

SOME LOWER BOUNDS IN THE B. AND M. SHAPIRO CONJECTURE FOR FLAG VARIETIES

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ABSTRACT. The B. and M. Shapiro conjecture stated that all solutions of the Schubert Calculus problems associated with real points on the rational normal curve should be real. For Grassmannians, it was proved by Mukhin, Tarasov and Varchenko. For flag varieties, Sottile found a counterexample and suggested that all solutions should be real under certain monotonicity conditions. In this paper, we compute lower bounds on the number of real solutions for some special cases of the B. and M. Shapiro conjecture for flag varieties, when Sottile's monotonicity conditions are not satisfied.

1. INTRODUCTION

Schubert Calculus is a recipe for counting geometric objects subject to certain incidence relations [17]. For example,

Problem 1.1. *Given $2d - 2$ lines in general position in $\mathbb{C}\mathbb{P}^d$. How many codimension 2 subspaces of $\mathbb{C}\mathbb{P}^d$ meet all $2d - 2$ lines?*

The answer is $u_d = \frac{1}{d} \binom{2d-2}{d-1}$, the d -th Catalan number. Schubert Calculus was based on the semi-empirical principle of conservation of number. Its rigorous foundation, subject of Hilbert's 15-th problem, was established through the development of intersection theory. From the point of view of intersection theory, enumerative problems such as Problem 1.1 are solved by counting intersection multiplicities of Schubert varieties in the Grassmannian (see, e.g., [1]).

Over the reals, the principle of conservation of number fails, and solving enumerative problems becomes considerably more complicated. In [7], W. Fulton says "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."

For example, suppose that all $2d - 2$ lines in Problem 1.1 are real. How many codimension 2 subspaces that meet all these lines are real?

Sottile [23] proved that all of them can be real for some choice of the $2d - 2$ lines. Boris and Michael Shapiro conjectured that all these subspaces are real whenever the given lines are tangent to the rational normal curve $\gamma(t) = [1 : t : t^2 : \dots : t^d]$ at distinct real points. Eremenko and Gabrielov [4] (see also [5]) proved the following equivalent theorem.

Theorem 1.2. *If the critical points of a rational function f are all real then f is equivalent to a real rational function, i.e., there exists a fractional linear transformation l such that $l \circ f$ is a real rational function.*

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A more general version of the B. and M. Shapiro conjecture, proved in [12]¹, claims a similar result for higher dimensional subspaces.

Theorem 1.3. *Let $P_1, P_2, \dots, P_{k(d-k+1)}$ be $(k-1)$ -dimensional planes in $\mathbb{C}\mathbb{P}^d$ osculating the rational normal curve at distinct real points. Any $(d-k)$ -dimensional plane that meets all P_i , $1 \leq i \leq k(d-k+1)$, must be real.*

B. and M. Shapiro suggested an extension of their conjecture to flag varieties, replacing osculating planes by osculating flags. A special case of that conjecture would imply that all solutions to the following problem are real.

Problem 1.4. *Let $y_1, y_2, \dots, y_{2d-3}, r, s$ be distinct points in \mathbb{R} . For $1 \leq i \leq 2d-3$, let T_i be the line tangent to γ at $\gamma(y_i)$ and let T_{2d-2} be the line through the points $\gamma(r)$ and $\gamma(s)$. Among all the codimension 2 subspaces of $\mathbb{C}\mathbb{P}^d$ that meet all the lines $T_1, T_2, \dots, T_{2d-2}$, how many are real?*

Sottile [20] found examples when the B. and M. Shapiro conjecture for flag varieties fails, in particular, Problem 1.4 does have non-real solutions. Computer experiments suggested that the conjecture might hold whenever a certain monotonicity condition is met [16]. For the case of Problem 1.4, Sottile's monotonicity condition simply means that the interval I with endpoints r and s contains either all or none of the points $y_1, y_2, \dots, y_{2d-3}$. A proof of this special case was given in [6]. For a complete survey of the B. and M. Shapiro conjecture and its modifications, see [21] or [22].

In this paper we study Problem 1.4 when the monotonicity condition does not hold, i.e., when the interval I contains k of the points $y_1, y_2, \dots, y_{2d-3}$ for some k with $1 \leq k < 2d-3$. We give an algorithm to compute a (non-strict) lower bound² for the number of real subspaces for any pair of integers d and k with $1 \leq k < 2d-3$. We do this by giving a lower bound for the number of solutions to the following equivalent problem.

Problem 1.5. *Let $y_1, y_2, \dots, y_{2d-3}, r, s$ be distinct points in \mathbb{R} and suppose that the interval I having endpoints r and s contains k of the points $y_1, y_2, \dots, y_{2d-3}$ for some $k \in \{1, 2, \dots, 2d-4\}$. How many equivalence classes of real rational functions f of degree d having critical points at $y_1, y_2, \dots, y_{2d-3}$ and satisfying $f(r) = f(s)$ are there?*

Remark 1.6. *The answer to Problem 1.5 does not change if we replace k by $2d-3-k$.*

Lower bounds have been computed for several other real enumerative problems. Eremenko and Gabrielov [3] showed that the number of real solutions to certain systems of real polynomial equations is bounded below by the topological degree of the corresponding Wronski map, and computed this degree. Soprunova and Sottile [19] extended the definition of the real Wronski map to certain sparse polynomial systems associated with partially ordered sets, computed its degree and derived the corresponding lower bounds on the number of real solutions. Degtyarev and Kharlamov [2] showed that there are at least 8 real rational cubics passing through 8 general real points in the plane. Welschinger [26] discovered an invariant that gives a lower bound for the number of real rational curves of degree d passing

¹This result dates back to 2005.

²The lower bound is strict for $d=4, 5$ and 6 .

through $3d - 1$ real points. Itenberg, Kharlamov and Shustin [10] showed that this invariant is at least $d!/2$ using Mikhalkin's tropical correspondence theorem [11]. Welschinger discovered other invariants for similar enumerative problems (see [25] and the references there), some of which have been computed by Pandharipande, Solomon and Walcher [13]. Solomon [18] showed how the Welschinger invariant can be interpreted as the degree of a map. An excellent review of these results can be found in [21, 22].

While the previous results are based on topological degree invariants, our method uses a different line of reasoning that does not seem to be associated with a topological degree. The lower bound for the number of real solutions of Problem 1.5 is obtained by studying a one-parameter family of real rational functions with critical points $y_1, \dots, y_{2d-3}, \theta$, and the dependence on θ of the increments on $[r, s]$ of the functions in that family.

2. THE WRONSKI MAP AND ONE-PARAMETER FAMILIES OF RATIONAL FUNCTIONS.

In this section, we represent the points of the Grassmannian G of 2-dimensional planes in the space of polynomials of degree d by pencils (p, q) of linearly independent polynomials, define the Wronski map $\mathcal{W} : G \rightarrow \mathbb{C}\mathbb{P}^{2d-2}$ and prove that \mathcal{W} is not ramified over the space Q of polynomials with all real roots of multiplicity at most 2. We study the properties of a one-parameter family $f_t = p_t/q_t$ of rational functions obtained by lifting to G a path $t \mapsto w_{\theta(t)}$ in Q . Here $\theta(t)$ is a path in $\mathbb{R}\mathbb{P}^1$ and w_θ is a polynomial with roots $y_1, \dots, y_{2d-3}, \theta$. Our main tool is a net of a real rational function with real critical points (see [4], [5]).

Let G be the Grassmannian of 2-dimensional planes in the space of complex polynomials of degree at most d . An element of G can be defined by a pencil $L = L(p, q) = \{\alpha p + \beta q\}$ where p and q are two linearly independent polynomials of degree at most d and $[\alpha : \beta] \in \mathbb{C}\mathbb{P}^1$. If $L \in G_{\mathbb{R}}$, the Grassmannian of real two-dimensional planes, we always assume that p and q are real and call L a real pencil. The Wronskian of (p, q) , given by $W(p, q) = p'q - q'p$, is a nonzero polynomial of degree at most $2d - 2$. If the degree of $W(p, q)$ is $2d - 2 - k$ for some $k > 0$, we say that $W(p, q)$ has a root at infinity of multiplicity k . If (p_1, q_1) is another basis for L , then $W(p_1, q_1)$ differs from $W(p, q)$ by a nonzero multiplicative constant. For a nonzero polynomial $r = \sum a_i z^i$ of degree at most $2d - 2$, let $h(r) = [a_0 : a_1 : \dots : a_{2d-2}]$.

Definition 2.1. *The Wronski map $\mathcal{W} : G \rightarrow \mathbb{C}\mathbb{P}^{2d-2}$ is defined by $\mathcal{W}(L) = h(W(p, q))$, where (p, q) is a basis for L .*

The map \mathcal{W} is well-defined and finite. Its degree is $u(d)$, the d -th Catalan number. This was originally computed by Schubert in 1886. A proof can be found in [8] or [9].

Definition 2.2. *Let $Q = h(Q_0)$, where Q_0 is the set of all polynomials r satisfying*

- (i) $2d - 4 \leq \deg(r) \leq 2d - 2$,
- (ii) *all roots of r belong to $\mathbb{R}\mathbb{P}^1$, and*
- (iii) *all roots of r have multiplicity at most 2.*

In Theorem 2.13, we shall show that \mathcal{W} is unramified over Q and hence, $\mathcal{W}|_{\mathcal{W}^{-1}(Q)}$ is a covering map. Thus, for any path ρ in Q with initial point ρ_0 , given $L_0 \in G$ with $\mathcal{W}(L_0) = \rho_0$, ρ can be lifted in a unique way to a path in G starting at L_0 .

Definition 2.3. Let $D = \{y_1, y_2, \dots, y_{2d-3}\}$ be a set of points in \mathbb{R} . For $y \in \mathbb{R}$, let $\omega_y = (x - y_1)(x - y_2) \dots (x - y_{2d-3})(x - y)$, and let $\omega_\infty = (x - y_1)(x - y_2) \dots (x - y_{2d-3})$.

The map $\mathbb{R}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^{2d-2}$ given by $y \rightarrow h(w_y)$ is continuous. In fact, it extends to an algebraic map $\mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^{2d-2}$. Let \mathcal{I} be an interval in \mathbb{R} and let $t_0 \in \mathcal{I}$. A path $\theta : \mathcal{I} \rightarrow \mathbb{R}\mathbb{P}^1$ induces a path $\rho : \mathcal{I} \rightarrow Q$ given by $\rho(t) = h(\omega_{\theta(t)})$. Given $L_0 \in G$ with $\mathcal{W}(L_0) = h(\omega_{\theta(t_0)})$, ρ can be lifted to a unique path χ in G satisfying $\chi(t_0) = L_0$. It was shown in [4] that $\mathcal{W}^{-1}(\mathbb{R}\mathbb{P}^{2d-2}) \subset G_{\mathbb{R}}$. This implies that $\chi(\mathcal{I})$ is contained in $G_{\mathbb{R}}$.

Lemma 2.4. Let \mathcal{R}_d be the space of real rational functions of degree at most d all of whose critical points are real. If θ is analytic, all the lifts defined above are analytic, and polynomials p_t and q_t with coefficients analytic in t can be selected to form the bases of $\chi(t)$, $t \in \mathcal{I}$. Let f_t be the real rational function p_t/q_t viewed as a holomorphic function from $\mathbb{C}\mathbb{P}^1$ to $\mathbb{C}\mathbb{P}^1$. The path $\eta : \mathcal{I} \rightarrow \mathcal{R}_d$ given by $t \rightarrow f_t$ has the following properties.

- (i) The Wronskian $W(p_t, q_t)$ is equal to $\rho(t)$.
- (ii) If $x \in \mathbb{C}\mathbb{P}^1 \setminus D$ then the map $t \xrightarrow{\phi_x} f_t(x)$ is continuous.
- (iii) For each $x \in D$, the map $t \xrightarrow{\phi_x} f_t(x)$ is continuous at all values t for which p_t and q_t do not have a common root at x .

Proof. The first property follows from the construction of f_t . To prove (ii) and (iii), first observe that if $x \in \mathbb{C}\mathbb{P}^1 \setminus D$ then, for any $t \in \mathcal{I}$, $p_t(x)$ and $q_t(x)$ cannot both be zero. Let $(t_0, x) \in \mathcal{I} \times \mathbb{C}\mathbb{P}^1$ be such that at least one of $p_{t_0}(x)$ and $q_{t_0}(x)$ is not equal to 0. This implies that there exists a neighborhood N of t_0 such that either $f_t(x)$ or $1/f_t(x)$ is a continuous complex-valued function on N and hence, $f_t(x) : N \rightarrow \mathbb{C}\mathbb{P}^1$ is continuous. \square

For pairs (t, x) such that $p_t(x) = q_t(x) = 0$ we shall see in the following lemma that for small $\epsilon > 0$, $f_{t+\epsilon}$ is monotone on $(x, x + \epsilon)$ and the image of $(x, x + \epsilon)$ covers $\mathbb{R}\mathbb{P}^1$ except for an interval of length $O(\epsilon)$.

Lemma 2.5. Suppose that $y_1 = 0$ and that $\{(p_\epsilon, q_\epsilon) : \epsilon \geq 0\}$ is a family of ordered pairs of linearly independent real polynomials of degree at most d satisfying:

- (i) the coefficients of p_ϵ and q_ϵ are analytic in ϵ ,
- (ii) the roots of $w_\epsilon = W(p_\epsilon, q_\epsilon)$ are $y_1 = 0, y_2, \dots, y_{2d-3}, \epsilon$,
- (iii) p_0 and q_0 have a common root at $x = 0$, and
- (iv) $f_0(0) = 0$.

Then for small $\epsilon > 0$, $f_\epsilon(0) = O(\epsilon)$, $f_\epsilon(\epsilon) = O(\epsilon)$ and f_ϵ has a pole in $(0, \epsilon)$. Moreover, $f_\epsilon : [0, \epsilon] \rightarrow \mathbb{R}\mathbb{P}^1$ is one-to-one and its image is $\mathbb{R}\mathbb{P}^1 \setminus (\alpha, \beta)$ where (α, β) is an open interval of size $O(\epsilon)$ with endpoints at $f_\epsilon(0)$ and $f_\epsilon(\epsilon)$.

Proof. By our assumption, p_0 and q_0 have a common root at $x = 0$, and $f_0(0) = 0$, so there exist $k_1 \neq 0$ and $k_2 \neq 0$ such that $p_0 = k_1 x^2 + O(x^3)$ and $q_0 = k_2 x + O(x^2)$. The requirement that f_ϵ has a fixed critical point at $x = 0$ for all $\epsilon > 0$ implies that $p_\epsilon(x) = k_1(x^2 + a\epsilon x + ab\epsilon^2) + O(x^3, \epsilon x^2, \epsilon^2 x, \epsilon^3)$ and $q_\epsilon(x) = k_2(x + b\epsilon) + O(x^2, \epsilon x, \epsilon^2)$. Therefore, $w_\epsilon(x) = k_1 k_2(x^2 + 2b\epsilon x) + O(x^3, \epsilon x^2, \epsilon^2 x, \epsilon^3)$. In particular, $w_\epsilon(\epsilon) = (1 + 2b)k_1 k_2 \epsilon^2 + O(\epsilon^3)$.

Since for all $\epsilon > 0$, f_ϵ has a critical point at $x = \epsilon$, we should have $b = -\frac{1}{2}$. Hence q_ϵ has a root at $x_\epsilon = \frac{1}{2}\epsilon + O(\epsilon^2)$, between the critical points 0 and ϵ of f_ϵ , and

$p_\epsilon(x_\epsilon) \neq 0$. Since $p_\epsilon(0) = O(\epsilon^2)$ and $p_\epsilon(\epsilon) = O(\epsilon^2)$, we have $f_\epsilon(0) = O(\epsilon)$ and $f_\epsilon(\epsilon) = O(\epsilon)$.

For small enough $\epsilon > 0$, the interval $(0, \epsilon)$ does not contain any points of D . If $f'_\epsilon < 0$ on $(0, \epsilon)$ then when x moves from 0 to ϵ , the value of $f_\epsilon(x)$ decreases from $O(\epsilon)$, tends to $-\infty$ as x approaches x_ϵ , and returns from ∞ to $O(\epsilon)$ at $x = \epsilon$. If $f'_\epsilon > 0$ on $(0, \epsilon)$ then when x moves from 0 to ϵ , the value of $f_\epsilon(x)$ increases from $O(\epsilon)$, tends to ∞ as x approaches x_ϵ , and returns from $-\infty$ to $O(\epsilon)$ at $x = \epsilon$. \square

Suppose $t \rightarrow g_t = p_t^*/q_t^*$ is another path satisfying $L(p_t^*, q_t^*) = \chi(t) = L(p_t, q_t)$. There exists a real fractional linear transformation $h_t = (a_t x + b_t)/(c_t x + d_t)$ such that $g_t = h_t \circ f_t$. In particular, for each t , f_t and g_t have the same critical points. Observe that continuity of p_t, q_t, p_t^*, q_t^* implies that the sign of $a_t d_t - b_t c_t$ does not change as t varies in \mathcal{I} . Post-composition with a real fractional linear transformation defines an equivalence relation on \mathcal{R}_d . This proves the following.

Theorem 2.6. *With the above notation, let $f_{t_0} \in \mathcal{R}_d$ be a function whose Wronskian is equal to $\rho(t_0)$. Let \mathcal{R}_d^0 be the family of equivalence classes of functions in \mathcal{R}_d of degree exactly d and let $[f_{t_0}] \in \mathcal{R}_d^0$ be the equivalence class of f_{t_0} . There exists a unique map $\hat{\eta}: \mathcal{I} \rightarrow \mathcal{R}_d^0$ with $\hat{\eta}(t_0) = [f_{t_0}]$ that projects to ρ .*

The set \mathcal{R}_d^0 is in one-to-one correspondence with the set of nondegenerate real pencils. A pencil $L(p, q) \in G$ is nondegenerate if $\deg(p/q) = d$, in other words, if $\max(\deg p, \deg q) = d$ and p and q have no common zeros. Otherwise L is said to be degenerate.

Notation 2.7. *We shall identify $\mathbb{R}\mathbb{P}^1$ with the unit circle \mathbf{S}^1 via a fractional linear transformation $\vartheta: \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ mapping $\mathbb{R}\mathbb{P}^1$ to \mathbf{S}^1 , 0 to 1 and preserving orientation. A real rational function f will be identified with the function $\tilde{f} = \vartheta \circ f \circ \vartheta^{-1}$. The critical points of f are the images of the critical points of \tilde{f} under ϑ . In particular, all critical points of \tilde{f} belong to the unit circle. Moreover, for any two points a and b in $\mathbb{R}\mathbb{P}^1$, $f(a) = f(b)$ if and only if $\tilde{f}(\vartheta(a)) = \tilde{f}(\vartheta(b))$.*

Let $\mathcal{S}_d = \{\tilde{f} = \vartheta \circ f \circ \vartheta^{-1} : f \in \mathcal{R}_d\}$. We shall write f instead of \tilde{f} and specify whether we are considering f as a function in \mathcal{R}_d or \mathcal{S}_d . With this identification, any construction or proof given for \mathcal{R}_d also applies to \mathcal{S}_d and vice versa.

Definition 2.8. *Let $L = L(p, q)$ be a real pencil whose Wronskian has $2d - 2$ real roots counted with multiplicity and let $f = p/q$. The set $\Gamma = f^{-1}(\mathbb{R}\mathbb{P}^1)$ is called the net of f [4]. Alternatively, let $\tilde{f} = \vartheta \circ f \circ \vartheta^{-1}$. Then $\Gamma = \tilde{f}^{-1}(\mathbf{S}^1)$.*

Observe that Γ is independent of the choice of basis (p, q) , so we can also refer to Γ as the net of L . Unless explicitly stated otherwise, we shall consider Γ to be defined as $\tilde{f}^{-1}(\mathbf{S}^1)$ rather than $f^{-1}(\mathbb{R}\mathbb{P}^1)$. Clearly, $\mathbf{S}^1 \subset \Gamma$ and Γ is invariant under the reflection $s(z) = 1/\bar{z}$ with respect to the unit circle.

The net Γ defines a cell decomposition of $\mathbb{C}\mathbb{P}^1$ as follows:

- 1) the vertices (0-cells) of the cell decomposition correspond to the zeros of $W(p, q)$,
- 2) the edges (1-cells) are the components of $\Gamma \setminus V$ where V is the set of vertices,
- 3) the faces (2-cells) are the components of $\mathbb{C} \setminus \Gamma$.

This cell decomposition has the following properties. The closure of each cell is homeomorphic to a closed ball of the corresponding dimension. If $W(p, q)$ has distinct real roots, then each vertex of Γ has degree 4. If $W(p, q)$ has multiple roots we call Γ a degenerate net.

Remark 2.9. Let $L(p, q)$ be a real pencil whose net Γ is degenerate and whose Wronskian has only simple or double roots, all real. A vertex of Γ of degree 2 corresponds to a point x such that $p(x) = q(x) = 0$ while a vertex of degree 6 corresponds to a double root x of $W(p, q)$ such that at least one of $p(x)$ and $q(x)$ is not 0.

Definition 2.10. The edges of Γ that lie inside the unit disc are called chords or interior edges of Γ . Since Γ is symmetric with respect to \mathbf{S}^1 , it suffices to consider its chords. Given a vertex v , the pair $\gamma = (\Gamma, v)$ is called the net of f with respect to v , or simply the net of f if v is clear. The vertex v is called the distinguished vertex of γ .

The standard orientation of \mathbf{S}^1 induces a cyclic order \prec on the vertices of Γ . We shall label these vertices $v_1, v_2, \dots, v_{2d-2}$ so that $v_1 \prec v_2 \prec \dots \prec v_{2d-2} \prec v_1$ and we shall set $v_{2d-1} = v_1$. For a net $\gamma = (\Gamma, v_1)$, we define a linear order on the vertices by $v_1 < v_2 < \dots < v_{2d-2}$.

Two nets (Γ, v_1) and (Γ', v'_1) are said to be equivalent if there exists a homeomorphism $\mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ mapping Γ to Γ' , v_1 to v'_1 , preserving orientation of both $\mathbb{C}\mathbb{P}^1$ and \mathbf{S}^1 and commuting with the reflection s . By abuse of notation, we shall often refer to an equivalence class of nets by one of its representatives.

Definition 2.11. For a net $\gamma = (\Gamma, v)$, let $\text{Shift}(\gamma) = (\Gamma, v')$ where v' is the predecessor of v under \prec .

Remark 2.12. Without loss of generality, we may assume that the vertices of Γ are equally spaced on the circle. The net $\text{Shift}(\gamma)$ is equivalent to the net (Γ', v) where the ordered vertex set of Γ' coincides with that of Γ and the chords of Γ' are obtained by rotating the chords of Γ by $\pi/(d-1)$ counterclockwise.

Theorem 2.13. The Wronski map $\mathcal{W} : G \rightarrow \mathbb{C}\mathbb{P}^{2d-2}$ is unramified over the space Q of all polynomials with real roots of multiplicity at most 2.

Proof. Let $\pi_0 \in Q$, $L(p_0, q_0) \in \mathcal{W}^{-1}(\pi_0)$ and $f_0 = p_0/q_0$. The function f_0 can be chosen real [4]. From the definition of Q , the common roots r_1, \dots, r_m of p_0 and q_0 (if any) must be simple. Suppose that \mathcal{W} is ramified at $L(p_0, q_0)$, with ramification index ν . Let π_j be a sequence of polynomials with distinct real roots converging to π_0 as $j \rightarrow \infty$. Since the polynomials with distinct real roots form an open set in the space of all real polynomials, we can assume that \mathcal{W} is not ramified over π_j for all $j \geq 1$. Then $\mathcal{W}^{-1}(\pi_j)$ contains ν distinct points $L(p_{j,k}, q_{j,k})$ of the Grassmannian converging to $L(p_0, q_0)$ as $j \rightarrow \infty$. Theorem 1.2 implies that all pencils $L(p_{j,k}, q_{j,k})$ are real. With the proper normalization, we can assume that $p_{j,k}$ and $q_{j,k}$ are real polynomials, and that $(p_{j,k}, q_{j,k}) \rightarrow (p_0, q_0)$ as $j \rightarrow \infty$ th all real critical points, converge to f_0 as $j \rightarrow \infty$ uniformly on every compact set not containing the points r_1, \dots, r_m , and their nets converge to the (degenerate) net of f_0 . Since the roots of π_0 are at most double, the net of f_0 may have vertices of degree either 6 or 2. In both cases, the net of $f_{j,k}$ is uniquely determined by the net of f_0 for large enough j , independent of k . It follows from [4] that, for large enough j , pencils $L(p_{j,k}, q_{j,k})$ do not depend on k . Hence $\nu = 1$ and \mathcal{W} is not ramified over π_0 . \square

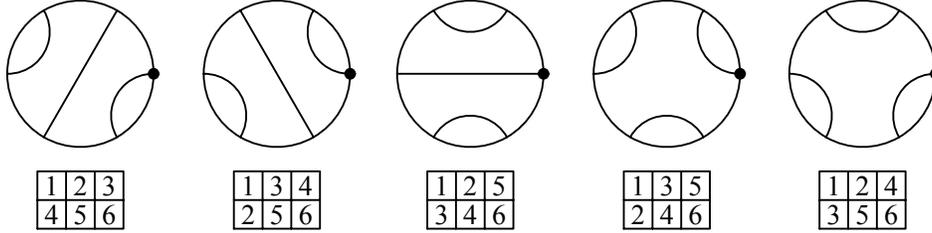
3. PROPERTIES OF THE NETS OF FUNCTIONS f_t .

In this section, we study dependence on the parameter t of the net γ_t of the rational function f_t defined in Section 2. In particular, we show that the net γ_t is shifted counterclockwise when $\theta(t)$ passes its first vertex y_1 . We describe also degenerate nets corresponding to polynomials w_θ with double roots.

Definition 3.1. A $2 \times (d-1)$ Young tableau is the distribution of the integers $1, 2, \dots, 2d-2$ on a $2 \times (d-1)$ rectangular array such that every integer is greater than the one above it and the one to its left, if any.

Equivalence classes of nets with $2d-2$ vertices can be identified with $2 \times (d-1)$ Young tableaux: to a net $\gamma = (\Gamma, v_1)$ with the ordered vertex set $\{v_1, v_2, \dots, v_{2d-2}\}$, we associate the tableau with an integer i belonging to the first row if and only if v_i is connected by a chord to a vertex v_j with $j > i$. In particular, the number of equivalence classes of nets having $2d-2$ vertices is the d -th Catalan number $u_d = \frac{1}{d} \binom{2d-2}{d-1}$ (see [24]).

Example 3.2. For $d = 4$, there are $u_4 = 5$ equivalence classes of nets with 6 vertices. These equivalence classes (with the distinguished vertex $v_1 = 1$) and their respective Young tableaux are shown in Fig. 1.


 FIGURE 1. Nets and their Young tableaux for $d = 4$.

Let $D = \{y_1, y_2, \dots, y_{2d-3}\}$ be an ordered set of distinct points in \mathbf{S}^1 all different from 1. Let $\theta : [0, 2d-2] \rightarrow \mathbf{S}^1$ be the path given by $\theta(t) = e^{2\pi it}$. By Lemma 2.4 and Notation 2.7, there exists a family $\{f_t \in \mathcal{S}_d : t \in [0, 2d-2]\}$ with the following properties:

- i) for each t , the critical points of f_t counted with multiplicity are $y_1, y_2, \dots, y_{2d-3}$ and $\theta(t)$;
- ii) given $(t_0, z_0) \in [0, 2d-2] \times \mathbb{C}\mathbb{P}^1$, $f_t(z_0)$ as a function of t is continuous at t_0 whenever $\deg f_{t_0} = d$. In particular, for any $z_0 \notin D$, $f_t(z_0)$ is a continuous function of t .

For $t \in [0, 2d-2]$, let $\gamma_t = (\Gamma_t, y_1)$ be the net of f_t with respect to y_1 . Let $\theta^{-1}(D) = \{t_1, t_2, \dots, t_{\kappa-1}\}$ with $0 < t_1 < t_2 < \dots < t_{\kappa-1} < 2d-2$ and let $t_0 = 0$ and $t_\kappa = 2d-2$. Here $\kappa - 1 = (2d-2)(2d-3)$ since θ makes $2d-2$ turns around \mathbf{S}^1 as t goes from 0 to $2d-2$. The sequence $\{s_1, s_2, \dots, s_{2d-2}\} = \theta^{-1}(y_1)$ is a subsequence of $\{t_i\}$. Let $s_0 = 0$ and $s_{2d-1} = 2d-2$.

Lemma 3.3. Let $l \in \{0, 1, \dots, 2d-2\}$. For $t \in (s_l, s_{l+1}) \setminus \theta^{-1}(D)$, the nets γ_t are equivalent.

Proof. Let a and b be two points in $(s_l, s_{l+1}) \setminus \theta^{-1}(D)$ with $a < b$. If $\theta([a, b]) \cap D = \emptyset$, then for each $z \in \mathbb{CP}^1$, $f_t(z)$ is a continuous function of t on $[a, b]$ and, for all $t \in [a, b]$ the nets γ_t are nondegenerate. Therefore $v_i v_j$ is a chord in γ_a if and only if it is also a chord in γ_b , and hence the two nets are equivalent.

If $\theta([a, b]) \cap D = \{y_i\}$, $i \neq 1$, then $\theta(a) = v_i$ and $y_i = v_{i+1}$ in the ordered vertex set of γ_a while $y_i = v_i$ and $\theta(a) = v_{i+1}$ in that of γ_b . For each $z \in \mathbb{CP}^1 \setminus \{y_i\}$, $f_t(z)$ is a continuous function of t on $[a, b]$ and, for all $t \in [a, b] \setminus \theta^{-1}(y_i)$ the nets γ_t are nondegenerate. Thus, for all $m, n \notin \{i, i+1\}$, $v_m v_n$ is a chord in γ_a if and only if it is also a chord in γ_b . We have to consider the following two cases.

Case 1: $v_i v_{i+1}$ is a chord in γ_a . In this case, every other chord $v_m v_n$ in γ_a satisfies $m, n \notin \{i, i+1\}$ and hence must also be a chord in γ_b . Therefore, $v_i v_{i+1}$ must be a chord in γ_b . An example of this case is given in Fig. 2a-c.

Case 2: $v_i v_{i+1}$ is not a chord in γ_a . Let y_j be the endpoint of the chord from v_i and y_k the endpoint of that from v_{i+1} . Continuity of $f_t(z)$ on $[a, b]$ for any $z \neq y_i$ implies that $y_j y_k$ is not a chord in γ_b . Determining Γ_b is thus reduced to finding a net with the vertex set $\{y_j \prec v_i \prec v_{i+1} \prec y_k\}$ such that y_j and y_k are not endpoints of the same chord. There are only two nets having four vertices, and exactly one of them does not have a chord joining y_j to y_k . For an example, see Fig. 2c-e. \square

Remark 3.4. *A similar argument can be used to determine the degenerate net Γ_{t_i} . If, for t close to t_i , there is a chord of Γ_t connecting $\theta(t)$ and $\theta(t_i)$ (see, for example Fig. 2a,c), then the vertex $\theta(t_i)$ is of degree 2 in Γ_{t_i} (Fig. 2b). Otherwise, $\theta(t_i)$ is of degree 6 in Γ_{t_i} (Fig. 2d).*

Lemma 3.5. *Let $[\gamma]_l = ([\Gamma]_l, y_1)$ denote the class $\{\gamma_t : t \in (s_l, s_{l+1}) \setminus \theta^{-1}(D)\}$. Then $[\gamma]_{l+1} = \text{Shift}([\gamma]_l)$.*

Proof. Choose representatives $\gamma_{t'}$ and $\gamma_{t''}$ of $[\gamma]_l$ and $[\gamma]_{l+1}$ respectively with $y_{2d-2} \prec \theta(t') \prec y_1 \prec \theta(t'') \prec y_2$. As t goes from t' to t'' , $\theta(t)$ crosses the distinguished vertex y_1 , so Lemma 3.3 cannot be applied to $\gamma_{t'}$ and $\gamma_{t''}$. Instead we shall consider $\gamma' = (\Gamma_{t'}, y_2)$ and $\gamma'' = (\Gamma_{t''}, y_2)$ which are related to $\gamma_{t'}$ and $\gamma_{t''}$ by $\gamma_{t'} = \text{Shift}(\gamma')$ and $\gamma_{t''} = \text{Shift}^2(\gamma'')$. Since $\theta(t)$ does not cross y_2 as t goes from t' to t'' , we can apply Lemma 3.3 to γ' and γ'' to get that $\gamma' = \gamma''$. Thus $\gamma_{t'} = \text{Shift}^2(\gamma'') = \text{Shift}^2(\gamma') = \text{Shift}(\gamma_{t'})$. \square

Example 3.6. *The nets a, c, e, g in Fig. 2 all belong to $[\gamma]_0$ while the net k belongs to $[\gamma]_1$ and is the shift of $[\gamma]_0$.*

An anonymous referee pointed out the correspondence between the (inverse of) the Shift operator and the Schutzenberger promotion p . If $\gamma' = \text{Shift}(\gamma)$ and Y and Y' are the Young tableaux corresponding to γ and γ' respectively, then $p(Y') = Y$. This correspondence is described in [14] where it is attributed to White. Furthermore, this correspondence, as well as lemma 3.3, follow from Remark 3.7 in [15].

Corollary 3.7. *If $t' \in (t_0, t_1)$ and $t'' \in (t_{\kappa-1}, t_\kappa)$ then $\gamma_{t'} = \gamma_{t''}$. Equivalently, $[\gamma]_0 = [\gamma]_{2d-2}$.*

Proof. As t goes from t' to t'' , $\theta(t)$ makes $2d - 2$ turns around the circle. By Lemma 3.5, each complete turn of θ corresponds to the shift operator (i.e., rotation by $\pi/(d-1)$) applied to the net. Applying the shift operator $2d - 2$ times results in a complete rotation of the net, resulting in the original net. \square

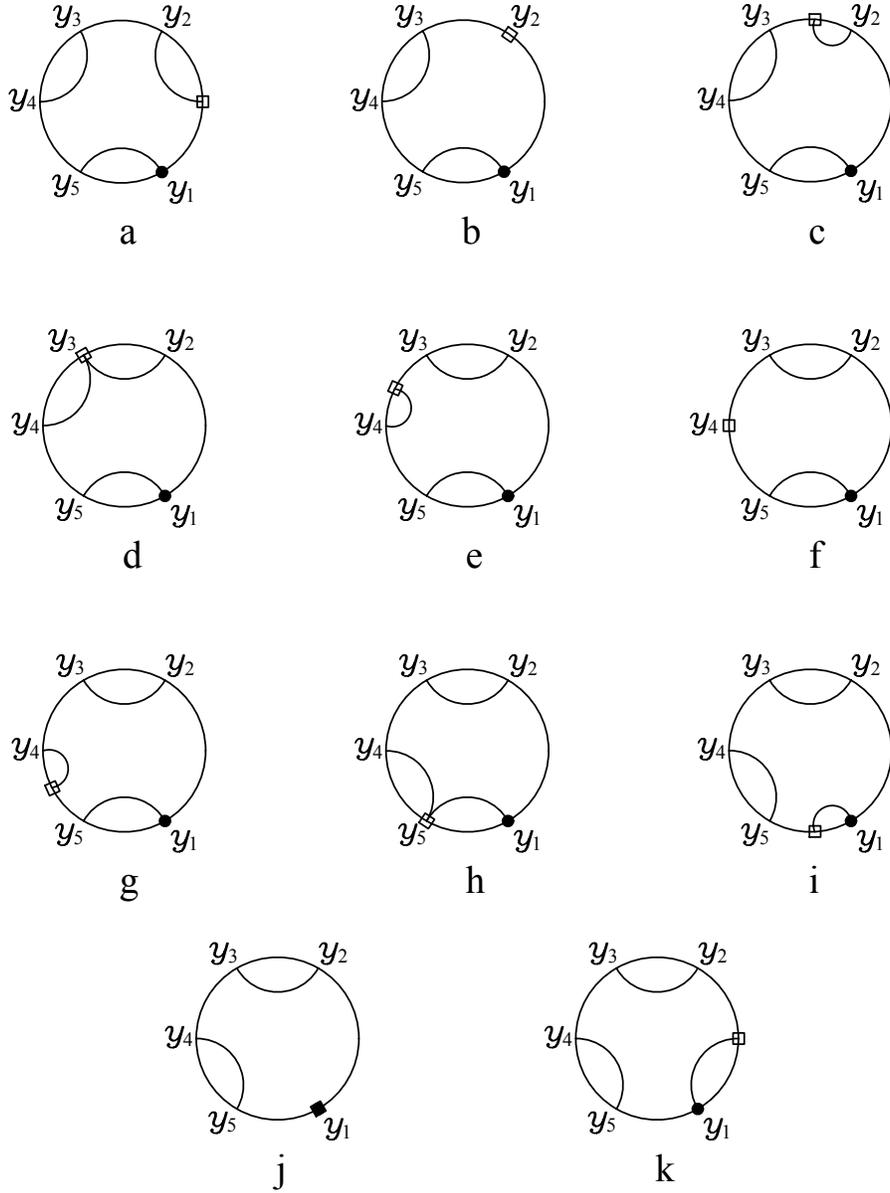


FIGURE 2. Dependence of the net γ_t on $\theta = e^{2\pi i t}$.

Remark 3.8. *It could happen that $[\gamma]_0 = [\gamma]_l$ for some positive integer $l < 2d - 2$. This occurs when $[\gamma]_0$ has rotational symmetry of order n for some integer $n > 1$, in which case $l = (2d - 2)/n$, a factor of $2d - 2$. For example, the net $[\gamma]_0$ in Fig. 2a has rotational symmetry of order 3, so $[\gamma]_0 = [\gamma]_2$.*

Example 3.9. Fig. 2 describes how a given net is modified as $\theta(t)$ makes a full turn around the circle. As $\theta(t)$ moves in (y_1, y_2) (Fig. 2a), the net remains unchanged. When $\theta(t)$ reaches y_2 (Fig. 2b), the net becomes degenerate with $\deg(y_2) = 2$. As $\theta(t)$ moves away from y_2 (Fig. 2c), we recover the original net until $\theta(t)$ reaches y_3 (Fig. 2d). At this point the net becomes degenerate again with $\deg(y_3) = 6$. As $\theta(t)$ moves away from y_3 (Fig. 2e), we recover the original net and the process continues in a similar manner until $\theta(t)$ crosses y_1 . The net obtained at this step (Fig. 2k) is the shift of the net in Fig. 2a.

4. LOWER BOUNDS FOR $k = 1$.

In this section, we derive lower bounds on the number of real solutions to Problem 1.5 in the special case when the interval (r, s) , or its complement, contains only one fixed critical point of f . To simplify notation, we shall assume that $y_1 = \infty$ is the only fixed vertex not contained in the interval (r, s) . Throughout this section, f is a real rational function of degree d and its net is given by $\Gamma = f^{-1}(\mathbb{R}\mathbb{P}^1)$.

Theorem 4.1. *Let $r < y_2 < y_3 < \dots < y_{2d-3} < s$ be points in \mathbb{R} and let $y_1 = \infty$. Then there are at least $u_d - 2u_{d-1}$ classes of real rational functions f of degree d having critical points at y_j , $1 \leq j \leq 2d - 3$ and satisfying $f(r) = f(s)$.*

Proof. Let \mathcal{C} be an equivalence class of the nets with $2d - 2$ vertices and no interior edges connecting the distinguished vertex y_1 to any of its two neighboring vertices. The number of such classes is $u_d - 2u_{d-1}$, since there are u_{d-1} classes of nets having two given neighboring vertices connected by an edge. It remains to show that for each such class \mathcal{C} there exists $y \in \mathbb{R}$ and a real rational function f_y having the net $\gamma_y \in \mathcal{C}$ with the vertex set $\{y_1, y_2, \dots, y_{2d-3}, y\}$, satisfying $f_y(r) = f_y(s)$. Let $y \in (s, \infty)$. There exists a unique class \mathcal{F}_y of real rational functions with the net belonging to \mathcal{C} and critical points $y_1, y_2, \dots, y_{2d-3}, y$ [4]. Choose $f_y \in \mathcal{F}_y$ so that f_y has a double pole at ∞ and $f_y(x) < 0$ for large $|x|$. Let $\rho: \mathbb{R} \rightarrow \mathbb{Q}$ be given by $\rho(y) = (x - y_2)(x - y_3) \dots (x - y_{2d-3})(x - y)$. Note that $\rho(y)$ is the Wronskian of f_y . By Lemma 2.4, ρ can be lifted to a path $\eta: \mathbb{R} \rightarrow \mathcal{R}_d$ with $\eta(y) = f_y$ satisfying properties (i)-(iii) of Lemma 2.4. By Lemma 3.3, $\gamma_y \in \mathcal{C}$ for all $y \in \mathbb{R} - \{y_j\}_{j=2}^{2d-3}$. The map $\phi_x: \mathbb{R} \rightarrow \mathbb{R}\mathbb{P}^1$ given by $\phi_x(y) = f_y(x)$ is continuous for all x in $\mathbb{R} - \{y_j\}_{j=2}^{2d-3}$. Continuity of $\phi_x(y)$ on (s, ∞) implies that for all $y \in (s, \infty)$, $f_y(x) < 0$ for large $|x|$. The following lemma completes the proof.

Lemma 4.2. *There exists $y \in \mathbb{R}$ such that $f_y(r) = f_y(s)$.*

Proof. Assume that $f_y(r) \neq f_y(s)$ for all $y \in [r, s]$. If $y \in [r, s]$ then f_y cannot have a pole at r nor at s since each of $[s, \infty)$ and $(-\infty, r]$ belongs to the boundary of a face of Γ_y and f_y has a pole at ∞ . Since both ϕ_r and ϕ_s are continuous, this implies that either $-\infty < f_y(r) < f_y(s) < \infty$ for all $y \in [r, s]$ or $-\infty < f_y(s) < f_y(r) < \infty$ for all $y \in [r, s]$. Without loss of generality, we may assume that $-\infty < f_y(r) < f_y(s) < \infty$ for all $y \in [r, s]$. In particular, $-\infty < f_s(r) < f_s(s) < \infty$. For $y \in (s, \infty)$, $f_y(y)$ is finite since f_y has a pole at ∞ and (y, ∞) belongs to the boundary of a face of Γ_y . Moreover, $f_y(s)$ cannot exceed $f_y(y)$ without first decreasing to $-\infty$ since ϕ_s is continuous and f_y has a local maximum at y and no critical points between s and y . On the other hand, since Γ_y has no interior edges connecting ∞ to y for all $y \in [s, \infty)$, $\lim_{y \rightarrow \infty} f_y(y) = -\infty$ and it follows that $f_y(s)$

must also decrease to $-\infty$ on a subinterval of (s, ∞) . In particular, there exists $y \in (s, \infty)$ such that $f_y(r) = f_y(s)$. \square

5. UPPER AND LOWER BOUNDS FOR THE ARC LENGTH OF $f([r, s])$

In this section, we assume that f maps \mathbf{S}^1 to \mathbf{S}^1 . We derive lower and upper bounds for the arc length of $f([r, s])$, in terms of the net of f and the position of the points r and s relative to the vertices of the net.

Consider \mathbf{S}^1 with the standard orientation. Given any two points r and s in \mathbf{S}^1 , we define $[r, s]$ and (r, s) respectively to be the closed and open positively oriented arcs in \mathbf{S}^1 starting at r and ending at s . The standard orientation on \mathbf{S}^1 induces an order on $[r, s]$ given by $a \leq b$ if and only if $a \in [r, b]$.

Definition 5.1. *Let Γ be a net such that all its vertices $y_1 < y_2 < \dots < y_n$ inside (r, s) are simple. The Young tableau of Γ corresponding to (r, s) , denoted by $Y_\Gamma(r, s)$, has 2 rows and n entries defined as follows. The integer i is placed in the first row of $Y_\Gamma(r, s)$ if and only if y_j is not connected by a chord of Γ to y_i , $i > j$.*

Note that $Y_\Gamma(r, s)$ is part of the Young tableau of (Γ, y_1) . See examples in Fig. 3 and Fig. 4.

Definition 5.2. *Given a rational function f and an oriented segment or a closed loop c in $\mathbb{C}\mathbb{P}^1$ with $f(c) \subset \mathbf{S}^1$, let $L(c) = L_f(c)$ be the argument increment of f on c .*

Let Γ be a net with simple vertices $y_1 < \dots < y_n$ inside (r, s) . Let f be a rational function with the net Γ . Assume that f is orientation preserving on (r, y_1) . Let $Y = Y_\Gamma(r, s)$.

Definition 5.3. *Let E and O be the numbers of even and odd entries in the second row of Y . Let m be the number of vertices of Γ inside (r, s) connected by a chord to vertices outside (r, s) .*

Lemma 5.4. *The number $m = n - 2(O + E)$ is the difference between the length of the first row of Y and the length of its second row.*

Consider first the case $m = 0$.

Lemma 5.5. *The following are equivalent:*

- (i) $m = 0$
- (ii) Y is rectangular
- (iii) no vertex in (r, s) is connected by a chord of Γ to a vertex outside (r, s) .

Definition 5.6. *Assuming $m = 0$, let $V = \{y_1, \dots, y_n\}$ be the set of vertices of Γ inside (r, s) . Let C be the set of chords of Γ having both endpoints in V and let $\mathcal{F} = \mathcal{F}(r, s)$ be the set of faces F of Γ satisfying $\partial F \cap \mathbf{S}^1 \subset [r, s]$. A chord in C will be denoted by $y_i y_j$ where y_i and y_j are its endpoints, with $i < j$. Given a chord $y_i y_j \in C$, let F_j be the face in \mathcal{F} having y_j as its largest vertex. Let F_0 be the face of Γ whose boundary contains $[r, y_1]$.*

There is a one-to-one correspondence between the sets C and \mathcal{F} . Condition (iii) of Lemma 5.5 implies that ∂F_0 must also contain $[y_n, s]$.

Example 5.7. *For the net Γ and arc (r, s) in Fig. 3, $n = 8$, $E = 2$, $O = 2$, $m = 0$, $C = \{y_1 y_2, y_3 y_8, y_4 y_5, y_6 y_7\}$ and $\mathcal{F} = \{F_2, F_5, F_7, F_8\}$.*

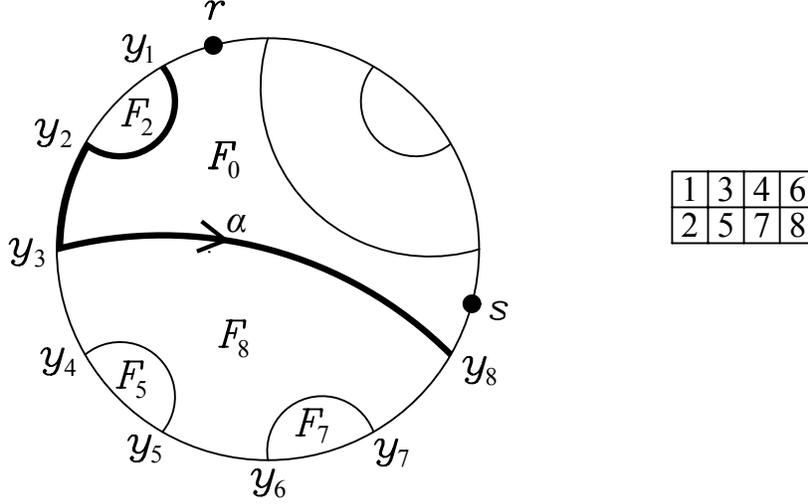


FIGURE 3. A net Γ with $m = 0$ and the corresponding Young tableau $Y_\Gamma(r, s)$.

Definition 5.8. Assuming $m = 0$, a chord $y_i y_j$ in C is even (resp., odd) if j is even (resp., odd).

By the construction of $Y_\Gamma(r, s)$ we have the following.

Lemma 5.9. The numbers E and O in Definition 5.3 are, respectively, the numbers of even and odd chords in C .

Definition 5.10. A face F of Γ is positive if f is orientation preserving on ∂F , otherwise F is negative. Let P and N be the numbers of positive and negative faces in \mathcal{F} . The face F_0 is always positive since f is orientation preserving on (r, y_1) .

Since f maps the boundary of each face $F \in \mathcal{F}$ bijectively onto \mathbf{S}^1 , it follows that $L(\partial F) = 2\pi$ if F is positive and $L(\partial F) = -2\pi$ if F is negative.

Lemma 5.11. Assuming $m = 0$, a chord $y_i y_j$ in C is odd if and only if F_j is positive. In particular, $O = P$ and $E = N$.

Proof. For $y_i y_j \in C$, the arc $[y_{j-1}, y_j]$ of \mathbf{S}^1 belongs to ∂F_j , and its orientation induced by F_j agrees with the standard orientation on \mathbf{S}^1 . Thus, since f preserves orientation on $[r, y_1]$ and has only simple critical points in (r, s) , it preserves orientation on $[y_{j-1}, y_j]$ if and only if j is odd. \square

Lemma 5.12. Let $m = 0$. Then $L([r, s]) \in (2\pi(O - E), 2\pi(O - E + 1))$.

Proof. Let Ω be the region bounded by the arc $[y_1, y_n] \subset \mathbf{S}^1$ and α , the positively oriented curve on ∂F_0 from y_1 to y_n (see Fig. 3). Since $\Omega \setminus \Gamma$ is the union of the faces in \mathcal{F} ,

$$L(\partial\Omega) = \sum_{F \in \mathcal{F}} L(\partial F) = 2\pi(P - N) = 2\pi(O - E).$$

On the other hand, $L([r, y_1]) + L(\alpha) + L([y_n, s]) \in (0, 2\pi)$ since $[r, y_1] \cup \alpha \cup [y_n, s]$ is part of the boundary of the positive face F_0 having positive orientation. Therefore,

$$\begin{aligned} L([r, s]) &= L([r, y_1]) + L([y_1, y_n]) + L([y_n, s]) \\ &= L([r, y_1]) + L(\alpha) + L(\partial\Omega) + L([y_n, s]) \\ &\in (2\pi(O - E), 2\pi(O - E + 1)). \end{aligned}$$

□

Remark 5.13. Assume $m = 0$. If f is orientation reversing on (r, y_1) then $g = 1/f$ is orientation preserving on (r, y_1) and $L_g([r, s]) = -L_f([r, s])$. Replacing f by g we obtain from Lemma 5.12 $L_f([r, s]) \in (2\pi(E - O - 1), 2\pi(E - O))$.

Consider now the case $m > 0$.

Let $y_{j_1}, y_{j_2}, \dots, y_{j_m}$ be the subsequence of y_1, y_2, \dots, y_n consisting of the vertices joined by a chord of Γ to vertices outside (r, s) . Letting $j_0 = 0, y_{j_0} = r, j_{m+1} = n + 1$ and $y_{j_{m+1}} = s$, we can represent $[r, s]$ as the union $\bigcup_{i=0}^m [y_{j_i}, y_{j_{i+1}}]$. Let $i \in \{0, 1, \dots, m\}$ and let E_i and O_i be the numbers of even and odd entries in the second row of Y that are between j_i and j_{i+1} .

Example 5.14. For the net Γ and arc (r, s) in Fig. 4, $E = O = 2, m = 3, E_0 = 1, O_0 = 0, E_1 = O_1 = 0, E_2 = O_2 = 1, E_3 = 0, O_3 = 1$.

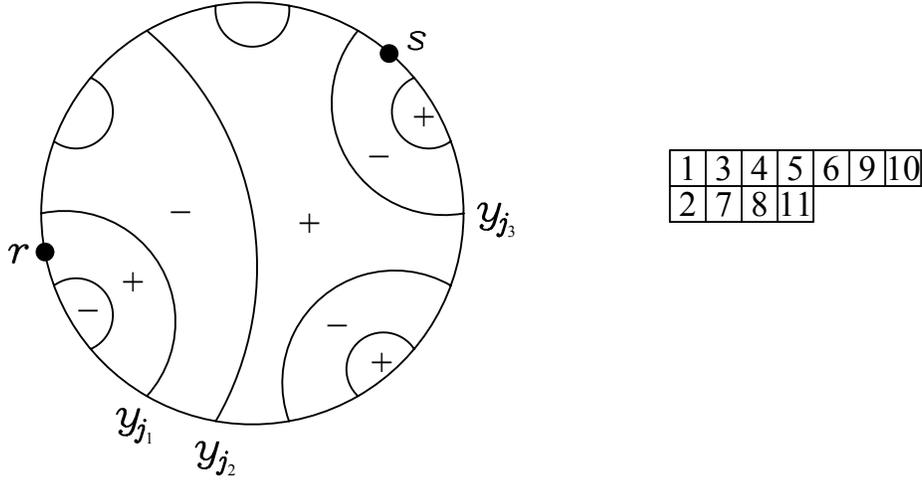


FIGURE 4. A net Γ with $m = 3$ and the corresponding Young tableau $Y_\Gamma(r, s)$.

For the arc (y_0, y_{j_1}) , f is orientation preserving on (y_0, y_1) , so by Lemma 5.12, $L([y_0, y_{j_1}]) \in (2\pi(O_0 - E_0), 2\pi(O_0 - E_0 + 1))$. For (y_{j_1}, y_{j_2}) , f is orientation reversing on (y_{j_1}, y_{j_1+1}) . Since j_1 is odd, the vertices that are even with respect to (y_{j_1}, y_{j_2}) belong to O_1 while the vertices that are odd with respect to (y_{j_1}, y_{j_2}) belong to E_1 . Accordingly, by Remark 5.13, $L_f([y_{j_1}, y_{j_2}]) \in (2\pi(O_1 - E_1 - 1), 2\pi(O_1 - E_1))$. Applying the same arguments to $(y_{j_i}, y_{j_{i+1}})$ we get the following.

Lemma 5.15. *Let $i \in \{0, 1, \dots, m\}$. If i is even then $L([j_i, j_{i+1}]) \in (2\pi(O_i - E_i), 2\pi(O_i - E_i + 1))$. If i is odd then $L([j_i, j_{i+1}]) \in (2\pi(O_i - E_i - 1), 2\pi(O_i - E_i))$.*

Summing up over $i = 0, \dots, m$ we obtain the following.

Theorem 5.16. *Let $I = I_\Gamma(r, s) =$*

$$([(n+1) \bmod 2 - n + 4O - 1]\pi, [(n+1) \bmod 2 + n - 4E + 1]\pi).$$

Then $L([r, s]) \in I$.

Now we consider the case when $W(p, q)$ has a double root y and the net Γ is degenerate. If $y \notin (r, s)$ this does not affect $L([r, s])$. If $y \in (r, s)$ then in labelling the vertices y_1, \dots, y_n in (r, s) , we assign to y two consecutive indices and define $Y_\Gamma(r, s)$ to be $Y_{\Gamma'}(r, s)$ where Γ' is a nondegenerate net with the vertices and chords close to those of Γ . With this agreement all the arguments above can be applied to Γ' and the statement of 5.16 remains true except for the case when the degree of y is 6 and both chords of Γ with the ends at y have other ends outside (r, s) . This means that y represents a segment $(y_{j_i}, y_{j_{i+1}})$ of length zero.

Theorem 5.17. *Let y be a double root of $W(p, q)$ inside (r, s) which is a vertex of Γ of degree 6 such that both chords of Γ with the ends at y have other ends outside (r, s) . Let $I = I_\Gamma(r, s)$ be*

$$([(n+1) \bmod 2 - n + 4O - 1]\pi, [(n+1) \bmod 2 + n - 4E - 1]\pi)$$

if the number of vertices between r and y is odd, and

$$([(n+1) \bmod 2 - n + 4O + 1]\pi, [(n+1) \bmod 2 + n - 4E + 1]\pi)$$

otherwise. Then $L([r, s]) \in I$.

We can similarly compute $I_\Gamma(s, r)$ and use it to improve $I_\Gamma(r, s)$ in some cases.

Let f be a function in \mathcal{S}_d all of whose critical points are simple except for possibly one double critical point. Let the critical points of f be y_1, \dots, y_{2d-2} (counted with multiplicity) with $y_1 \prec r \prec y_2 \preceq y_3 \preceq \dots \preceq y_{k+1} \prec s \prec y_{k+2} \preceq \dots \preceq y_{2d-2} \preceq y_1$. Assume f is orientation preserving on (y_1, y_2) . Let (Γ, y_1) be the net of f with respect to y_1 and let Y be the corresponding Young tableau. Let \mathcal{F} be the set of all faces of Γ . Let E and O be the numbers of even and odd entries in the second row of Y respectively. Then

$$L([r, s]) + L([s, r]) = \sum_{F \in \mathcal{F}} \partial F = 2\pi(E - O - 1).$$

So if $I_\Gamma(r, s) = (a, b)$, $I_\Gamma(s, r) = (a', b')$ and $L([r, s]) + L([s, r]) = c$, then $I_\Gamma(r, s)$ can be improved to $(a, b) \cap (c - b', c - a')$.

6. LOWER BOUNDS IN THE GENERAL CASE.

In this section, we apply the results of Section 5 to derive lower bounds on the number of real solutions to Problem 1.5. Without loss of generality, we may set $r = -1$ and $s = 1$. Let $D = \{y_1, y_2, \dots, y_{2d-3}\}$ be a set of distinct points in \mathbf{S}^1 all different from ± 1 . Assume that y_1 does not belong to the arc $[r, s]$. Let f_0 be a function in \mathcal{S}_d having critical set $D \cup \{1\}$. Let $\theta : \mathbb{R} \rightarrow \mathbf{S}^1$ be the path given by $\theta(t) = e^{2\pi it}$. By Lemma 2.4 and Notation 2.7, there exists a path $\eta : \mathbb{R} \rightarrow \mathcal{S}_d$ given by $\eta(t) = f_t$, where $f_t = p_t/q_t$ is a rational function having critical points $y_1, y_2, \dots, y_{2d-3}, \theta(t)$ counted with multiplicity, satisfying properties (i)-(iii) of Lemma 2.4.

Lemma 6.1. *The map $\mathbb{R} \rightarrow \mathbb{R}$ given by $t \mapsto L_{f_t}([r, s])$ is continuous everywhere except on a subset \mathcal{D} of $\theta^{-1}(D)$ given by $\mathcal{D} = \{t \in \theta^{-1}(D) : p_t(x) = q_t(x) = 0 \text{ for some } x \in D\}$. At any point of discontinuity t^* , $\lim_{t \rightarrow t^*} L_{f_t}([r, s])$ exists and $|L_{f_{t^*}}([r, s]) - \lim_{t \rightarrow t^*} L_{f_t}([r, s])| = 2\pi$.*

Proof. The first statement follows from Lemma 2.4. The second statement follows from Lemmas 2.4 and 2.5 and the fact that $f_{t^*}(x)$ is continuous at $\theta(t^*)$ as a function of x . \square

Let $L : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$L(t) = \begin{cases} L_{f_t}([r, s]) & \text{if } t \notin \mathcal{D} \\ \lim_{x \rightarrow t} L_{f_x}([r, s]) & \text{if } t \in \mathcal{D} \end{cases}$$

Corollary 6.2. *The map L is continuous on \mathbb{R} . For any $t \in \mathbb{R}$, $L(t) \in 2\pi\mathbb{Z}$ if and only if $f_t(r) = f_t(s)$.*

For $t \in \mathbb{R}$, let $\gamma_t = (\Gamma_t, y_1)$ be the net corresponding to f_t . The net Γ_t is nondegenerate if and only if $\theta(t) \notin D$. If $t \in \mathcal{D}$ then $\theta(t)$ is a vertex of degree 2 in Γ_t , and if $t \in \theta^{-1}(D) \setminus \mathcal{D}$ then $\theta(t)$ is of degree 6 in Γ_t . This follows from Lemma 2.4 and Remark 2.9.

Lemma 6.3. *The map $t \mapsto \gamma_t$ is periodic. The period T is a factor of $2d - 2$. It is equal to the number of distinct nondegenerate nets in the image.*

Proof. This follows from Corollaries 3.3 and 3.7 and Remark 3.4. \square

For $n \in \mathbb{Z}$, let $V_n = (n, n + \frac{1}{2})$ and $W_n = (n + \frac{1}{2}, n + 1)$ so that $\theta(V_n) = (s, r)$ and $\theta(W_n) = (r, s)$. For $t \in \mathbb{R}$, let $2\pi l_1(t)$ and $2\pi u_1(t)$ be the lower and upper endpoints of the interval $I_{\Gamma_t}(r, s)$ defined in Theorems 5.16 and 5.17, and let $2\pi l_2(t)$ and $2\pi u_2(t)$ be the lower and upper endpoints of the interval $I_{\Gamma_t}(s, r)$ defined similarly. Let $2\pi c = L([r, s]) + L([s, r])$. Then $L(t)/(2\pi) \in (l_1(t), u_1(t)) \cap (c - u_2(t), c - l_2(t))$.

Lemma 6.4. *The functions $l_1(t)$ and $u_1(t)$ are constant on each set V_n .*

Proof. The interval $I_{\Gamma_t}(r, s)$ depends only on the Young tableau of Γ_t corresponding to (r, s) which is independent of any changes outside (r, s) . \square

Lemma 6.5. *The functions $l_1(t)$ and $u_1(t)$ are constant on each set $W_n \setminus (\theta^{-1}(D) \setminus \mathcal{D})$ and assume only finitely many values on each set W_n .*

Proof. The arc (r, s) does not contain the distinguished vertex y_1 , so by Corollary 3.3, the nets γ_t , $t \in W_n \setminus \theta^{-1}(D)$ are equivalent. Since $\theta(t) \in (r, s) \setminus D$ for all $t \in W_n \setminus \theta^{-1}(D)$, Definition 5.1 implies that the Young tableaux $Y_{\Gamma_t}(r, s)$ are identical for all $t \in W_n \setminus \theta^{-1}(D)$. For $t \in W_n \cap \mathcal{D}$, the Young tableaux $Y_{\Gamma_t}(r, s)$ are identical to those for $t \in W_n \setminus \theta^{-1}(D)$, as defined after Theorem 5.16. This along with the fact that $W_n \cap (\theta^{-1}(D) \setminus \mathcal{D})$ is finite proves the second part of the statement. \square

If we let $\gamma_t = (\Gamma_t, y)$ for some fixed vertex $y \in (r, s)$ we get the following.

Corollary 6.6. *The functions $l_2(t)$ and $u_2(t)$ are constant on each set W_n and assume only finitely many values on each set V_n .*

Let $E = E(f_0)$ be the set of equivalence classes of functions f_t , $t \in \mathbb{R}$. The map $t \rightarrow [f_t]$ is periodic with period T . The value of T depends on the choice of f_0 , but it is always a factor of $2d - 2$. Therefore, instead of calculating T for each f_0 , we shall consider the interval $[0, 2d - 2]$ and deal with the issue of having counted some equivalence classes more than once at the end of this section.

Let \mathcal{U} and \mathcal{L} be two integer-valued functions defined by

$$\begin{aligned} \mathcal{U}(n) &= \inf \{u_1(t), c - l_2(t) : t \in V_n\} \\ \mathcal{U}(n + \frac{1}{2}) &= \inf \{u_1(t), c - l_2(t) : t \in W_n\} \\ \mathcal{L}(n) &= \sup \{l_1(t), c - u_2(t) : t \in V_n\} \\ \mathcal{L}(n + \frac{1}{2}) &= \sup \{l_1(t), c - u_2(t) : t \in W_n\} \end{aligned}$$

The functions \mathcal{U} and \mathcal{L} can be easily computed since l_1, u_1, l_2, u_2 assume only finitely many values on each V_n and W_n . In addition, continuity of $L(t)$ implies the existence of $t_n \in V_n$ and $t_{n+1/2} \in W_n$ such that $\frac{1}{2\pi}L(t_n) \in (\mathcal{L}(n), \mathcal{U}(n))$ and $\frac{1}{2\pi}L(t_{n+1/2}) \in (\mathcal{L}(n + 1/2), \mathcal{U}(n + 1/2))$. Let $S = \{k/2 : 0 \leq k < 4d - 4\}$. Let \prec be the cyclic order on S given by $0 \prec 1/2 \prec 1 \prec \dots \prec (4d - 5)/2 \prec 0$. For $i, j \in S$, let $(i, j) = \{k \in S \mid i \prec k \prec j\}$.

Definition 6.7. A point $i \in S$ is called a *max point* if

- i) $\exists k \in S$ such that $\mathcal{L}(i) \geq \mathcal{U}(k)$ and $\mathcal{L}(i) > \mathcal{L}(j)$, $\forall j \in (k, i)$, and
- ii) $\exists k \in S$ such that $\mathcal{L}(i) \geq \mathcal{U}(k)$ and $\mathcal{L}(i) \geq \mathcal{L}(j)$, $\forall j \in (i, k)$.

A point $i \in S$ is called a *min point* if

- i) $\exists k \in S$ such that $\mathcal{U}(i) \leq \mathcal{L}(k)$ and $\mathcal{U}(i) < \mathcal{U}(j)$, $\forall j \in (k, i)$, and
- ii) $\exists k \in S$ such that $\mathcal{U}(i) \leq \mathcal{L}(k)$ and $\mathcal{U}(i) \leq \mathcal{U}(j)$, $\forall j \in (i, k)$.

Lemma 6.8. Between any two max (resp. min) points of S there is a min (resp. max) point.

Proof. Let m and m' be two max points in S . Assume that $\mathcal{L}(m) \geq \mathcal{L}(m')$. There exists $k \in (m, m')$ such that $\mathcal{L}(m) \geq \mathcal{L}(m') \geq \mathcal{U}(k)$. Choose k to be the first value where \mathcal{U} attains a minimum on (m, m') (the order here is the order induced by the cyclic order). Then k is a min point. \square

Let $m_1 < m_2 < \dots < m_l$ be the max and min points of S and let $m_{l+1} = m_1$.

Lemma 6.9. Let $i \in \{1, 2, \dots, l\}$. If m_i is a max point then $\mathcal{U}(m_{i+1}) \leq \mathcal{L}(m_i)$. If m_i is a min point then $\mathcal{U}(m_i) \leq \mathcal{L}(m_{i+1})$.

Proof. Let m_i be a max point. Assume $\mathcal{U}(m_{i+1}) > \mathcal{L}(m_i)$. Since m_{i+1} is a min point, $\exists k \in (m_i, m_{i+1})$ such that $\mathcal{U}(m_{i+1}) \leq \mathcal{L}(k)$. So $\mathcal{L}(m_i) < \mathcal{L}(k) < \mathcal{U}(k)$. Since m_i is a max point, $\exists k' \in (m_i, k)$ such that $\mathcal{U}(k') \leq \mathcal{L}(m_i) < \mathcal{L}(k)$. Choose k' to be the first point where \mathcal{U} attains a minimum on (m_i, k) . Then k' is a min point between m_i and m_{i+1} , which is impossible. The proof of the second statement is similar. \square

To simplify notation, fix $i \in \{1, 2, \dots, l\}$ and let $j = m_i$ and $k = m_{i+1}$. If j is a max point then $L(t_k) < 2\pi\mathcal{U}(k) \leq 2\pi\mathcal{L}(j) < L(t_j)$, so the interval $(L(t_k), L(t_j))$ contains $\mathcal{L}(j) - \mathcal{U}(k) + 1$ multiples of 2π . Continuity of L implies that each of these multiples of 2π is attained by L at some value $t \in (t_j, t_k)$. Similarly, if j is a min point, then each of the $\mathcal{L}(k) - \mathcal{U}(j) + 1$ multiples of 2π in the interval $(L(t_j), L(t_k))$ is attained by L at some value $t \in (t_j, t_k)$. Summing up over i , we get a value $V(f_0)$ which is a lower bound for the number of times L crosses a multiple of 2π over the

interval $[0, 2d - 2]$. At each point t where this happens, the corresponding function f_t satisfies $f_t(r) = f_t(s)$.

Algorithm 6.10. *The algorithm described above to compute $V(f_0)$ depends on the net of f_0 and not on f_0 itself, so we may think of it as accepting a net γ and label its output $V(\gamma)$.*

Example 6.11. *Figure 5 shows some steps of Algorithm 6.10 applied to a net γ with 6 vertices and $c = 0$. The only min point of γ corresponds to W_2 where $\mathcal{U} = 0$, and the only max point corresponds to W_5 where $\mathcal{L} = 0$. So $V(\gamma) = 2$.*

Let $f_0^1, f_0^2, \dots, f_0^{u_d}$ be nonequivalent functions in \mathcal{S}_d having critical points at 1 and at $y_1, y_2, \dots, y_{2d-3}$. For each $i \in \{1, 2, \dots, u_d\}$ there exists a family $f_t^i, t \in [0, 2d - 2]$ satisfying properties (i)-(iii) of Lemma 2.4. By Theorem 2.6 and Notation 2.7, the map $t \rightarrow [f_t^i]$ is unique. Let $\gamma_t^i = (\Gamma_t^i, y_1)$ be the net corresponding to f_t^i and let T_i be the period of the map $t \rightarrow \gamma_t^i$. Uniqueness of the map $t \rightarrow [f_t^i]$ implies that the map $t \rightarrow \gamma_t^i$ is unique and that T_i is well-defined. Let $C_i = \{\gamma_j^i : j = 0, 1, \dots, T_i - 1\}$. The sets C_i are well-defined and partition the set C of nets with $2d - 2$ vertices. This follows from the fact that the map $t \rightarrow \gamma_t^i$ is unique and its period is a factor of $2d - 2$. Let k be the number of distinct sets C_i . For $i \in \{1, 2, \dots, k\}$, let N_{C_i} be the number of equivalence classes of functions f in \mathcal{S}_d whose net belongs to C_i and which satisfy $f(r) = f(s)$. For $\gamma \in C$, let $V(\gamma)$ be the output of Algorithm 6.10 when the input net is γ .

Theorem 6.12. *The number N of classes $[f] \in \mathcal{S}_d$ having critical points at the points $y_1, y_2, \dots, y_{2d-3}$ and satisfying $f(r) = f(s)$ is bounded below by $\sum_{\gamma \in C} \frac{V(\gamma)}{2d - 2}$.*

Proof. Since $C_i, i = 1, 2, \dots, k$ partition C , $N = \sum_{i=1}^k N_{C_i}$. For each $i \in \{1, 2, \dots, k\}$,

$$N_{C_i} = \frac{T_i}{2d - 2} V(\gamma_0^i).$$

Since $t \rightarrow \gamma_t^i$ is periodic, $V(\gamma_j^i) = N_{C_i}$ for all $j \in \{0, 1, \dots, T_i - 1\}$, hence,

$$T_i N_{C_i} = \sum_{j=0}^{T_i-1} \frac{T_i}{2d - 2} V(\gamma_j^i).$$

Dividing both sides by T_i and taking the sum over all classes C_i we get

$$\sum_{i=1}^k N_{C_i} = \sum_{i=1}^k \sum_{j=0}^{T_i-1} \frac{V(\gamma_j^i)}{2d - 2} = \sum_{\gamma \in C} \frac{V(\gamma)}{2d - 2}.$$

□

The lower bounds computed by the algorithm for $4 \leq d \leq 14$ and the corresponding values of k appear in Table 1. The first row and column give the values of d and k respectively. Computations in

www.math.tamu.edu/~sottile/research/pages/Flags/Data/F125/We2Xe5.5.html,

www.math.tamu.edu/~sottile/research/pages/Flags/Data/F126/Ve2We7.14.html,

and www.math.tamu.edu/~secant/monotone/monotoneindivproblemview.php?DB=monotoneSecant&result_id=957

show that the lower bounds for $d = 4, 5$, and 6 are sharp.

$t \in$	V_0	W_0	V_1	W_1
γ_t				
(l_1, u_1)	$(0, 1)$	$(0, 2)$	$(-1, 2)$	$(-2, 1)$
(l_2, u_2)	$(-1, 0)$	$(-2, 0)$	$(-2, 1)$	$(-2, 2)$
$(\mathcal{L}, \mathcal{U})$	$(0, 1)$	$(0, 2)$	$(-1, 1)$	$(-1, 1)$
$t \in$	V_2	W_2	V_3	W_3
γ_t				
(l_1, u_1)	$(-2, 1)$	$(-2, 0)$	$(-1, 0)$	$(-2, 0)$
(l_2, u_2)	$(-1, 2)$	$(0, 2)$	$(0, 1)$	$(0, 2)$
$(\mathcal{L}, \mathcal{U})$	$(-1, 1)$	$(-2, 0)$	$(-1, 0)$	$(-2, 0)$
$t \in$	V_4	W_4	V_5	W_5
γ_t				
(l_1, u_1)	$(-2, 1)$	$(-1, 2)$	$(-1, 2)$	$(0, 2)$
(l_2, u_2)	$(-1, 2)$	$(-2, 2)$	$(-2, 1)$	$(-2, 0)$
$(\mathcal{L}, \mathcal{U})$	$(-1, 1)$	$(-1, 1)$	$(-1, 1)$	$(0, 2)$

FIGURE 5. Some steps of Algorithm 6.10 applied to a net with $d = 4$ and $k = 2$.

	4	5	6	7	8	9	10	11	12	13	14
1	1	4	14	48	165	572	2002	7072	25194	90440	326876
2	1	2	6	18	57	186	622	2120	7338	25724	91144
3		4	12	36	113	366	1216	4122	14202	49592	175124
4			12	34	107	348	1156	3920	13514	47212	166788
5				36	115	372	1232	4166	14326	49950	176178
6					117	370	1232	4164	14326	49920	175978
7						370	1224	4104	14024	48610	170606
8							1244	4134	14176	49188	172660
9								4098	13948	48030	167690
10									14106	48348	169326
11										47904	166630
12											168000

TABLE 1. Lower bounds for $4 \leq d \leq 14$.7. COMBINATORIAL INTERPRETATION FOR $k = 1, 2$

In this section, we give a combinatorial interpretation of the algorithm in the cases (r, s) contains exactly one or exactly two points of D . The first case agrees with the result obtained in section 4. For $t \in W_n$, $(l_1(t), u_1(t)) \subseteq (c - u_2(t), c - l_2(t))$. For $t \in V_n$, $(l_1(t), u_1(t)) \subseteq (c - u_2(t), c - l_2(t))$ except when $\theta(t)$ coincides with a fixed vertex y and the two edges with endpoint y have their other endpoint outside (s, r) . When $k = 1$ and $t \in V_n$, there is only one fixed vertex outside (s, r) . So for $k = 1$, it is enough to consider (l_1, u_1) .

Case 1: (r, s) contains one point

Suppose there is only one fixed vertex y in (r, s) . The Young tableaux corresponding to a net with one fixed vertex in (r, s) can have one or two squares. The only one having one square gives rise to a unique $(\mathcal{L}, \mathcal{U})$ -interval, namely $(-1, 1)$. Each of the two tableaux with two squares gives rise to two intervals. These intervals are $(-1, 0)$, $(0, 1)$, $(-2, 0)$ and $(0, 2)$. All of these 5 intervals appear in Figure 6.

The three sequences $(0, 1)(-1, 1)(-1, 0)$, $(0, 1)(-1, 1)(-2, 0)$, $(0, 2)(-1, 1)(-1, 0)$, and their reversals cannot occur in the sequence of $(\mathcal{L}, \mathcal{U})$ -intervals of any net with $d > 2$. So $(-1, 0)$ and $(0, 1)$ do not yield any max or min points. The $(\mathcal{L}, \mathcal{U})$ -interval $(0, 2)$ gives a max point if and only if it is preceded by $(-2, 0)(-1, 1)$. In this case the net corresponding to $(-1, 1)$ has the property that the moving vertex is outside (r, s) and the fixed vertex in (r, s) is not connected to any of its neighboring vertices. The same holds when $(-2, 0)$ is a min point. Also, if the net has the property that the moving vertex is outside (r, s) and the fixed vertex in (r, s) is not connected to any of its neighboring vertices then its $(\mathcal{L}, \mathcal{U})$ -interval must be preceded by $(0, 2)$ and followed by $(-2, 0)$ or vice versa.

So given a net Γ with $2d - 2$ vertices, $V(\Gamma)$ is precisely the number of vertices y of Γ not connected to any of their neighboring vertices. The lower bound for a given d and $k = 1$ is the number of distinct nets (Γ, y) such that Γ has $2d - 2$ vertices and y is not connected to any of its neighboring vertices. This agrees with the result obtained in section 4.

Example 7.1. Figure 6 shows some steps of Algorithm 6.10 applied to a net γ with 6 vertices and $k = 1$. The degenerate nets with degenerate vertex outside (r, s) are not shown. Below each net is its Young tableau corresponding to (r, s) . The last row gives the $(\mathcal{L}, \mathcal{U})$ -interval for each V_i and W_i , $i = 0, 1, \dots, 5$. The only min point of γ corresponds to W_2 where $\mathcal{U} = 0$, and the only max point corresponds to W_5 where $\mathcal{L} = 0$. So $V(\gamma) = 2$, which is the number of vertices of Γ not connected to any of the neighboring vertices.

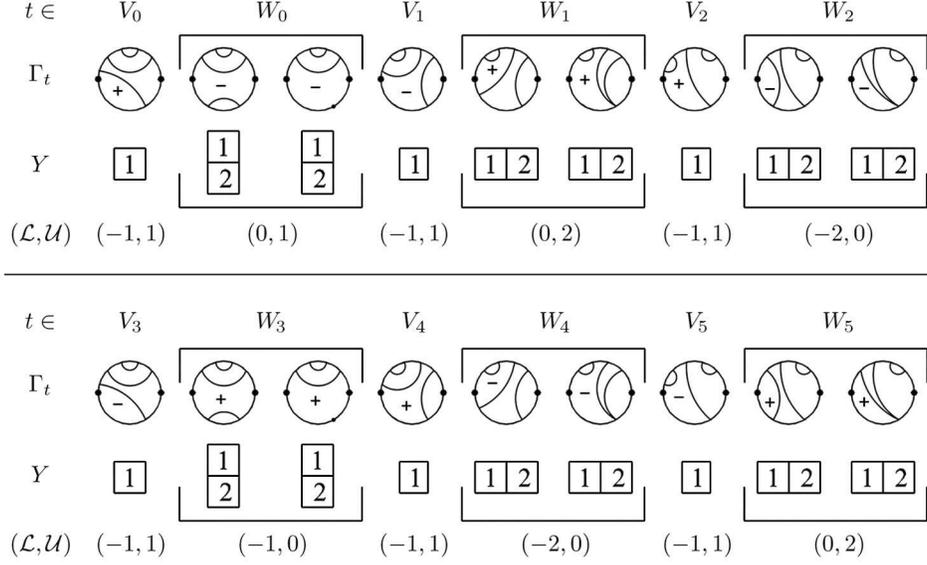


FIGURE 6. Algorithm 6.10 applied to a net with $d = 4$ and $k = 1$.

Case 2: (r, s) contains two points

Suppose there are two fixed vertices y_1 and y_2 in (r, s) . All possible $(\mathcal{L}, \mathcal{U})$ -intervals appear in Figure 5. Any max point must correspond to an $(\mathcal{L}, \mathcal{U})$ -interval with $\mathcal{L} = 0$. More specifically, any max point must correspond to an $(\mathcal{L}, \mathcal{U})$ -interval equal to $(0, 2)$ since the interval $(0, 1)$ must always be preceded by $(0, 2)$. Similarly, a min point must correspond to an $(\mathcal{L}, \mathcal{U})$ -interval equal to $(-2, 0)$.

Any interval giving rise to a max (resp. min) point must be immediately followed by the interval $(0, 1)$ (resp. $(-1, 0)$). The interval $(0, 1)$ (resp. $(-1, 0)$) is obtained when the moving critical point is outside (r, s) , the two critical points in (r, s) are connected by a chord and the function is orientation reversing (resp. preserving) in a neighborhood of r .

So the min/max points correspond to chords in the net connecting adjacent critical points and separated by an odd number of critical points. The lower bound is \mathcal{N}_{2d-2} , the number of nets $\gamma' = (\Gamma', y_1)$ with $2d - 2$ vertices satisfying $y_1 y_2$ is a chord of Γ' and the next chord connecting two consecutive vertices is of the form $y_{2k} y_{2k+1}$ with $1 < k < d - 1$.

Counting the number of such nets for a fixed k is equivalent to finding the number N_k of nets $\gamma = (\Gamma, v_1)$ having $2d - 6$ vertices $v_1, v_2, \dots, v_{2d-6}$ such that if $v_i v_{i+1}$ is

a chord of Γ then $i \geq 2k - 3$. Let S be the set of all nondegenerate nets γ having $2d - 6$ vertices. Given a net γ , let C_γ be the set of chords of γ .

$N_2 = u_{d-2}$ since, when $k = 2$ there is no condition on the chords of γ .

N_3 is the number of nets γ such that v_1v_2 and v_2v_3 are not chords in γ . So $N_3 = u_{d-2} - 2u_{d-3}$.

N_4 is the number of nets γ such that v_iv_{i+1} is not a chord in γ for $i = 1, 2, 3, 4$.

$$\begin{aligned} N_4 &= |S| - \sum_{i=1}^4 |\{\gamma \in S : v_iv_{i+1} \in C_\gamma\}| \\ &\quad + |\{\gamma \in S : \{v_1v_2, v_3v_4\} \text{ or } \{v_1v_2, v_4v_5\} \text{ or } \{v_2v_3, v_4v_5\} \subset C_\gamma\}| \\ &= u_{d-2} - 4u_{d-3} + 3u_{d-4}. \end{aligned}$$

Similar computations yield $N_5 = u_{d-2} - 6u_{d-3} + 10u_{d-4} - 4u_{d-5}$ and $N_6 = u_{d-2} - 8u_{d-3} + 21u_{d-4} - 20u_{d-5} + 5u_{d-6}$, etc. Using these values, we can compute \mathcal{N}_{2d-2} . The values of \mathcal{N}_{2d-2} for some small values of d are

$$\begin{aligned} \mathcal{N}_6 &= N_2 = u_2 = 1 \\ \mathcal{N}_8 &= N_2 + N_3 = 2u_3 - 2u_2 = 2 \\ \mathcal{N}_{10} &= N_2 + N_3 + N_4 = 3u_4 - 6u_3 + 3u_2 = 6 \\ \mathcal{N}_{12} &= N_2 + N_3 + N_4 + N_5 = 4u_5 - 12u_4 + 13u_3 - 4u_2 = 18 \\ \mathcal{N}_{14} &= N_2 + N_3 + N_4 + N_5 + N_6 = 5u_6 - 20u_5 + 34u_4 - 24u_3 + 5u_2 = 57. \end{aligned}$$

REFERENCES

- [1] F. Browder, editor. *Mathematical developments arising from Hilbert problems*. Proceedings of Symposia in Pure Mathematics, Vol. XXVIII. American Mathematical Society, Providence, R. I., 1976.
- [2] A. I. Degtyarev and V. M. Kharlamov. Topological properties of real algebraic varieties: Rokhlin's way. *Uspekhi Mat. Nauk*, 55(4(334)):129–212, 2000.
- [3] A. Eremenko and A. Gabrielov. Degrees of real wronski maps. *Discrete Comput. Geom.*, 28:331–347, 2002.
- [4] A. Eremenko and A. Gabrielov. Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry. *Ann. of Math. (2)*, 155(1):105–129, 2002.
- [5] A. Eremenko and A. Gabrielov. Elementary proof of the B. and M. Shapiro conjecture for rational functions, 2005.
- [6] A. Eremenko, A. Gabrielov, M. Shapiro, and A. Vainshtein. Rational functions and real Schubert calculus. *Proc. Amer. Math. Soc.*, 134(4):949–957, 2006.
- [7] W. Fulton. *Introduction to intersection theory in algebraic geometry*, volume 54 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1984.
- [8] L. Goldberg. Catalan numbers and branched coverings by the Riemann sphere. *Adv. Math.*, 85(2):129–144, 1991.
- [9] W. Hodge and D. Pedoe. *Methods of algebraic geometry. Vol. II*. Cambridge University Press, Cambridge, 1952.
- [10] Ilia Itenberg, Viatcheslav Kharlamov, and Eugenii Shustin. Welschinger invariant and enumeration of real rational curves. *Int. Math. Res. Not.*, (49):2639–2653, 2003.
- [11] Grigory Mikhalkin. Counting curves via lattice paths in polygons. *C. R. Math. Acad. Sci. Paris*, 336(8):629–634, 2003.
- [12] E. Mukhin, V. Tarasov, and A. Varchenko. The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz. *Annals of Math.*, 170(2):863–881, 2009.
- [13] R. Pandharipande, J. Solomon, and J. Walcher. Disk enumeration on the quintic 3-fold. *J. Amer. Math. Soc.*, 21(4):1169–1209, 2008.

- [14] T. Kyle Petersen, Pavlo Pylyavskyy, and Brendon Rhoades. Promotion and cyclic sieving via webs. *J. Algebraic Combin.*, 30(1):19–41, 2009.
- [15] Kevin Purbhoo. Jeu de taquin and a monodromy problem for Wronskians of polynomials. *Adv. Math.*, 224(3):827–862, 2010.
- [16] J. Ruffo, Y. Sivan, E. Soprunova, and F. Sottile. Experimentation and conjectures in the real Schubert calculus for flag manifolds. *Experiment. Math.*, 15(2):199–221, 2006.
- [17] H. Schubert. *Kalkül der abzählenden Geometrie*. Springer-Verlag, Berlin, 1979. Reprint of the 1879 original.
- [18] J. Solomon. Intersection theory on the moduli space of holomorphic curves with lagrangian boundary conditions. arXiv:math/0606429v1, 2006.
- [19] Evgenia Soprunova and Frank Sottile. Lower bounds for real solutions to sparse polynomial systems. *Adv. Math.*, 204(1):116–151, 2006.
- [20] F. Sottile. Real Schubert calculus: polynomial systems and a conjecture of Shapiro and Shapiro. *Experiment. Math.*, 9(2):161–182, 2000.
- [21] F. Sottile. Frontiers of reality in Schubert calculus. *Bulletin of the AMS*, 47(1):31–71, 2010.
- [22] F. Sottile. Real solutions to equations from geometry. arXiv:math/0609829v2, 2010.
- [23] Frank Sottile. Enumerative geometry for the real Grassmannian of lines in projective space. *Duke Math. J.*, 87(1):59–85, 1997.
- [24] R. Stanley. *Enumerative combinatorics. Vol. 2*. Cambridge University Press, Cambridge, 1999.
- [25] J.-Y. Welschinger. Invariants entiers en géométrie énumérative réelle. arXiv:1003.2707v1, 2010.
- [26] Jean-Yves Welschinger. Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry. *C. R. Math. Acad. Sci. Paris*, 336(4):341–344, 2003.

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