PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 7, July 2010, Pages 2489–2492
S 0002-9939(10)10328-1
Article electronically published on February 12, 2010

TANGENCIES BETWEEN HOLOMORPHIC MAPS AND HOLOMORPHIC LAMINATIONS

A. EREMENKO AND A. GABRIELOV

(Communicated by Mario Bonk)

Abstract. We prove that the set of leaves of a holomorphic lamination of codimension one that are tangent to a germ of a holomorphic map is discrete.

Let F be a *holomorphic lamination* of codimension one in an open set V in a complex Banach space B. In this paper, this means that $V = W \times \mathbf{C}$, where W is a neighborhood of the origin in some Banach space, and the leaves L_λ of the lamination are disjoint graphs of holomorphic functions $\mathbf{w} \mapsto f(\lambda, \mathbf{w})$, $W \to \mathbf{C}$. For holomorphic functions in a Banach space we refer to [5]. Here λ is a parameter and we assume that the dependence of f on λ is continuous. A natural choice of this parameter is such that $\lambda = f(\lambda, 0)$, in which case the continuity with respect to λ follows from the so-called λ-lemma of Mane-Sullivan-Sad and Lyubich; see, for example, [5]. With this choice of the parameter, our definition of a lamination coincides with that of a holomorphic motion of $\mathbf{C}$ parametrized by W.

Let $\gamma : U \to V$ be a holomorphic map with $U \subset \mathbf{C}^n$. We say that γ is *tangent* to the lamination at a point $\mathbf{z}_0 \in U$ if the image of the derivative $\gamma'(\mathbf{z}_0)$ is contained in the tangent space $T_L(\gamma(\mathbf{z}_0))$, where L is the leaf passing through $\gamma(\mathbf{z}_0)$. A leaf for which this holds is called a *tangent leaf* to γ.

Theorem. *Let K be a compact subset of U. Then the set of leaves tangent to γ at the points of K is finite.*

For the case of holomorphic curves ($n = 1$) this result is contained in [1, Lemma 9.1] where it is credited to Douady. Artur Avila, in a conversation with the authors, proposed to extend this result to arbitrary holomorphic maps. According to Avila, this generalization has several applications to holomorphic dynamics [1].

Proof. We assume without loss of generality that

$$f(0, \mathbf{w}) \equiv 0$$

and that L_0 is tangent to γ at $\mathbf{z}_0 = 0$.

We have to show that other tangent leaves cannot accumulate to L_0. Suppose the contrary; that is, suppose that there is a sequence $\lambda_k \to 0$ such that L_{λ_k} are tangent to γ, and let L_{λ_k} be the graphs of the functions $f_k(\mathbf{w}) = f(\lambda_k, \mathbf{w})$. We may assume that tangency points $\mathbf{z}_k \to 0$.

Received by the editors October 25, 2008, and, in revised form, October 28, 2009.
2010 *Mathematics Subject Classification.* Primary 32B99; Secondary 37F99.
The first author was supported by NSF grant DMS-0555279.
The second author was supported by NSF grant DMS-0801050.

We make several preliminary reductions. □

1. Let

$$\gamma(\mathbf{z}) = (\phi(\mathbf{z}), \psi(\mathbf{z})) \in W \times \mathbf{C}.$$

Consider the new lamination in $U \times \mathbf{C}$ whose leaves are the graphs of $f^*(\lambda, \mathbf{z}) = f(\lambda, \phi(\mathbf{z}))$ and the new map $\gamma^*(\mathbf{z}) = (\mathbf{z}, \psi(\mathbf{z}))$. Then γ^* is tangent to a leaf L^* if and only if γ is tangent to L. This reduces our problem to the case where W is an open set in $\mathbf{C}^n$ and the map γ is a graph of a function ψ of the same variable as the functions f_k. From now on we assume that $U = W$ and $\gamma(\mathbf{w}) = (\mathbf{w}, \psi(\mathbf{w}))$.

2. Now we reduce the problem to the case where ψ is a monomial. For this we use the desingularization theorem of Hironaka [4, 2, 3, 6].

Let X be a complex analytic manifold, and let ψ be an analytic function on X. Then there exists a complex analytic manifold M and a proper surjective map $\pi : M \to X$ such that the restriction of π onto the complement of the π-preimage of the set $\{\psi = 0,\ \psi' = 0\}$ is injective and for each point $\mathbf{z}_0 \in \pi^{-1}(\{\psi = 0\})$ there is a local coordinate system with the origin at $\mathbf{z}_0$ such that $\psi \circ \pi$ is a monomial $z_1^{m_1} \dots z_n^{m_n}$. We choose such M corresponding to $X = W$.

Let $Y = W \times \mathbf{C}$, and let $S \subset Y$ be the set of points $(\mathbf{w}, t)$ with $t \neq 0$ such that the graph of $t = \psi(\mathbf{w})$ is tangent to the lamination. In our proof by contradiction, we assume that the origin belongs to the closure of S. Let $N = M \times \mathbf{C}$, and let $\rho : N \to Y$ be the map defined by $\rho(\mathbf{z}, t) = (\pi(\mathbf{z}), t)$. Then $\rho^{-1}(F)$ is the lamination whose leaves are the components of the ρ-preimages of the leaves of F, and the set $T = \rho^{-1}(S)$ has a limit point $(\mathbf{z}_0, 0)$ with $\pi(\mathbf{z}_0) = 0$ since π is proper. Also, the set T is exactly the set of those points in N where the graph $t = \psi \circ \pi(\mathbf{z})$ is tangent to the lamination $\rho^{-1}(F)$ since ρ is injective in a neighborhood of each point of T. (Any point $(\mathbf{z}, t)$ where ρ is not injective satisfies $\psi(\pi(\mathbf{z})) = 0$, while at every point of T we have $\psi(\pi(\mathbf{z})) \neq 0$.) This reduces our problem to the case where ψ is a monomial.

3. We may assume now that $W = \{\mathbf{z} : |\mathbf{z}| < 2\}$. Thus, we are in the situation

$$\psi(z) = z_1^{m_1} \dots z_n^{m_n},$$

and $\{f_k\}$ is a family of holomorphic functions on W with disjoint graphs, $f_k(\mathbf{z}) \neq 0$ for $\mathbf{z} \in W$, and $f_k \to 0$ as $k \to \infty$ uniformly on W. Moreover, for some sequence $\mathbf{z}_k \to 0$ we have

$$f_k(\mathbf{z}_k) = \prod_{j=1}^{n} z_{j,k}^{m_j}, \tag{1}$$

$$\text{grad } f_k(\mathbf{z}_k) = (m_1 z_{1,k}^{m_1-1} z_{2,k}^{m_2} \dots,\ m_2 z_1^{m_1} z_{2,k}^{m_2-1} \dots), \tag{2}$$

assuming zero values for the components with $m_j = 0$. We may assume that $|f_k| \leq 1$ in W. Setting $f_k = \exp g_k$ we obtain that $\Re g_k \leq 0$. Now we put

$$h_k(\mathbf{z}) = g_k(\mathbf{z} + \mathbf{z}_k) - g_k(\mathbf{z}_k).$$

Then the h_k are defined in the unit ball and satisfy

$$\Re h_k(\mathbf{z}) \leq \sum_{j:m_j>0} m_j \log |z_{j,k}|^{-1}, h_k(0) = 0.$$

From this we conclude that

$$\left|\frac{\partial h_k}{\partial z_j}(0)\right| \le 2 \sum_{j:m_j>0} m_j \log |z_{j,k}|^{-1}. \tag{3}$$

This follows from the following lemma.

Lemma. *Let h be an analytic function in the unit disc, $h(0) = 0$, and $\Re h \le A$, where $A > 0$. Then $|h'(0)| \le 2A$.*

Proof. The function

$$\phi(z) = \frac{z}{2Az - z}$$

maps the half-plane $\{\Re z < A\}$ onto the unit disc, $\phi(0) = 0$, and $\phi'(0) = 1/(2A)$. Now the statement follows from the Schwarz lemma applied to $\phi \circ h$ and the chain rule:

$$|(\phi \circ h)'(0)| = |\phi'(0)h'(0)| = |h'(0)/2A)| \le 1.$$

□

On the other hand, (1) and (2) imply that, for $m_j > 0$,

$$\left|\frac{\partial h_k}{\partial z_j}(0)\right| = \frac{m_j}{|z_{j,k}|}. \tag{4}$$

Assume without loss of generality that $m_1 > 0$ and

$$|z_{1,k}| = \min_{j:m_j>0} |z_{j,k}|.$$

Then the right-hand side of (3) is at most

$$\operatorname{const} \log |z_{1,k}|^{-1},$$

while the right-hand side of (4) for $j = 1$ is

$$\frac{m_1}{|z_{1,k}|}.$$

As $|z_{1,k}| \to 0$, we obtain a contradiction which proves our theorem.

Acknowledgment

We thank the referee for useful comments.

References

[1] A. Avila, M. Lyubich and W. de Melo, Regular or stochastic dynamics in real analytic families of unimodal maps, Invent. Math. 154 (2003), no. 3, 451–550. MR2018784 (2006i:37083)

[2] E. Bierstone and P.D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. 128 (1997), 207-302. MR1440306 (98e:14010)

[3] S. Encinas and O. Villamayor, A new proof of desingularization over fields of characteristic zero, Revista Matematica Iberoamericana 19 (2003), 339-353. MR2023188 (2004m:14017)

[4] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I, Ann. of Math. (2) 79 (1964), 109-203; II, ibid. (2) 79 (1964), 205-326. MR0199184 (33:7333)

[5] J. Hubbard, Teichmüller theory and applications to geometry, topology, and dynamics, Vol. 1, Matrix Editions, Ithaca, NY, 2006. MR2245223 (2008k:30055)

[6] J. Wlodarczyk, Simple Hironaka resolution in characteristic zero, J. Amer. Math. Soc. 18 (4) (2005), 779-822. MR2163383 (2006f:14014)

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907
E-mail address: eremenko@math.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907
E-mail address: agabriel@math.purdue.edu