A HELLY-TYPE THEOREM FOR SEMI-MONOTONE SETS AND MONOTONE MAPS

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Abstract. We consider sets and maps defined over an o-minimal structure over the reals, such as real semi-algebraic or globally subanalytic sets. A *monotone map* is a multi-dimensional generalization of a usual univariate monotone continuous function on an open interval, while the closure of the graph of a monotone map is a generalization of a compact convex set. In a particular case of an identically constant function, such a graph is called a *semi-monotone set*. Graphs of monotone maps are, generally, non-convex, and their intersections, unlike intersections of convex sets, can be topologically complicated. In particular, such an intersection is not necessarily the graph of a monotone map. Nevertheless, we prove a Helly-type theorem, which says that for a finite family of subsets of $\mathbb{R}^n$, if all intersections of subfamilies, with cardinalities at most $n+1$, are non-empty and graphs of monotone maps, then the intersection of the whole family is non-empty and the graph of a monotone map.

1. Introduction

In [1, 2] the authors introduced a certain class of definable subsets of $\mathbb{R}^n$ (called *semi-monotone sets*) and definable maps $f : \mathbb{R}^n \to \mathbb{R}^k$ (called *monotone maps*) in an o-minimal structure over $\mathbb{R}$. These objects are meant to serve as building blocks for obtaining a conjectured cylindrical cell decomposition of definable sets into topologically regular cells, without changing the coordinate system in the ambient space $\mathbb{R}^n$ (see [1, 2] for a more detailed motivation behind these definitions).

The semi-monotone sets, and more generally the graphs of monotone maps, have certain properties which resemble those of classical convex subsets of $\mathbb{R}^n$. Indeed, the intersection of any definable open convex subset of $\mathbb{R}^n$ with an affine flat (possibly $\mathbb{R}^n$ itself) is the graph of a monotone map. In this paper, we prove a version of the classical theorem of Helly on intersections of convex subsets of $\mathbb{R}^n$.

We first fix some notation that we are going to use for the rest of the paper.

Notation 1.1. For every positive integer p, we will denote by $[p]$ the set $\{1, \dots, p\}$. We fix an integer $s > 0$, and we will henceforth denote by I the set $[s]$. For any family, $\mathcal{F} = (\mathbf{F}_i)_{i \in I}$, of subsets of $\mathbb{R}^n$ and $J \subset I$, we will denote by $\mathcal{F}_J$ the set $\bigcap_{j \in J} \mathbf{F}_j$.

Theorem 1.2 (Helly's Theorem [5, 8])**.** *Let $\mathcal{F} = (\mathbf{F}_i)_{i \in I}$ be a family of convex subsets of $\mathbb{R}^n$, such that for each subset $J \subset I$ such that* card $J \le n+1$*, the intersection $\mathcal{F}_J$ is non-empty. Then, $\mathcal{F}_I$ is non-empty.*

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In this paper we prove an analogue of Helly's theorem for semi-monotone sets as well as for graphs of monotone maps. One important result in [2, Theorem 13] is that the graph of a monotone map is a topologically regular cell. However, unlike the case of a family of convex sets, the intersection of a finite family of graphs of monotone maps need not be a graph of a monotone map, or even be connected. Moreover, such an intersection can have an arbitrarily large number of connected components. Because of this lack of a good intersectional property, one would not normally expect a Helly-type theorem to hold in this case. Nevertheless, we are able to prove the following theorem.

Theorem 1.3. *Let $\mathcal{F} = (\mathbf{F}_i)_{i \in I}$ be a family of definable subsets of $\mathbb{R}^n$ such that for each $i \in I$ the set $\mathbf{F}_i$ is the graph of a monotone map, and for each $J \subset I$, with* card $J \leq n+1$*, the intersection $\mathcal{F}_J$ is non-empty and the graph of a monotone map. Then, $\mathcal{F}_I$ is non-empty and the graph of a monotone map as well.*

Moreover, if $\dim\ \mathcal{F}_J \geq d$ *for each $J \subset I$, with* card $J \leq n+1$, *then* $\dim\ \mathcal{F}_I \geq d$.

Remark 1.4. Katchalski [6] (see also [4]) proved the following generalization of Helly's theorem which took into account dimensions of the various intersections.

Theorem 1.5 ([6, 4])**.** *Define the function $g(j)$ as follows:*

$$g(0) = n+1,$$
$$g(j) = \max(n+1, 2(n-j+1)) \text{ for } 1 \leq j \leq n.$$

Fix any j such that $0 \leq j \leq n$. Let $\mathcal{F} = (\mathbf{F}_i)_{i \in I}$ be a family of convex subsets of $\mathbb{R}^n$, with card $I \geq g(j)$, *such that for each $J \subset I$, with* card $J \leq g(j)$, *the dimension* $\dim\ \mathcal{F}_J \geq j$. *Then, the dimension* $\dim\ \mathcal{F}_I \geq j$.

Notice that in the special case of definable convex sets in $\mathbb{R}^n$ that are open subsets of flats, Theorem 1.3 gives a slight improvement over Theorem 1.5 in that $n+1 \leq g(j)$ for all j, $0 \leq j \leq n$, where $g(j)$ is the function defined in Theorem 1.5. The reason behind this improvement is that convex sets that are graphs of monotone maps (i.e., definable open convex subsets of affine flats) are rather special and easier to deal with, since we do not need to control the intersections of their boundaries.

Also note that, while it follows immediately from Theorem 1.5 (using the same notation) that

$$\dim\ \mathcal{F}_I = \min(\dim\ \mathcal{F}_J | J \subset I, \text{card } J \leq 2n),$$

Katchalski [7] proved the stronger statement that

$$\dim\ \mathcal{F}_I = \min(\dim\ \mathcal{F}_J | J \subset I, \text{card } J \leq n+1).$$

In the case of graphs of monotone maps, the analogue of the latter statement is an immediate consequence of Theorem 1.3.

2. Proof of Theorem 1.3

We begin with a few preliminary definitions.

Definition 2.1. Let $L_{j,\sigma,c} := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n |\ x_j \sigma c\}$ for $j = 1, \ldots, n$, $\sigma \in \{<, =, >\}$, and $c \in \mathbb{R}$. Each intersection of the kind

$$C := L_{j_1,\sigma_1,c_1} \cap \cdots \cap L_{j_m,\sigma_m,c_m} \subset \mathbb{R}^n,$$

where $m = 0, \ldots, n$, $1 \leq j_1 < \cdots < j_m \leq n$, $\sigma_1, \ldots, \sigma_m \in \{<, =, >\}$, and $c_1, \ldots, c_m \in \mathbb{R}$, is called a *coordinate cone* in $\mathbb{R}^n$.

Each intersection of the kind

$$S := L_{j_1,=,c_1} \cap \cdots \cap L_{j_m,=,c_m} \subset \mathbb{R}^n,$$

where $m = 0, \ldots, n$, $1 \leq j_1 < \cdots < j_m \leq n$, and $c_1, \ldots, c_m \in \mathbb{R}$, is called an *affine coordinate subspace* in $\mathbb{R}^n$.

In particular, the space $\mathbb{R}^n$ itself is both a coordinate cone and an affine coordinate subspace in $\mathbb{R}^n$.

Definition 2.2 ([1])**.** An open (possibly, empty) bounded set $X \subset \mathbb{R}^n$ is called *semi-monotone* if for each coordinate cone C the intersection $X \cap C$ is connected.

Remark 2.3. In fact, in Definition 2.2 above, it suffices to consider intersections with only affine coordinate subspaces (see Definition 2.5 below).

We refer the reader to [1, Figure 1] for some examples of semi-monotone subsets of $\mathbb{R}^2$, as well as some counter-examples. In particular, it is clear from the examples that the intersection of two semi-monotone sets in plane is not necessarily connected and hence not semi-monotone.

Notice that any convex open subset of $\mathbb{R}^n$ is semi-monotone.

We now define *monotone maps*. The definition below is not the one given in [2], but equivalent to it as shown in [2, Theorem 9].

We first need a preliminary definition.

Definition 2.4. Let a bounded continuous map $\mathbf{f} = (f_1, \ldots, f_k)$ defined on an open bounded non-empty set $X \subset \mathbb{R}^n$ have the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$. We say that $\mathbf{f}$ is *quasi-affine* if for any coordinate subspace T of $\mathbb{R}^{n+k}$, the projection $\rho_T : \mathbf{F} \to T$ is injective if and only if the image $\rho_T(\mathbf{F})$ is n-dimensional.

Definition 2.5. Let a bounded continuous quasi-affine map $\mathbf{f} = (f_1, \ldots, f_k)$ defined on an open bounded non-empty set $X \subset \mathbb{R}^n$ have the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$. We say that the map $\mathbf{f}$ is monotone if for each affine coordinate subspace S in $\mathbb{R}^{n+k}$ the intersection $\mathbf{F} \cap S$ is connected.

The following two statements were proved in [2].

Theorem 2.6. [2, Corollary 7] *Let $\mathbf{f} : X \to \mathbb{R}^k$ be a monotone map having the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$. Then for every coordinate z in $\mathbb{R}^{n+k}$ and every $c \in \mathbb{R}$, each of the intersections $\mathbf{F} \cap \{z\, \sigma\, c\}$, where $\sigma \in \{<, >, =\}$, is either empty or the graph of a monotone map.*

Theorem 2.7. [2, Theorem 10] *Let $\mathbf{f} : X \to \mathbb{R}^k$ be a monotone map defined on a semi-monotone set $X \subset \mathbb{R}^n$ and having the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$. Then for any coordinate subspace T in $\mathbb{R}^{n+k}$ the image $\rho_T(\mathbf{F})$ under the projection map $\rho_T : \mathbf{F} \to T$ is either a semi-monotone set or the graph of a monotone map.*

Remark 2.8. In view of Definition 2.5, it is natural to identify any semi-monotone set $X \subset \mathbb{R}^n$ with the graph of the constant function $f : X \to \mathbb{R}^0 = \mathbf{0}$. Also, note that in this case the function f is trivially quasi-affine (cf. Definition 2.4).

We need two preliminary lemmas before we prove Theorem 1.3.

Lemma 2.9. *Suppose that $\mathcal{F} = (\mathbf{F}_i)_{i \in I}$ is a family of definable subsets of $\mathbb{R}^n$ such that for each $i \in I$ the set $\mathbf{F}_i$ is the graph of a monotone map. Then, there exists a family of definable sets, $\mathcal{F}' = (\mathbf{F}'_i)_{i \in I}$ such that:*

(1) *for each $i \in I$ the set $\mathbf{F}'_i$ is closed and bounded;*

(2) *for each* $J \subset I$ *we have*

$$\mathrm{H}_*(\mathcal{F}'_J, \mathbb{Z}) \cong \mathrm{H}_*(\mathcal{F}_J, \mathbb{Z}),$$

where $\mathrm{H}_*(X, \mathbb{Z})$ *denotes the singular homology of* X.

Proof. Since, according to [1, Theorem 5.1], the graph of a monotone map is a regular cell, we have for each $i \in I$ a definable homeomorphism

$$\phi_i : (0,1)^{\dim(\mathbf{F}_i)} \to \mathbf{F}_i.$$

For each real $\varepsilon > 0$ small enough, and for each $i \in I$ consider the image

$$\mathbf{F}_i^{(\varepsilon)} = \phi_i([\varepsilon, 1-\varepsilon]^{\dim(\mathbf{F}_i)}).$$

Consider the family $\mathcal{F}^{(\varepsilon)} = \left(\mathbf{F}_i^{(\varepsilon)}\right)_{i \in I}$. Observe for each $J \subset I$ and each $\varepsilon > 0$ the intersection $\mathcal{F}_J^{(\varepsilon)}$ is compact, and the increasing family $\left(\mathcal{F}_J^{(\varepsilon)}\right)_{\varepsilon > 0}$ is co-final in the directed system (under the inclusion maps) of the compact subsets of $\mathcal{F}_J$. Since, the singular homology group of any space is isomorphic to the direct limit of the singular homology groups of its compact subsets [9, Sec. 4, Theorem 6], we have

$$\varinjlim \mathrm{H}_*(\mathcal{F}_J^{(\varepsilon)}, \mathbb{Z}) \cong \mathrm{H}_*(\mathcal{F}_J, \mathbb{Z}).$$

Finally, by Hardt's triviality theorem [3] there exists $\varepsilon_0 > 0$, such that

$$\varinjlim \mathrm{H}_*(\mathcal{F}_J^{(\varepsilon)}, \mathbb{Z}) \cong \mathrm{H}_*(\mathcal{F}_J^{(\varepsilon_0)}, \mathbb{Z}).$$

For each $i \in I$, we let $\mathbf{F}'_i = \mathbf{F}_i^{(\varepsilon_0)}$. □

Lemma 2.10. *Let* $\mathcal{F} = (\mathbf{F}_i)_{i \in I}$ *be a family of definable subsets of* $\mathbb{R}^n$ *such that for each* $i \in I$ *the set* $\mathbf{F}_i$ *is the graph of a monotone map, and for each* $J \subset I$*, with* $\operatorname{card} J \le n+1$ *the intersection* $\mathcal{F}_J$ *is non-empty and the graph of a monotone map. Suppose that* $\dim\ \mathcal{F}_I = p$*, where* $0 \le p \le n$*. Then, there exists a subset* $J \subset I$ *with* $\operatorname{card} J \le n-p$ *such that* $\dim\ \mathcal{F}_J = p$*. (Note that if* $J = \emptyset$ *then* $\mathcal{F}_J = \mathbb{R}^n$ *by convention, and* $\dim \mathcal{F}_J = n$*.)*

Proof. The proof is by induction on p. If $p = n$, then the lemma trivially holds.

Suppose that the claim holds for all dimensions strictly larger than p. Then, there exist a subset $J' \subset I$ (possibly empty), with $\dim\ \mathcal{F}_{J'} > p$ (noting that if $J' = \emptyset$, then $\dim\ \mathcal{F}_{J'} = n$), and $i \in I$ such that $\dim\ \mathcal{F}_{J' \cup \{i\}} = p$. By the induction hypothesis there exists a subset $J'' \subset J'$, with $\operatorname{card} J'' < n-p$, such that $\dim\ \mathcal{F}_{J''} = \dim\ \mathcal{F}_{J'}$.

Since $\mathcal{F}_{J'} \subset \mathcal{F}_{J''}$, there exists an open definable subset $U \subset \mathbb{R}^n$ such that

(1) $U \cap \mathcal{F}_{J'} = U \cap \mathcal{F}_{J''}$ and
(2) $\dim\ (U \cap \mathcal{F}_{J'} \cap \mathbf{F}_i) = p$.

Then taking $J = J'' \cup \{i\}$, $\operatorname{card} J \le n-p$, and, since $n-p \le n+1$, the intersection $\mathcal{F}_J$ is the graph of a monotone map by the conditions of the theorem. This proves the lemma because $\mathcal{F}_J$, being a regular cell, has the same local dimension at each point. □

Proof of Theorem 1.3. The proof is by a double induction on n and s. For $n = 1$ the theorem is true for all s, since it is just Helly's theorem in dimension 1.

Now assume that the statement is true in dimension $n-1$ for all s. In dimension n, for $s \le n+1$, there is nothing to prove. Assume that the theorem is true in dimension n for at least $s-1$ sets.

We first prove that $\mathcal{F}_I$ is non-empty. The proof of this fact is adapted from the classical proof of the topological version of Helly's theorem (also due to Helly [5]).

According to Lemma 2.9 there exists a family $\mathcal{F}' = (\mathbf{F}'_i)_{i \in I}$ consisting of closed and bounded definable sets, such that for each $J \subset I$ we have

$$\mathrm{H}_*(\mathcal{F}'_J, \mathbb{Z}) \cong \mathrm{H}_*(\mathcal{F}_J, \mathbb{Z}).$$

Thus, it suffices to prove that $\mathcal{F}'_I$ is non-empty. Suppose that $\mathcal{F}'_I$ is empty. Then, there exists a smallest sub-family, $(\mathbf{F}'_j)_{j \in [p]}$, for some p with $n+2 \le p \le s$, such that $\mathcal{F}'_{[p]}$ is empty, and for each proper subset $J \subset [p]$ the intersection $\mathcal{F}'_J$ is non-empty.

Using the induction hypothesis on s, applied to the family $(\mathbf{F}_j)_{j \in J}$, for each $J \subset I$ with card $J <$ card $I = s$, we conclude that $\mathcal{F}_J$ is the graph of a monotone map and hence acyclic. But then the set $\mathcal{F}'_J$ is also acyclic since it has the same singular homology groups as $\mathcal{F}_J$. Consider the nerve simplicial complex of the family $(\mathbf{F}'_j)_{j \in [p]}$. Clearly, it has the homology of the $(p-2)$-dimensional sphere $\mathbf{S}^{p-2}$ (being isomorphic to the simplicial complex of the boundary of a $(p-1)$-dimensional simplex). Therefore, the union $\bigcup_{i \in [p]} \mathbf{F}'_i$ also has the homology of $\mathbf{S}^{p-2}$, which is impossible since $p - 2 \ge n$. Thus, $\mathcal{F}'_I$ is non-empty, and hence $\mathcal{F}_I$ is non-empty as well.

We next prove that $\mathcal{F}_I$ is connected. If not, let $\mathcal{F}_I = B_1 \cup B_2$, where the sets B_1, B_2 are non-empty, disjoint and closed in $\mathcal{F}_I$.

For any $c \in \mathbb{R}$ the intersection $\mathcal{F}_I \cap \{x_1 = c\}$, where x_1 is a coordinate in $\mathbb{R}^n$, is either empty or connected, by Theorem 2.6 and the induction hypothesis for dimension $n-1$. Hence, B_1 and B_2 must lie on the opposite sides of a hyperplane $\{x_1 = c\}$ for some $c \in \mathbb{R}$, with

$$B_1 \cap \{x_1 = c\} = B_2 \cap \{x_1 = c\} = \emptyset.$$

Now, for every $J \subset I$, such that card $J \le n$, the intersection $\mathcal{F}_J$ is the graph of a monotone map by the conditions of the theorem, and contains both B_1 and B_2. Hence $\mathcal{F}_J$ meets the hyperplane $\{x_1 = c\}$, and, by Theorem 2.6, the intersection $\mathcal{F}_J \cap \{x_1 = c\}$ is a graph of a monotone map. Applying the induction hypothesis in dimension $n-1$, to the family $(\mathbf{F}_i \cap \{x_1 = c\})_{i \in I}$ we obtain that $\mathcal{F}_I \cap \{x_1 = c\}$ is non-empty, which is a contradiction.

We next prove that $\mathcal{F}_I$ is the graph of a quasi-affine map.

If $p = \dim\, \mathcal{F}_I = n$, then $\mathcal{F}_I$ is an non-empty, open, bounded, definable set and is automatically the graph of a quasi-affine map (cf. Remark 2.8). So we can assume that $p < n$.

Let $\dim\, \mathcal{F}_I = p$. By Lemma 2.10, there exists $J \subset I$ with card $J \le n - p$ such that $\dim\, \mathcal{F}_J = p$. By the assumption of the theorem, $\mathcal{F}_J$ is the graph of a monotone map, in particular, that map is quasi-affine. Since $p < n$, there exists $i \in J$ such that $m := \dim\, \mathbf{F}_i < n$. Assume $\mathbf{F}_i$ to be the graph of a monotone map defined on the semi-monotone subset of the coordinate subspace T. Then $\dim\, T = m < n$.

Let $\rho_T : \mathbb{R}^n \to T$ be the projection map. Consider the family

$$\mathcal{F}'' := (\rho_T(\mathbf{F}_j \cap \mathbf{F}_i))_{j \in I}.$$

Every intersection of at most $m+1$ members of $\mathcal{F}''$ is the image under ρ_T of the intersection of at most $m+2 \le n+1$ members of $\mathcal{F}$. By the assumption of the theorem, each intersection of at most $m+2 \le n+1$ members of $\mathcal{F}$ is the non-empty graph of a monotone map. Then, by Theorem 2.7, every intersection of at most

$m+1$ elements of $\mathcal{F}''$ is non-empty and is either the graph of a monotone map or a semi-monotone set. The case when all intersections are semi-monotone sets is trivial, so assume that some of them are graphs of a monotone maps. Applying the induction hypothesis (with respect to n) to the family $\mathcal{F}''$ we obtain that the intersection, $\mathcal{F}''_I$ is a graph of a monotone map defined on some semi-monotone subset $U \subset L$ where L is a coordinate subspace of T, and hence $\mathcal{F}_I$ is the graph of a definable map defined on U. This, together with the fact that $\mathcal{F}_I$ is contained in the graph $\mathcal{F}_J$ of a quasi-affine map, having the same dimension, implies that $\mathcal{F}_I$ is also the graph of a quasi-affine map.

It now follows from Definition 2.5 that $\mathcal{F}_I$ is the graph of a monotone map.

Finally, we prove the claim that if dim $\mathcal{F}_J \geq d$ for each $J \subset I$, with card $J \leq n+1$, then dim $\mathcal{F}_I \geq d$. This is clearly true if $d = n$, since in this case $\mathcal{F}_I$ is non-empty and open, and hence dim $\mathcal{F}_I = d$. So we can assume that $d < n$.

Since $d < n$, there exists $i \in I$, such that $m := \dim \mathbf{F}_i < n$. Let $T \subset \mathbb{R}^n$ be a coordinate subspace such that $\mathbf{F}_i$ is a graph over a non-empty semi-monotone subset of T, and let dim $T = m$. Let $\rho_T : \mathbb{R}^n \to T$ be the projection map, and consider the family $(\rho_T(\mathbf{F}_j \cap \mathbf{F}_i))_{j \in I}$. By assumption of the theorem and Theorem 2.7 we have that for every subset $J \subset I$, with card $J \leq n$, the family $(\rho_T(\mathbf{F}_j \cap \mathbf{F}i))_{j \in I}$ consists of graphs of monotone maps, and every finite intersection of at most $m+1 \leq n$ of these sets is non-empty and also the graph of monotone map having dimension at least d. Using the induction hypothesis with respect to n, we conclude that

$$\dim \bigcap_{j \in I} \rho_T(\mathbf{F}_j \cap \mathbf{F}_i) \geq d.$$

It follows that dim $\mathcal{F}_I \geq d$. □

Remark 2.11. It is possible to generalize Theorem 1.3 slightly by requiring only that all members of the family $\mathcal{F}$ be contained in the graph of some fixed monotone map of dimension n (rather than in $\mathbb{R}^n$ as in Theorem 1.3). However, since this would unduly complicate the statement of the theorem we preferred not to make this slight extension.

References

[1] Saugata Basu, Andrei Gabrielov, and Nicolai Vorobjov. Semi-monotone sets. *J. Eur. Math. Soc. (JEMS)*, 15(2):635–657, 2013.

[2] Saugata Basu, Andrei Gabrielov, and Nicolai Vorobjov. Monotone functions and maps. *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, 107(1):5–33, 2013.

[3] Michel Coste. *An introduction to o-minimal geometry.* Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000. Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica.

[4] Branko Grünbaum. The dimension of intersections of convex sets. *Pacific J. Math.*, 12:197–202, 1962.

[5] Eduard Helly. Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten. *Monatsh. Math. Phys.*, 37(1):281–302, 1930.

[6] Meir Katchalski. The dimension of intersections of convex sets. *Israel J. Math.*, 10:465–470, 1971.

[7] Meir Katchalski. Reconstructing dimensions of intersections of convex sets. *Aequationes Math.*, 17(2-3):249–254, 1978.

[8] Johann Radon. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. *Math. Ann.*, 83(1-2):113–115, 1921.

[9] Edwin H. Spanier. *Algebraic topology.* McGraw-Hill Book Co., New York, 1966.

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