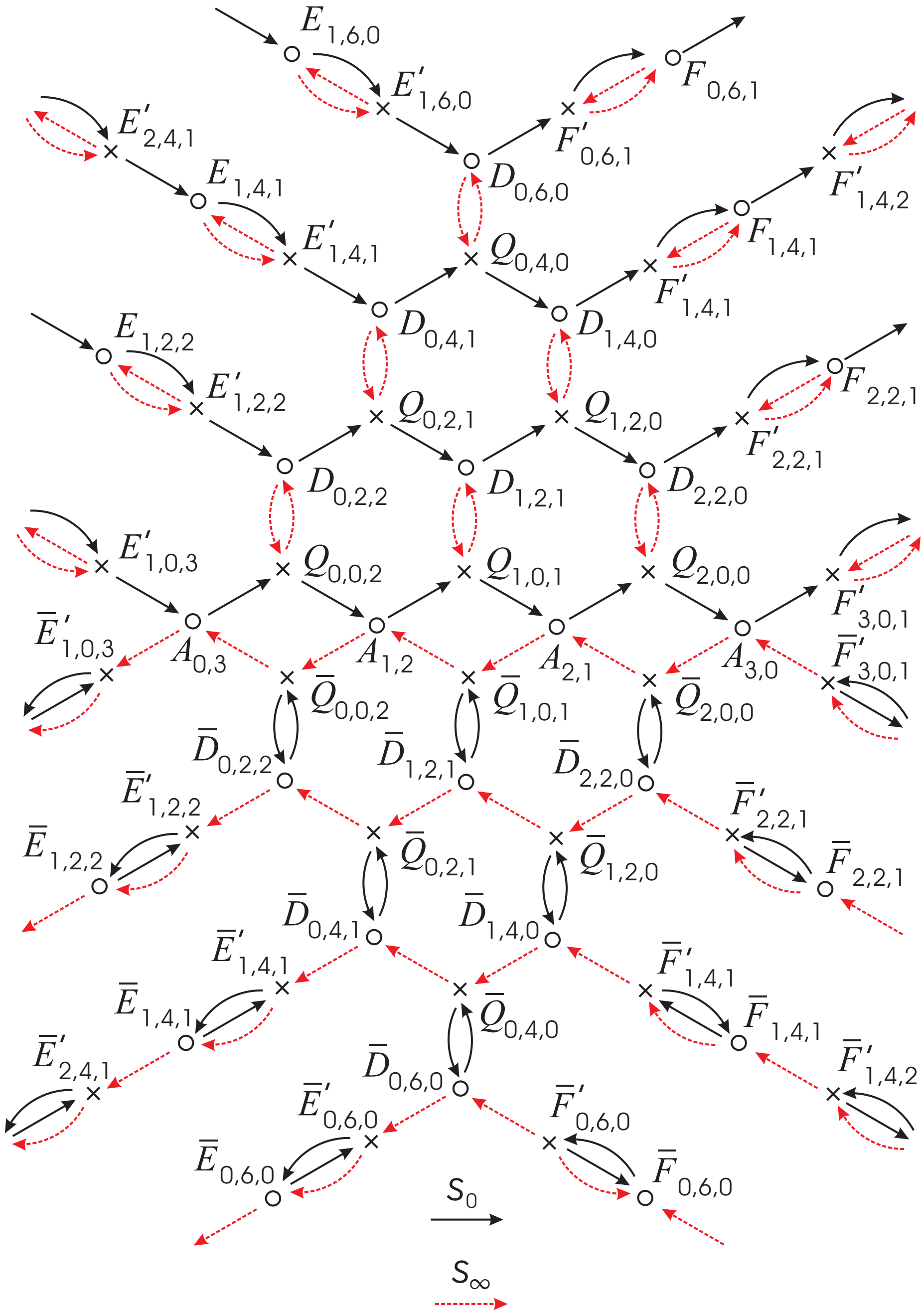


Fig S4. Monodromy action for an even QES sextic, $m = 3,\ p = 0$

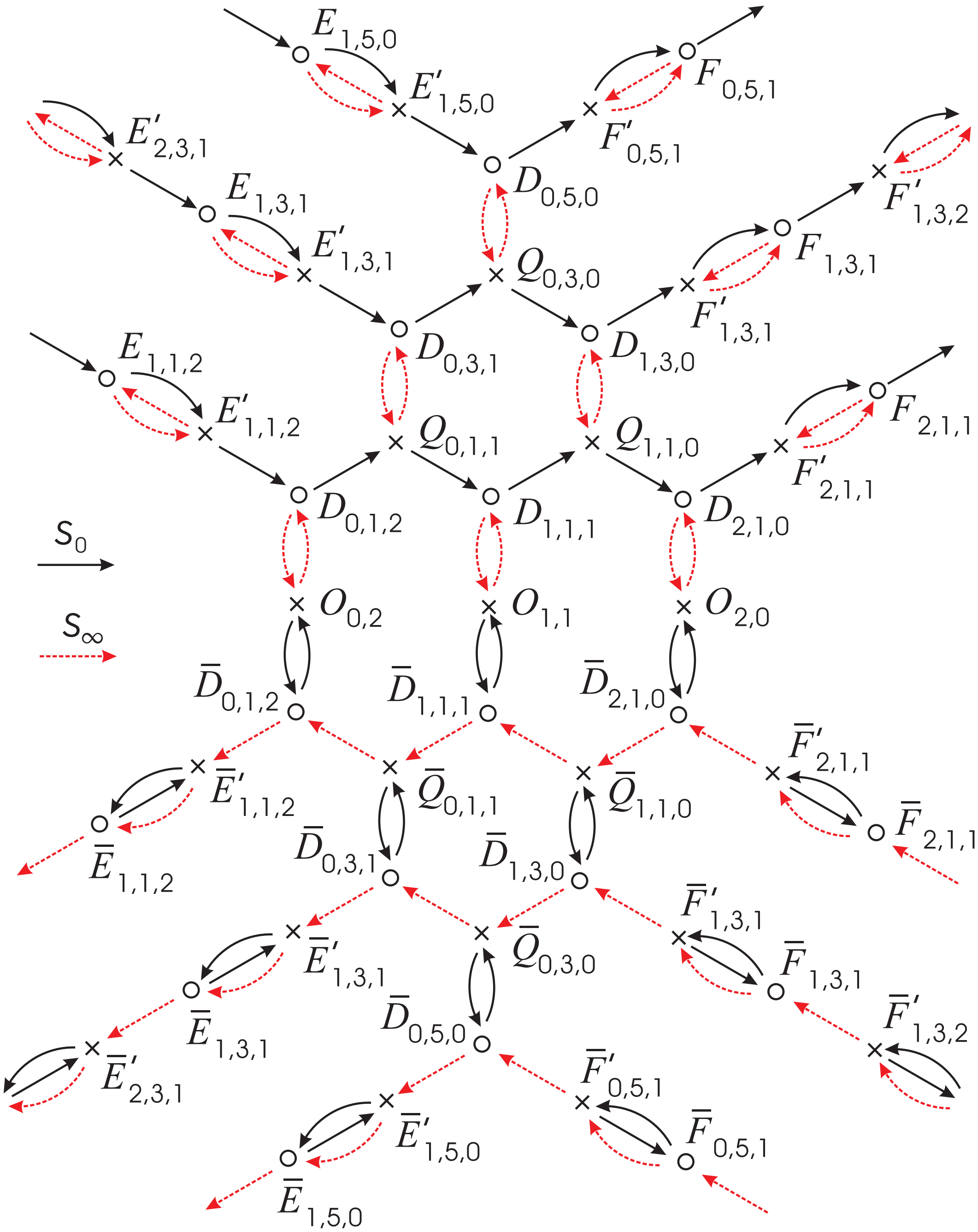


Fig S5. Monodromy action for an odd QES sextic, $m = 2,\ p = 1$

The set $Z_{m,p}$ is an algebraic curve in $\mathbf{C}^2$, and the next picture shows the critical values of its projection onto the α-plane (=branch points of the multi-valued function $\lambda(\alpha)$). This is a computer generated picture, and computation shows that these branch points are simple. The same applies to the branch points of the function $\lambda(\alpha)$ generated by the even quartic considered above. We don't know any rigorous proof of these facts about branch points.

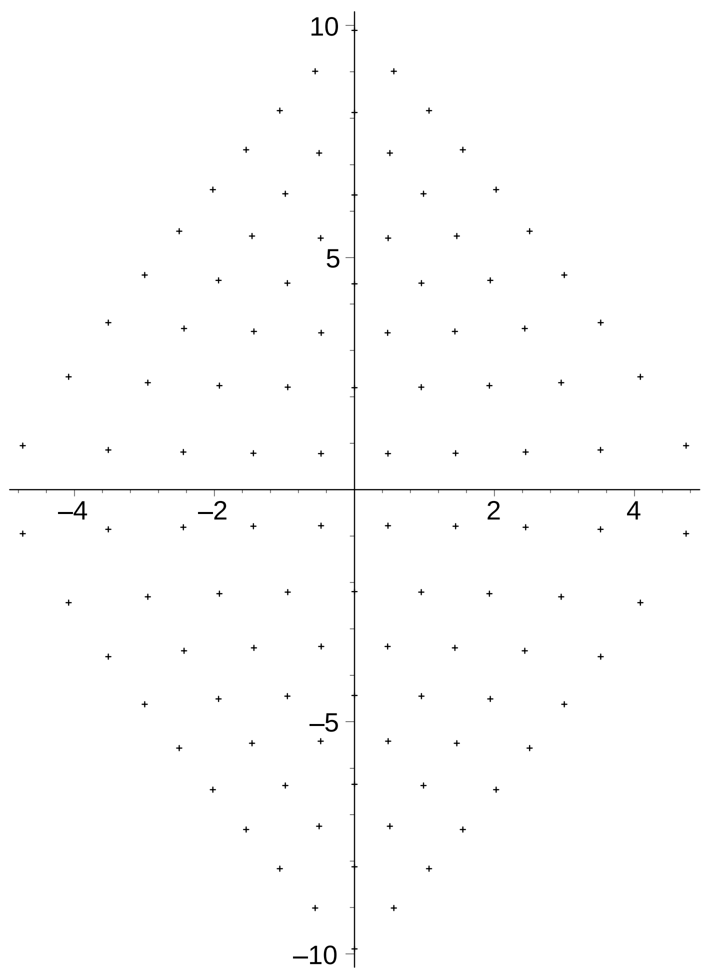


Fig S6. Branching points for a QES sextic with $m = 10,\ p = 0$

Rescaling

When $m \to \infty$ in (3), the problem approximates in certain sense the even quartic oscillator (1), (2) [16]. To see this we put $n = 4m + 2p + 3$. Then the quasi-exactly solvable equation (3) is related to

$$-y''(x) + [a^2x^6 + 2abx^4 + (b^2 - an)x^2]y(x) = \mu y(x) \tag{5}$$

by the scaling $z = a^{1/4}x,\ b = a^{1/2}\alpha,\ \lambda = a^{1/2}\mu$. To approximate the quartic potential $2x^4 + \beta x^2$ by the rescaled quasi-exactly solvable sextic potentials in (5) as $m \to \infty$, let $b = n^{1/3}(1 + sn^{-2/3}),\ a = n^{-1/3}(1 + tn^{-2/3})$. Then $\alpha = b/a^{1/2} = n^{1/2}(1 + (s - t/2)n^{-2/3} + O(n^{-4/3})$. Substituting expression for a and b into (5), we get the potential

$$n^{-2/3}(1+O(n^{-2/3}))x^6+2(1+(s+t)n^{-2/3}+stn^{-4/3})x^4+((2s-t)+s^2n^{-2/3})x^2.$$

Hence $\beta = 2s - t = 2(n^{-1/2}\alpha - 1)n^{2/3} + O(n^{-2/3})$.

Figure S7 shows location of the branch points of $\lambda(\alpha)$ for the rescaled sextic, and Figure S8, which is taken from [4] shows the same for the quartic oscillator.

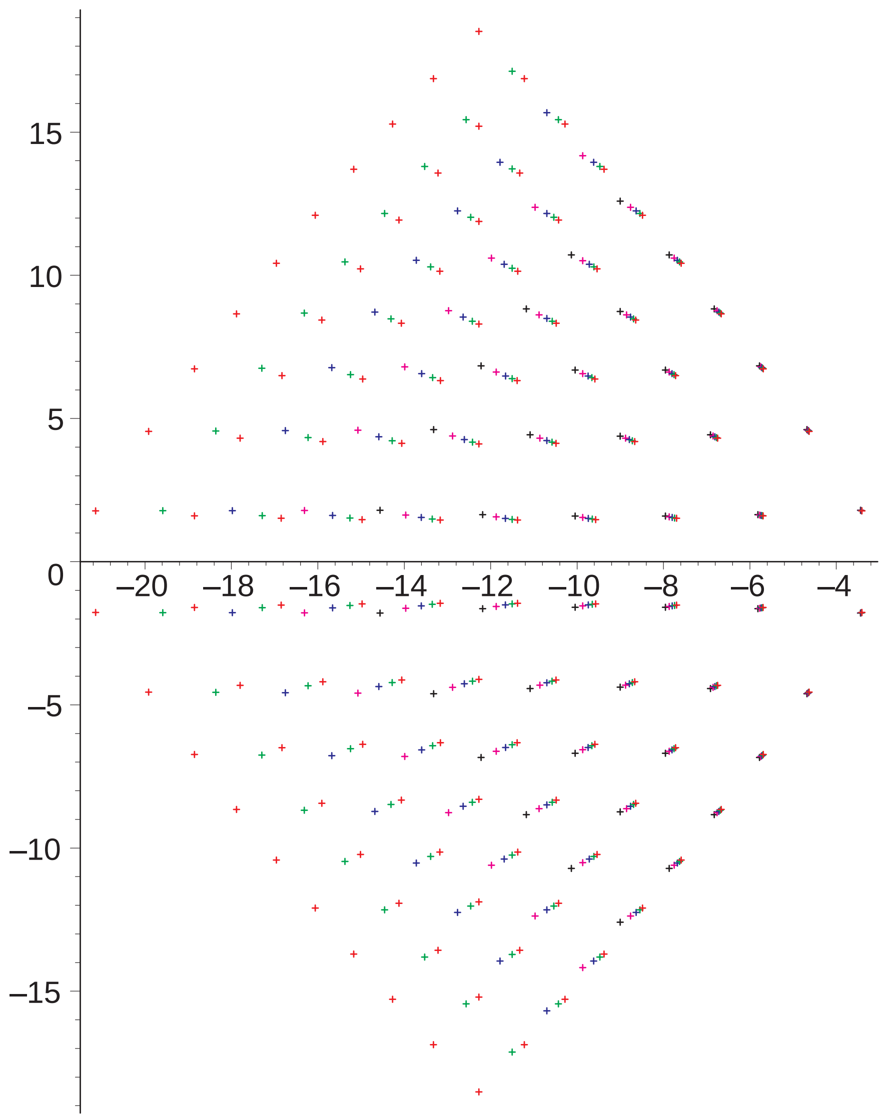


Fig S7. Branching points for rescaled QES sextics with $m = 6\text{-}10,\ p = 0$

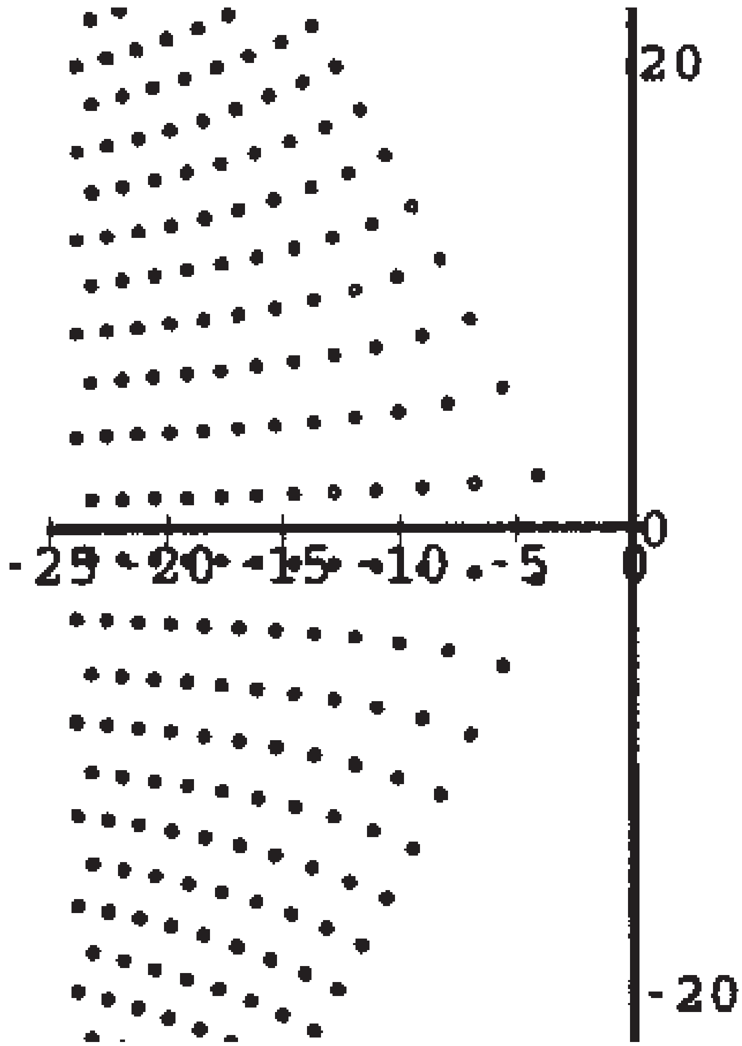


Fig S8. Branching points for the quartic oscillator (Delabaere–Pham).

PT-symmetric cubic (Delabaere-Trinh [5])

Polynomial potential $P(z)$ is PT-symmetric if $P(-\bar{z}) = \overline{P(z)}$.

Eigenvalue problem

$$-y'' + (iz^3 + i\alpha z)\, y = \lambda y, \tag{6}$$

$$y(\pm\infty) = 0 \text{ on the real line,} \tag{7}$$

is PT-symmetric for real α.

Let Z be the set of all pairs (α, λ) such that λ is an eigenvalue of (6), (7).

Theorem 3 *The set Z is irreducible.*

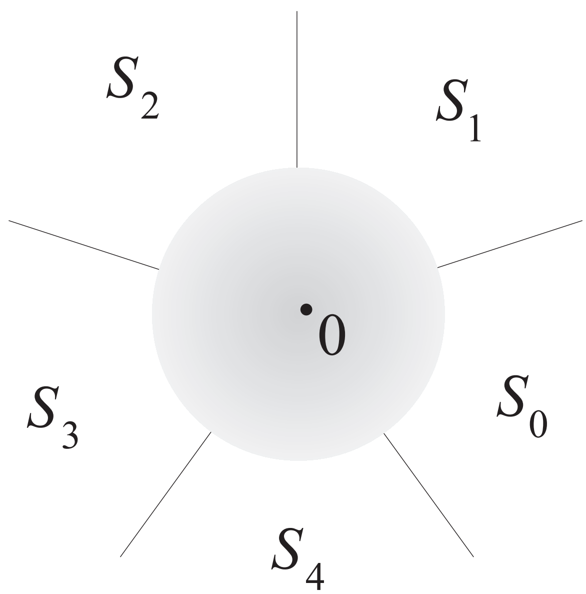


Fig C0. Stokes sectors for the cubic potential

Line complexes

Let f be a Nevanlinna function with q asymptotic values. Consider a cell decomposition of the sphere with two vertices which we denote by $\times$ and $\circ$, with q edges each connecting these two vertices, and such that each of the faces contains exactly one asymptotic value. The f-preimage of such cell decomposition is called a line complex (see [10]) All possible line complexes of Nevanlinna functions can be simply characterized: they are bipartite graphs embedded in the plane such that the degree of each vertex is q and all faces have either two or infinitely many boundary edges. Replacing multiple edges of a cell complex by single edges we obtain a tree. The line complex can be recovered from this tree.

These properties make line complexes very convenient. The reason why we used other cell decompositions of the sphere for quartics and sextics is that there are no line complexes for these cases having all symmetries present in the problems.

In the case of PT-symmetric cubic we can move the asymptotic values to the following positions: $(0, 1, -1, 0, \infty)$ (listed counterclockwise starting from S_0. The line complex is the preimage of the following cell decomposition of the Riemann sphere:

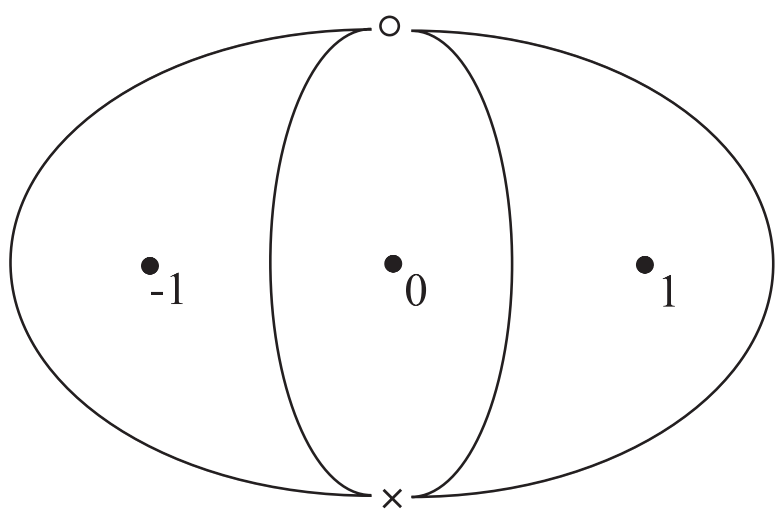


Fig C1. Cell decomposition of the sphere for the cubic

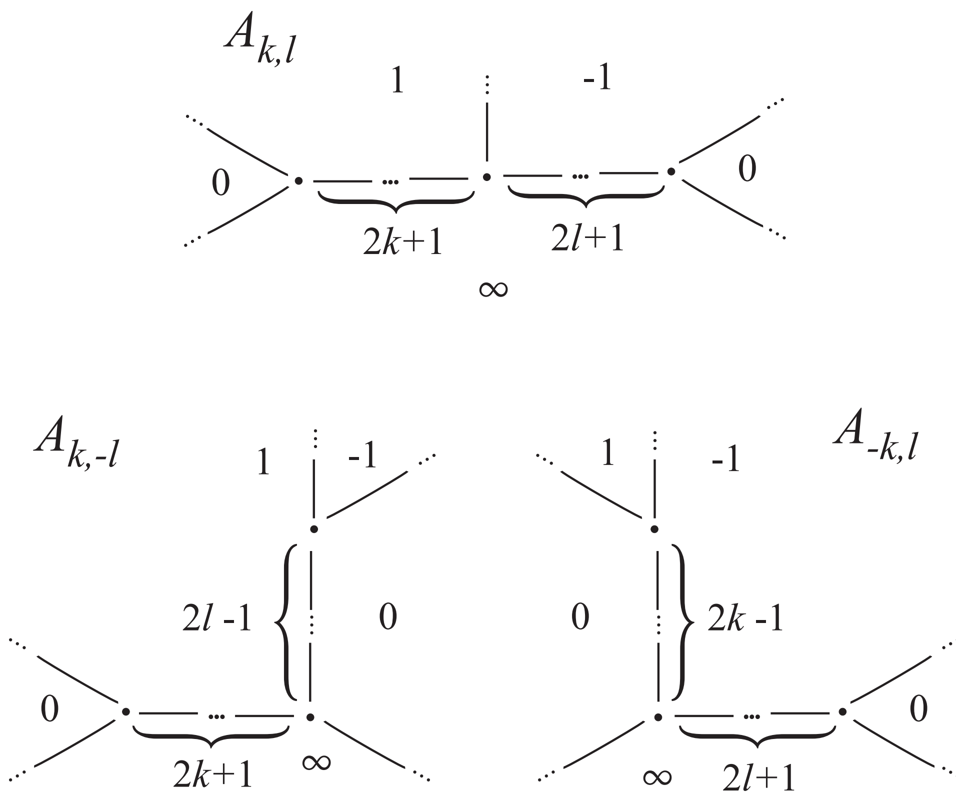


Fig C2. Trees for the cubic. Type A

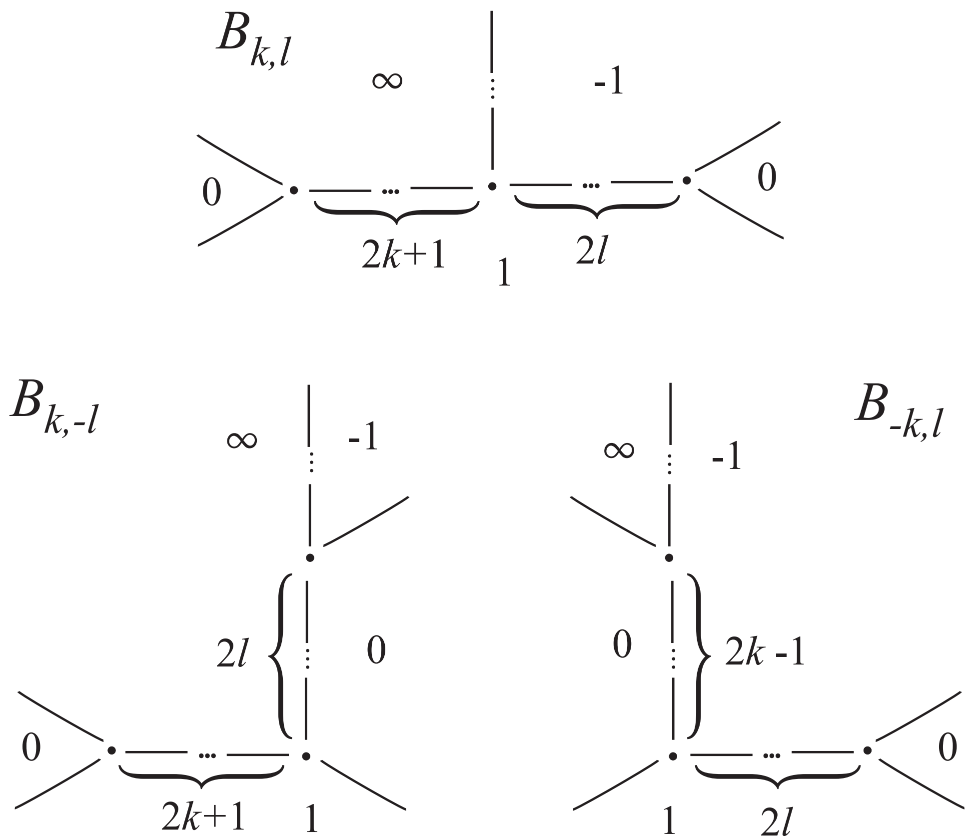


Fig C3. Trees for the cubic. Type B

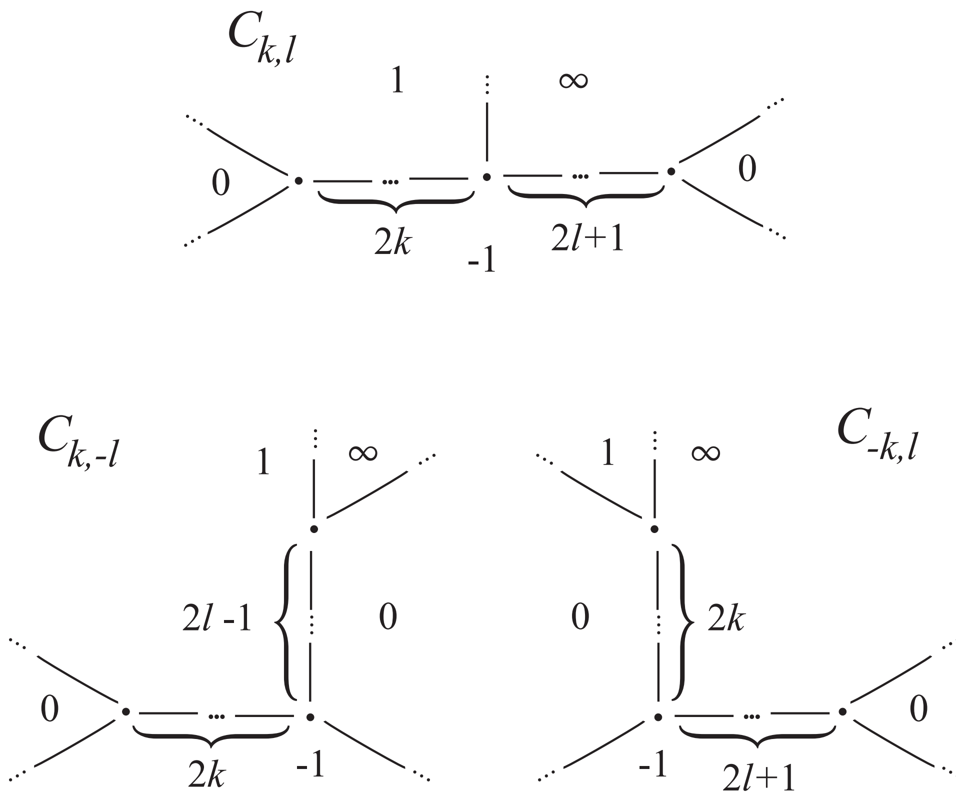


Fig C4. Trees for the cubic. Type C

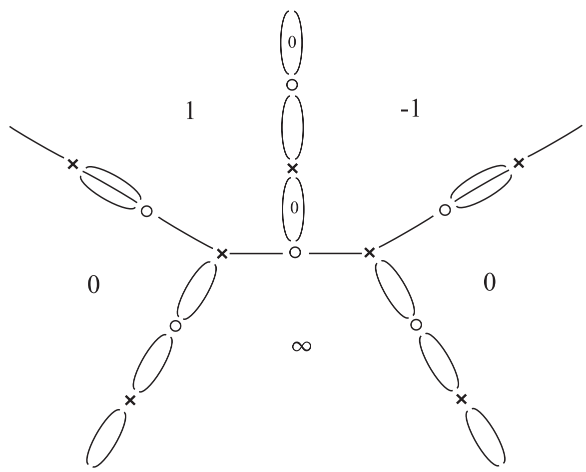


Fig C5. Line complex of type $A_{0,0}$

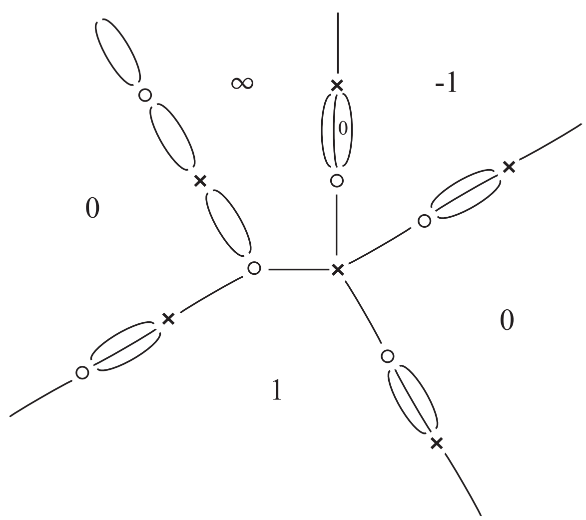


Fig C6. Line complex of type $B_{0,0}$

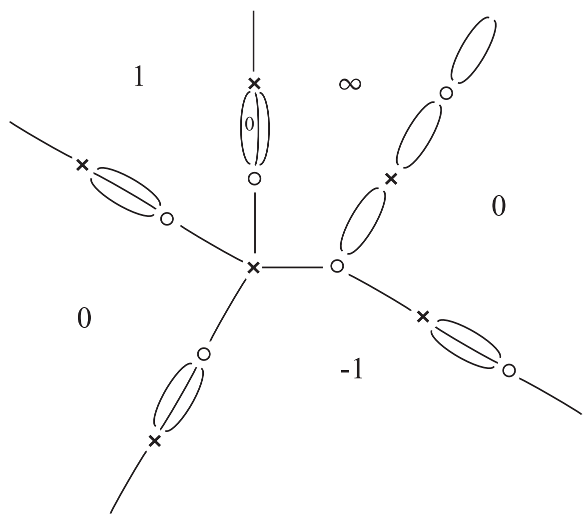


Fig C7. Line complex of type $C_{0,0}$

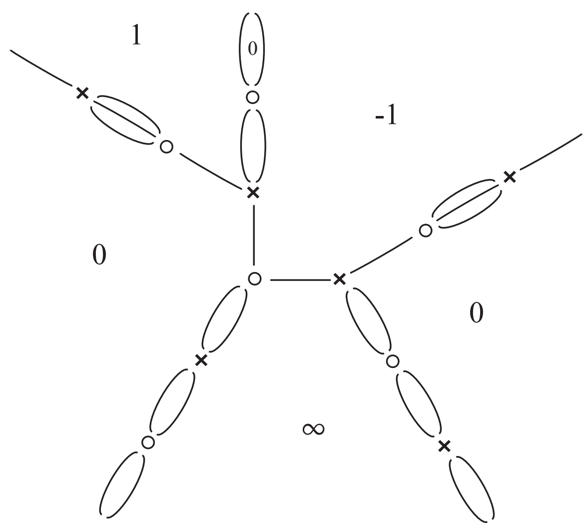


Fig C8. Line complex of type $A_{-1,0}$

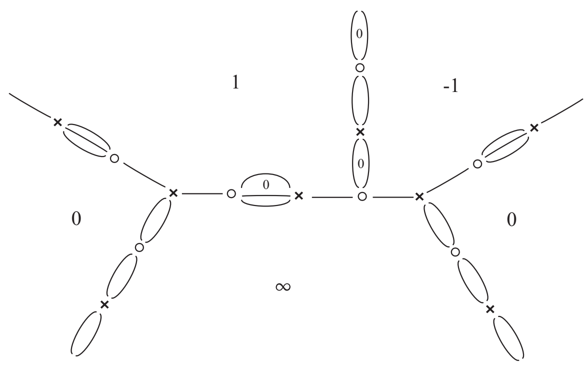


Fig C9. Line complex of type $A_{1,0}$

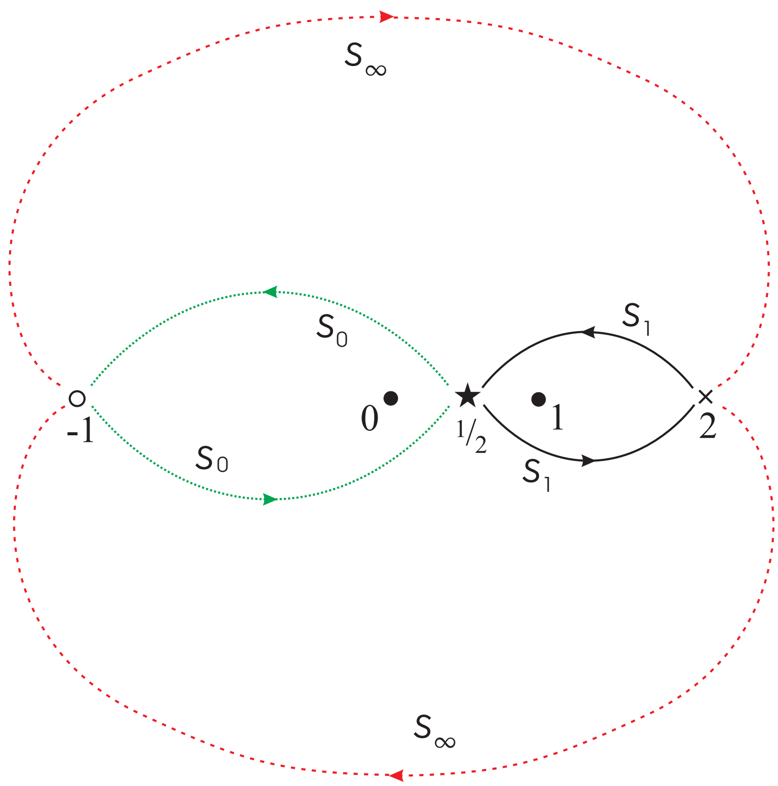


Fig C10. Paths s_0, s_1 and s_∞ used in the deformation of the Nevanlinna function corresponding to the cubic.

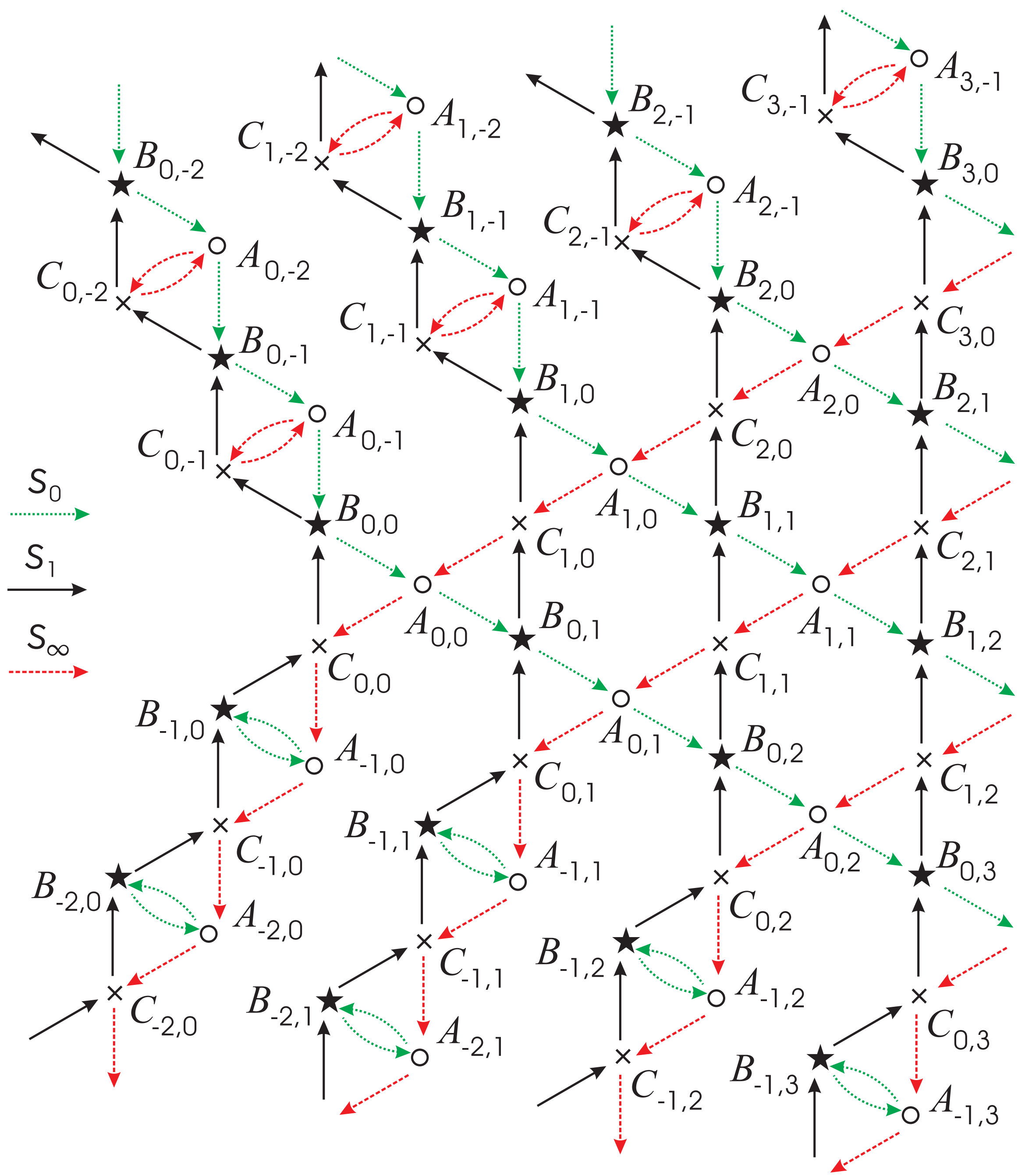


Fig C11. Monodromy action for the cubic

Quasi-Exactly Solvable quartics (Bender-Boettcher [1])

The problem

$$-y'' + (-z^4 - 2\alpha z^2 - 2imz)\, y = \lambda z, \quad m \geq 1 \tag{8}$$

$$y(r\, e^{i\theta}) \to 0 \text{ as } r \to \infty, \quad \theta \in \{-\pi/6, -\pi + \pi/6\} \tag{9}$$

is PT-symmetric for real α.

There are m "elementary" eigenfunctions

$$P(z) \exp(-iz^3/3 - ibz)$$

P polynomial of degree $m-1$.

Let Z_m be the set of all pairs (α, λ) such that λ is an eigenvalue of (8), (9) corresponding to an elementarey eigenfunction.

Theorem 4 *For each m, the set Z_m is irreducible.*

There are six Stokes sectors, with $S_0 = \{-\pi/3 < \arg z < 0\}$ and the asymptotic values can be placed at the points $(0, -1, 0, 1, 0, \infty)$, listed anti-clockwise, starting from S_0.

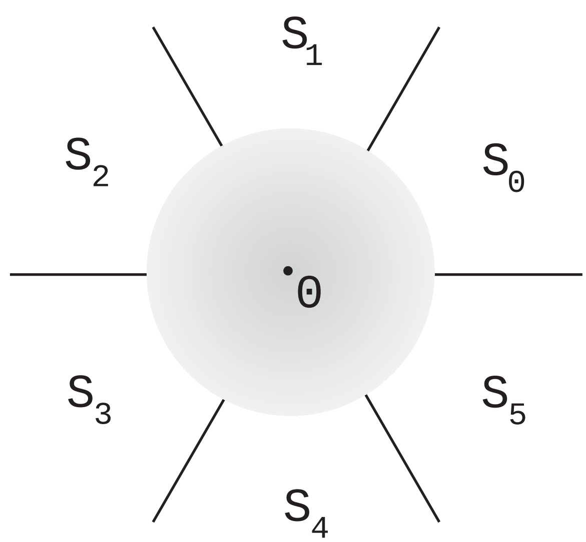


Fig Q0. Stokes lines for the PT quartic.

The cell decomposition of the Riemann sphere is the same as in Fig. C1, and its preimage is a line complex. The associated trees have five ends, and the faces labeled with 0 have disjoint boundaries. Such trees are classified into types A, B and C. The deformation paths are the same as for the PT-symmetric cubic, Fig. C10.

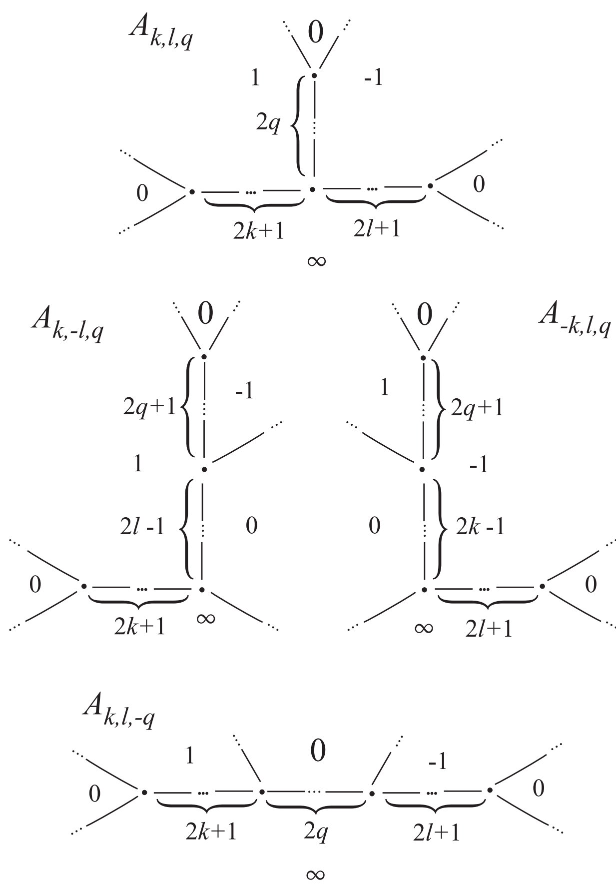


Fig Q1. Trees for the QES quartic, type A

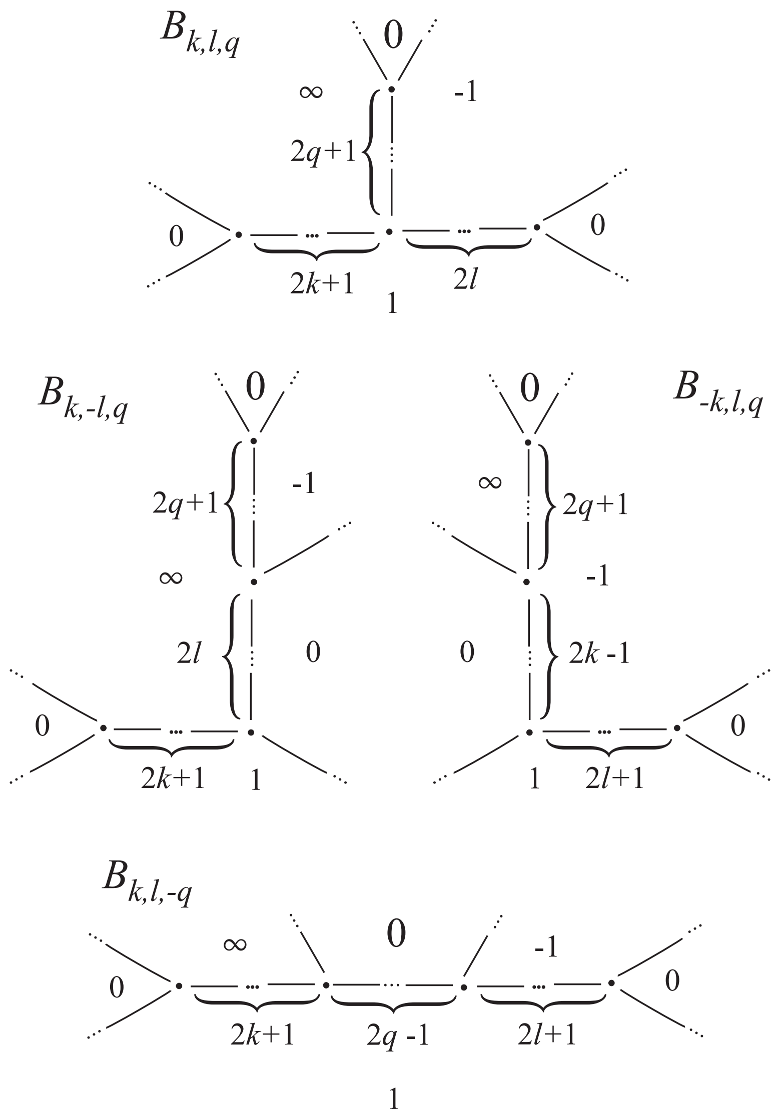


Fig Q2. Trees for the QES quartic, type B

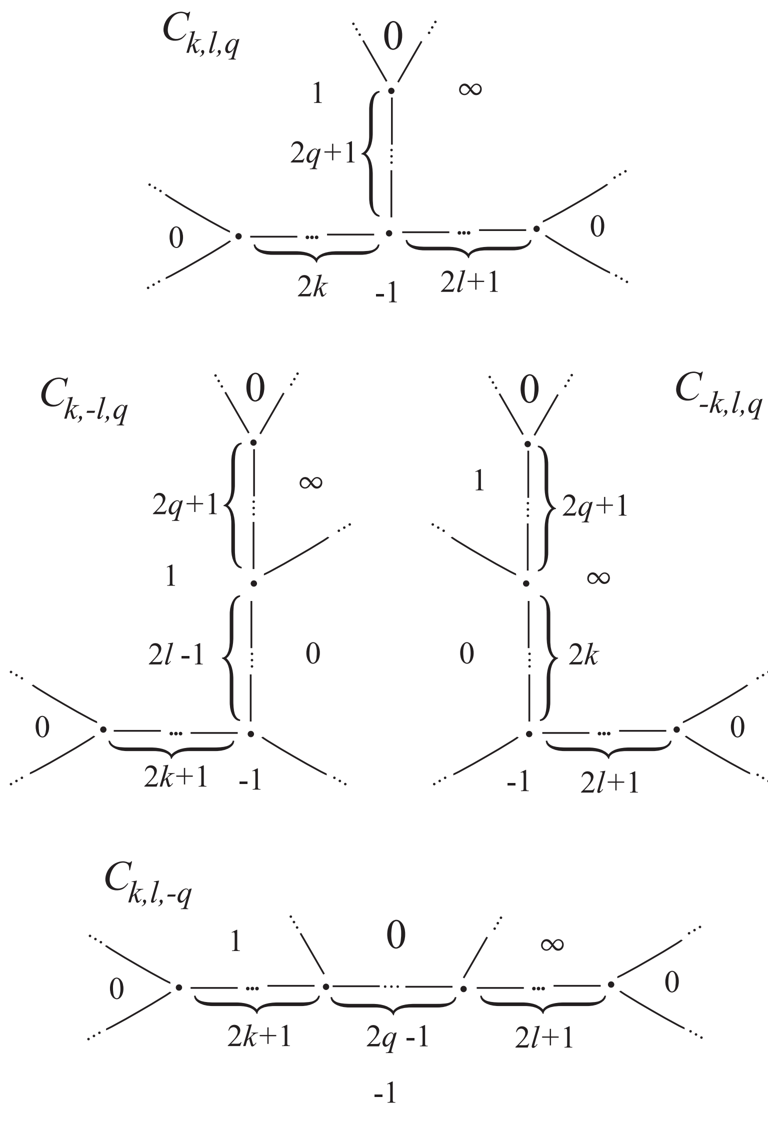


Fig Q3. Trees for the QES quartic, type C

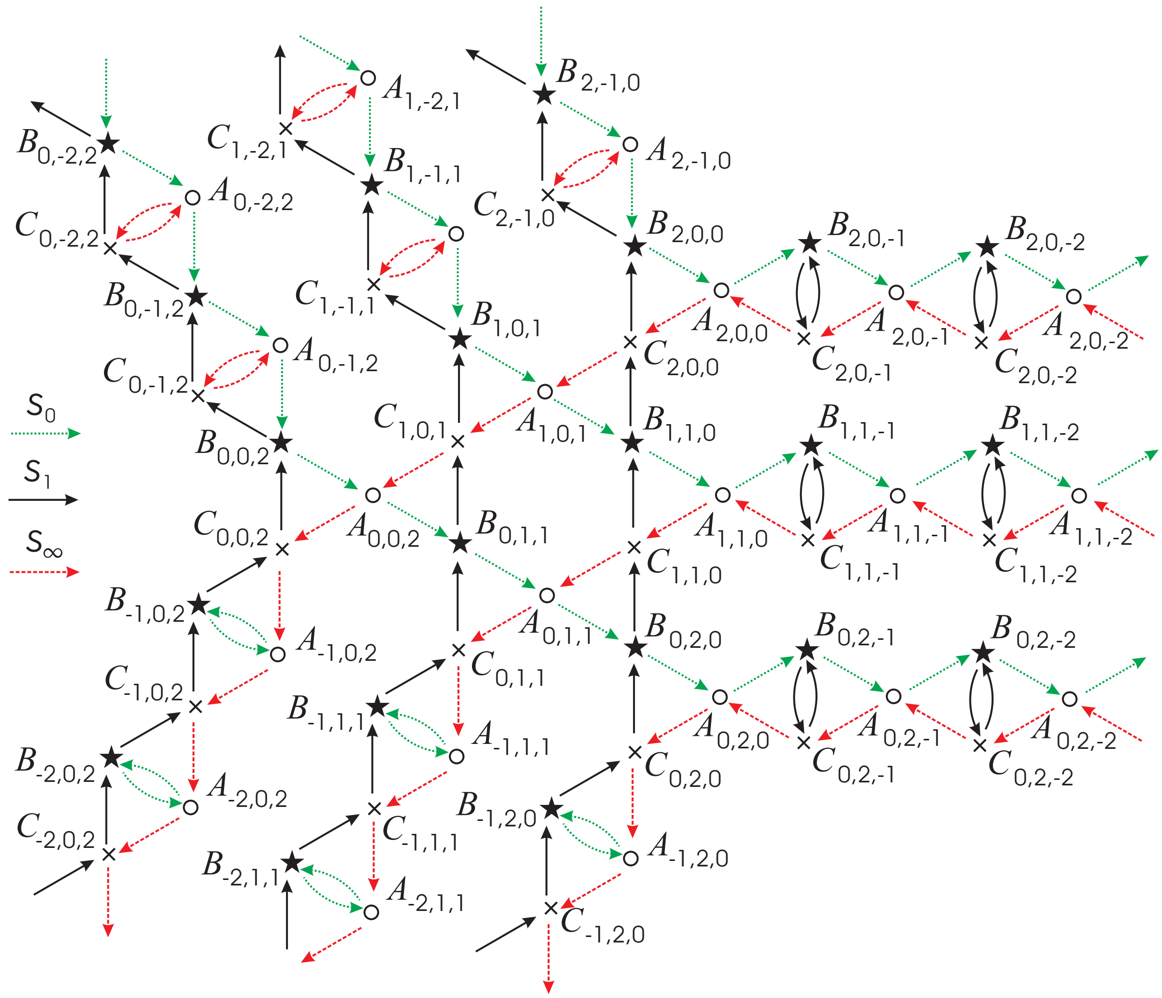


Fig Q4. Monodromy action for the QES quartic, $m = 2$

When $m \to \infty$ in (8), the quasi-exactly solvable quartics approximate the PT symmetric cubic (6), (7) in the same sense and the quasi-exactly solvable sextics (3), (4) approximate the quartic (1), (2).

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