MONOTONE FUNCTIONS AND MAPS

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To Professor Heisuke Hironaka on the occasion of his 80th birthday

Abstract. In [1] we defined semi-monotone sets, as open bounded sets, definable in an o-minimal structure over the reals (e.g., real semialgebraic or subanalytic sets), and having connected intersections with all translated coordinate cones in $\mathbb{R}^n$. In this paper we develop this theory further by defining *monotone functions* and *maps*, and studying their fundamental geometric properties. We prove several equivalent conditions for a bounded continuous definable function or map to be monotone. We show that the class of graphs of monotone maps is closed under intersections with affine coordinate subspaces and projections to coordinate subspaces. We prove that the graph of a monotone map is a topologically regular cell. These results generalize and expand the corresponding results obtained in [1] for semi-monotone sets.

Introduction

This paper is a continuation of the work initiated in an earlier paper [1] where the authors introduced a particular class of sets, called semi-monotone, being open, bounded and definable in an o-minimal structure over the reals (e.g., real semialgebraic or subanalytic). One of the main results in [1] is that semi-monotone sets are topologically regular cells. Here we generalize this result to the sets of any codimension. The immediate motivation for defining this class of definable sets was to prove the existence of definable triangulations "compatible" with a given definable function – more precisely, the following conjecture.

Conjecture 1 ([1])**.** Let $f : K \to \mathbb{R}$, be a definable function on a compact definable set $K \subset \mathbb{R}^m$. Then there exists a definable triangulation of K such that, for each $n \le \dim K$ and for each open n-simplex Δ of the triangulation,

(1) the graph $\Gamma := \{(\mathbf{x}, t) |\ \mathbf{x} \in \Delta,\ t = f(\mathbf{x})\}$ of the restriction of f on Δ is a topologically regular n-cell (see Definition 12);
(2) either f is a constant on Δ or each non-empty level set $\Gamma \cap \{t = \text{const}\}$ is a topologically regular $(n-1)$-cell.

Conjecture 1 is part of a larger program of obtaining combinatorial classification of monotone families of definable sets discussed in [1]. The triangulation described in the conjecture can be viewed as a topological resolution of singularities of definable functions.

The role of semi-monotone sets in the proposed proof of the above conjecture is as follows. We first hope to prove the existence of a definable, regular cell decomposition of K, such that the properties (i) and (ii) are satisfied for each

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cell of the decomposition. The triangulation will then be obtained by generalized barycentric subdivision of these cells. In order for such an approach to work, one needs a good supply of definable cells guaranteed to be regular.

The semi-monotone sets fit this requirement. Non-empty semi-monotone sets are topologically regular cells [1]. Moreover, the class of semi-monotone sets is stable under maps that permute the coordinates of $\mathbb{R}^n$. However, non-empty semi-monotone sets are open (and hence full dimensional) definable subsets of $\mathbb{R}^n$. In this paper, we introduce a certain class of definable maps $\mathbf{f} : X \to \mathbb{R}^k$ where X is a semi-monotone subset of $\mathbb{R}^n$. We call these maps *monotone maps* (see Definition 8 below). We give several characterizations of monotone maps. Our main result (Theorem 13 below) states that the graphs of monotone maps are topologically regular cells. This implies the topological regularity of toric cubes (Corollary 11) and settles a conjecture proposed in [4].

We also prove that monotone maps satisfy a suitable generalization of the coordinate exchange property satisfied by semi-monotone sets – namely, if $\mathbf{F} \subset \mathbb{R}^n \times \mathbb{R}^k$ is a graph of a monotone map $\mathbf{f} : X \to \mathbb{R}^k$, then for any subset of n coordinates such that the image X' of $\mathbf{F}$ under projection to the span of these coordinates is n-dimensional, X' is a semi-monotone set, and $\mathbf{F}$ is the graph of a monotone map on X' (see Theorem 7 below).

For $k = 0$, we recover the main statements about semi-monotone sets proved in [1]. Moreover, the proof here is simpler than in [1]. As a result we now have a full supply of regular cells (of all dimensions), and hence we are a step nearer to the proof of Conjecture 1.

Note that Conjecture 1 does not follow from results in the literature on the existence of definable triangulations adapted to a given finite family of definable subsets of $\mathbb{R}^n$ (such as [8, 3]), since all the proofs use a preparatory linear change of coordinates in order for the given definable sets to be in a good position with respect to coordinate projections. Since we are concerned with the graphs and the level sets of a function, in order to prove Conjecture 1 we are not allowed to make any change of coordinates which involves the last coordinate. Pawłucki [5] has considered the problem of obtaining a regular cell decomposition with a restriction on the allowed change in coordinates – namely, only permutations of the coordinates are allowed. In this setting Pawłucki obtains a decomposition whose full dimensional cells are regular. Note that even if this decomposition can be carried through so that all cells (including those of positive codimension) are regular, it would not be enough for our purposes since we cannot allow a change of the last coordinate.

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1. Semi-monotone sets

In what follows we fix an o-minimal structure over $\mathbb{R}$, and consider only sets and maps that are definable in this structure (unless explicitly stated otherwise). The reader may assume that all sets and maps in this paper are either real semialgebraic or subanalytic.

Definition 1. Let $L_{j,\sigma,c} := \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n | \ x_j \sigma c\}$ for $j = 1, \dots, n$, $\sigma \in \{<, =, >\}$, and $c \in \mathbb{R}$. Each intersection of the kind

$$C := L_{j_1,\sigma_1,c_1} \cap \cdots \cap L_{j_m,\sigma_m,c_m} \subset \mathbb{R}^n,$$

where $m = 0, \dots, n$, $1 \le j_1 < \cdots < j_m \le n$, $\sigma_1, \dots, \sigma_m \in \{<, =, >\}$, and $c_1, \dots, c_m \in \mathbb{R}$, is called a *coordinate cone* in $\mathbb{R}^n$.

Each intersection of the kind

$$S := L_{j_1,=,c_1} \cap \cdots \cap L_{j_m,=,c_m} \subset \mathbb{R}^n,$$

where $m = 0, \dots, n$, $1 \le j_1 < \cdots < j_m \le n$, and $c_1, \dots, c_m \in \mathbb{R}$, is called an *affine coordinate subspace* in $\mathbb{R}^n$.

In particular, the space $\mathbb{R}^n$ itself is both a coordinate cone and an affine coordinate subspace in $\mathbb{R}^n$.

Throughout the paper we assume that the empty set is connected.

Definition 2 ([1])**.** An open (possibly, empty) bounded set $X \subset \mathbb{R}^n$ is called *semi-monotone* if for each coordinate cone C the intersection $X \cap C$ is connected.

Proposition 1 ([1], Lemma 1.2, Corollary 1.4)**.** *The projection of a semi-monotone set X on any coordinate subspace, and the intersection $X \cap C$ with a coordinate cone C are semi-monotone sets.*

The following necessary and sufficient condition of semi-monotonicity shows that in the definition it is enough to consider the intersections of X with affine coordinate subspaces.

Theorem 1. *An open (possibly, empty) bounded set $X \subset \mathbb{R}^n$ is semi-monotone if and only if for each affine coordinate subspace S the intersection $X \cap S$ is connected.*

Lemma 1. *If $X \subset \mathbb{R}^n$ is any connected definable set such that for some $j \in \{1, \dots n\}$ and each $b \in \mathbb{R}$ the intersection $X \cap \{x_j = b\}$ is connected, then the sets $X \cap \{x_j < c\}$ and $X \cap \{x_j > c\}$ are connected for all $c \in \mathbb{R}$.*

Proof. Observe that connectedness is equivalent to path-connectedness for definable sets. Consider any two points $\mathbf{y}$, $\mathbf{z} \in X \cap \{x_j < c\}$, then there is a path $\gamma \subset X$ connecting them. Suppose, for definiteness, that $y_j \le z_j$. Let $\mathbf{w}$ be the point in $\gamma \cap X \cap \{x_j = z_j\}$ which is closest to $\mathbf{y}$ in γ. Then the union of the segment of γ between $\mathbf{y}$ and $\mathbf{w}$, and a path in $X \cap \{x_j = z_j\}$, that connects $\mathbf{w}$ with $\mathbf{z}$, is a path in $X \cap \{x_j < c\}$ connecting $\mathbf{y}$ with $\mathbf{z}$.

The similar argument shows that $X \cap \{x_j > c\}$ is path-connected. □

Proof of Theorem1. If X is semi-monotone, then $X \cap S$ is always connected by the definition.

To prove the converse, observe that since X is connected, and $X \cap \{x_j = b\}$ is connected for every $j = 1, \dots, n$ and every $b \in \mathbb{R}$, the intersections $X \cap \{x_j < c\}$ and $X \cap \{x_j > c\}$ are connected for every $c \in \mathbb{R}$, by Lemma 1. The theorem follows, by the induction on the number of half-spaces which form a coordinate cone, since these intersections can then be taken as X. □

Corollary 1. *An open (possibly, empty) bounded connected set $X \subset \mathbb{R}^n$ is semi-monotone if and only if the intersection $X \cap L_{j,=,c}$ is semi-monotone for every $j = 1, \dots, n$ and every $c \in \mathbb{R}$.*

Proof. The statement easily follows from Theorem 1 by induction on n. □

Definition 3. A bounded upper semi-continuous function f defined on a non-empty semi-monotone set $X \subset \mathbb{R}^n$ is *submonotone* if, for any $b \in \mathbb{R}$, the set $\{\mathbf{x} \in X |\, f(\mathbf{x}) < b\}$ is semi-monotone. A function f is *supermonotone* if $(-f)$ is submonotone.

Notation 2. Let the space $\mathbb{R}^n$ have coordinate functions $x_1, \ldots, x_n$. Given a subset $I = \{x_{j_1}, \ldots, x_{j_m}\} \subset \{x_1, \ldots, x_n\}$, let W be the linear subspace of $\mathbb{R}^n$ where all coordinates in I are equal to zero. By a slight abuse of notation we will denote by $\mathrm{span}\{x_{j_1}, \ldots, x_{j_m}\}$ the quotient space $\mathbb{R}^n/W$. Similarly, for any affine coordinate subspace $S \subset \mathbb{R}^n$ on which all the functions $x_j \notin I$ are constant, we will identify S with its image under the canonical surjection to $\mathbb{R}^n/W$.

Lemma 2. *Let the function $f : X \to \mathbb{R}$ be submonotone (respectively, supermonotone), and let X' be the image of the projection of X to $\mathrm{span}\{x_1, \ldots, x_{n-1}\}$. Then the function $\inf_{x_n} f : X' \to \mathbb{R}$ (respectively, $\sup_{x_n} f : X' \to \mathbb{R}$) is submonotone (respectively, supermonotone).*

Proof. According to Proposition 1, the set X' is semi-monotone. Assume that f is submonotone. Then for any $b \in \mathbb{R}$ the image X'_b of the projection of $\{\mathbf{x} \in X |\, f(\mathbf{x}) < b\}$ to $\mathrm{span}\{x_1, \ldots, x_{n-1}\}$ coincides with $\{(x_1, \ldots, x_{n-1}) \in X' |\, \inf_{x_n} f(\mathbf{x}) < b\}$. Since, by Proposition 1, X'_b is semi-monotone, the function $\inf_{x_n} f$ satisfies the definition of submonotonicity.

The proof that $\sup_{x_n} f$ is supermonotone is analogous. □

Proposition 2 ([1], Theorem 1.7)**.** *An open non-empty bounded set $X \subset \mathbb{R}^n$ is semi-monotone if and only if it satisfies the following conditions. If $X \subset \mathbb{R}^1$ then X is an open interval. If $X \subset \mathbb{R}^n$ then*

$$X = \{(\mathbf{x}, y) |\, \mathbf{x} \in X',\ f(\mathbf{x}) < y < g(\mathbf{x})\}$$

for a submonotone function f and a supermonotone function g, both defined on a semi-monotone set $X' \subset \mathbb{R}^{n-1}$, with $f(\mathbf{x}) < g(\mathbf{x})$ for all $\mathbf{x} \in X'$.

The rest of the paper is organized as follows.

In Section 2, we define the class of monotone functions. These are a special type of definable functions $f : X \to \mathbb{R}$ where X is any non-empty semi-monotone set. We give several different characterizations of monotone functions (Lemma 4, Corollary 2, and Theorem 3). In particular, Lemma 4 should be compared with the Definition 2 above of semi-monotone sets, and Theorem 3 should be compared with the corresponding result, Corollary 1, for semi-monotone sets. We also prove a few useful topological results in this section. In particular, we prove a topological property of semi-monotone sets and graphs of monotone functions that could be viewed as an analog of Schönflies Theorem for semi-monotone sets (see Lemma 7 below).

In Section 3, we generalize the definition of monotone functions and define monotone maps $\mathbf{f} : X \to \mathbb{R}^k$, where $X \subset \mathbb{R}^n$ is a non-empty semi-monotone subset of $\mathbb{R}^n$ (see Definition 8 below). The definition is inductive (induction on n) and is more complicated than the definitions of semi-monotone sets and monotone functions. The combinatorial information regarding the dependence or independence of the map $\mathbf{f}$ with respect to the various coordinates is more subtle and is recorded in a matroid, $\mathbf{m}$, of rank n (see Theorem 6), which is associated with $\mathbf{f}$. We prove several important properties of monotone maps and their associated matroids in Section 3. In particular, we show that if $\mathbf{F} \subset \mathbb{R}^{n+k}$ is the graph of a monotone map $\mathbf{f}: X \to \mathbb{R}^k$, where $X \subset \mathbb{R}^n$, and $I \subset \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$ is a basis of its associated matroid, then the image of $\mathbf{F}$ under the projection to span I is a semi-monotone set, and $\mathbf{F}$ is also the graph of a monotone map defined on this set. We also identify a key property of monotone maps, of being *quasi-affine* (Definition 9, and Theorem 8) which will be used later in an essential way.

In Section 4, we prove several different characterizations of monotone maps including Theorem 9 (which generalizes Lemma 4 from functions to maps) and Theorem 12 (generalizing similarly Theorem 3 from functions to maps). We also prove a topological result namely Theorem 11 (generalizing Lemma 7).

It was proved in [1] that every semi-monotone set is a topologically regular cell. In Section 5 we generalize this theorem to graphs of monotone maps (see Theorem 13). The proof of Theorem 13 is new even in the case of semi-monotone sets, and simpler, as it avoids a more advanced machinery from PL topology that was used in [1].

In Section 6 we give an application of Theorem 13, proving that *toric cubes*, introduced in [4], are topologically regular cells.

Section 7 contains a digest of some propositions, mostly from PL topology, which are used in the proofs in the preceding sections.

2. Monotone functions

Definition 4 ([8]). A definable function f on a non-empty open set $X \subset \mathbb{R}^n$ is called *strictly increasing in the coordinate* x_j, where $j = 1, \ldots, n$, if for any two points $\mathbf{x}, \mathbf{y} \in X$ that differ only in the coordinate x_j, with $x_j < y_j$, we have $f(\mathbf{x}) < f(\mathbf{y})$. Similarly we define the notions of f *strictly decreasing in the coordinate* x_j and f *independent of the coordinate* x_j, the latter meaning that $f(\mathbf{x}) = f(\mathbf{y})$ whenever $\mathbf{x}, \mathbf{y} \in X$ differ only in the coordinate x_j.

Definition 5. A definable function f defined on a non-empty semi-monotone set $X \subset \mathbb{R}^n$ is called *monotone* if it is

(i) both sub- and supermonotone (in particular, bounded and continuous, see Definition 3);
(ii) either strictly increasing in, or strictly decreasing in, or independent of x_j, for each $j = 1, \ldots, n$.

Example 1. The function $x_1^2 + x_2^2$ on the semi-monotone set

$$X = \{x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\} \subset \mathbb{R}^2$$

satisfies (ii) in Definition 5, is submonotone but not supermonotone. Hence this function is not monotone. On the other hand, the function $x_1^2 + x_2^2$ on the semi-monotone set $(0,1)^2$ is monotone.

Remark 1. It follows from the definition that the restriction of a monotone function f to a non-empty set $X \cap \{x_j = c\}$ for any $j = 1, \ldots, n$ and $c \in \mathbb{R}$ is a monotone function in $n-1$ variables. Lemma 4 below implies that the restriction of f to a non-empty $X \cap C$, where C is a coordinate cone in $\mathbb{R}^n$, is also a monotone function. However, as exhibited in Example 2.3, the restriction of a monotone function $f : X \to \mathbb{R}$ to a semi-monotone subset $Y \subset X$ is not necessarily monotone.

Example 2. The function on the semi-monotone set $X = (0,1) \times (-1,1) \subset \mathbb{R}^2$ defined as:

$$x_1 x_2 \text{ when } x_2 \geq 0, \text{ and } (1-x_1)x_2 \text{ when } x_2 \leq 0,$$

is sub- and supermonotone, strictly increasing in x_2 on X, strictly increasing in x_1 on $X \cap \{x_2 \neq 0\}$, but is constant on $X \cap \{x_2 = 0\}$. Hence this function is not monotone.

Definition 6. We say that a monotone function f is *non-constant* in x_j if it is either strictly increasing or strictly decreasing in x_j.

Let a monotone function $f : X \to \mathbb{R}$ on a semi-monotone set $X \subset \mathbb{R}^n$ be non-constant in x_n. Let

$$F := \{(\mathbf{x}, y) |\, \mathbf{x} \in X,\ y = f(\mathbf{x})\} \subset \mathbb{R}^{n+1}$$

be the graph of f and U be the projection of F to $\operatorname{span}\{x_1, \dots, x_{n-1}, y\}$.

Lemma 3. *The set U is semi-monotone, and*

$$F = \{(\mathbf{x}, y) |\, \mathbf{x} \in X,\ x_n = g(x_1, \dots, x_{n-1}, y)\}$$

is the graph of a continuous function g on U.

Proof. Since the projection of U to $\operatorname{span}\{x_1, \dots, x_{n-1}\}$, coincides with the projection X' of X to the same space, it is a semi-monotone set by Proposition 1.

Observe that

$$U = \{(x_1, \dots, x_{n-1}, y) |\, (x_1, \dots, x_{n-1}) \in X',\ \inf_{x_n} f < y < \sup_{x_n} f\}.$$

According to Lemma 2, $\inf_{x_n} f$ is submonotone and $\sup_{x_n} f$ is supermonotone. Moreover, $\sup_{x_n} f(x_1, \dots, x_{n-1}) < \inf_{x_n} f(x_1, \dots, x_{n-1})$, for each $(x_1, \dots, x_{n-1}) \in X'$, since f is non-constant in x_n. Therefore the set U is semi-monotone, by Proposition 2.

The function g is defined, since f is non-constant in x_n, and continuous since f is continuous and these functions have the same graph F. □

Lemma 4. *Let f be a bounded continuous function defined on an open bounded non-empty set $X \subset \mathbb{R}^n$, either strictly increasing in, strictly decreasing in, or independent of x_j, for each $j = 1, \dots, n$. Let F be the graph of f. The following three statements are equivalent.*

(i) *The function f is monotone.*
(ii) *For each coordinate cone C in $\mathbb{R}^{n+1}$ the intersection $C \cap F$ is connected.*
(iii) *For each affine coordinate subspace S in $\mathbb{R}^{n+1}$ the intersection $S \cap F$ is connected.*

Proof. We first prove that (i) is equivalent to (ii).

Let f be monotone (in particular, X is semi-monotone), and let C be a coordinate cone in $\mathbb{R}^{n+1}$. It is sufficient to consider the cases when C is defined by a sign condition on the variable y, otherwise, by Proposition 1, the situation is reduced to f defined on a smaller semi-monotone set, the intersection of X with a coordinate cone in $\operatorname{span}\{x_1, \dots, x_n\}$. If $C = \{y < c\}$ for some $c \in \mathbb{R}$, then, since f is submonotone, the projection of $C \cap F$ to $\operatorname{span}\{x_1, \dots, x_n\}$ is semi-monotone, hence connected. Since f is continuous, the pre-image of this projection in F is connected. Similar argument applies in the case when $C = \{y > c\}$.

Suppose that $C = \{y = c\}$. Due to Lemma 3, the intersection $U \cap \{y = c\}$ is semi-monotone, hence connected, and since the function g is continuous, the pre-image of $U \cap \{y = c\}$ in F is connected.

Conversely, let for each coordinate cone C in $\mathbb{R}^{n+1}$ the intersection $C \cap F$ be connected. Let C' be a coordinate cone in $\operatorname{span}\{x_1, \dots, x_n\}$. Then the intersection $F \cap (C' \times \mathbb{R})$ is connected, hence the image of its projection, $C' \cap X$, is connected. It follows that X is semi-monotone. We need to prove that f is both sub- and supermonotone. Let $c \in \mathbb{R}$. Then the set $C' \cap \{f < c\}$ is the image under the projection to $\mathbb{R}^n$ of the connected set $C \cap F$ where $C := (C' \times \mathbb{R}) \cap \{y < c\}$. Since f is continuous, $C' \cap \{f < c\}$ is connected, hence f is submonotone. The similar arguments show that f is supermonotone.

Now we prove that (ii) is equivalent to (iii).

If (ii) is satisfied, then for each S the intersection $S \cap F$ is connected, since S ia a particular case of the coordinate cone. The converse follows from Lemma 1 by a straightforward induction on the number of strict inequalities defining the coordinate cone. □

Corollary 2. *Under the conditions of Lemma 4 and assuming X connected, the non-constant function f is monotone if and only if*

(i) *for every x_j and every $c \in \mathbb{R}$ the intersection $F \cap \{x_j = c\}$ is either empty, or the graph of a monotone function in $n-1$ variables, and*
(ii) *for every $b \in \mathbb{R}$ the intersection $F \cap \{y = b\}$ is either empty, or the graph of a monotone function in $n-1$ variables.*

Proof. The statement easily follows from Lemma 4 by induction on n. □

Remark 2. In Lemma 8 we will prove that the requirement (ii) alone in Corollary 2 is a necessary and sufficient for f to be monotone. In Theorem 3 we will show that *fixing* any j in the part (i) of Corollary 2, and adding the requirement for $\inf_{x_j} f$ and $\sup_{x_j} f$ to be sub- and supermonotone functions respectively, makes (i) also a necessary and sufficient condition for f to be monotone.

Corollary 3. *Let $f : X \to \mathbb{R}$ be a monotone function having the graph $F \subset \mathbb{R}^{n+1}$. Then for every $z \in \{x_1, \dots, x_n, y\}$ and every $c \in \mathbb{R}$ each of the intersections $F \cap \{z < c\}$ and $F \cap \{z > c\}$ is either empty or the graph of a monotone function.*

Proof. According to part (ii) of Lemma 4, for any affine coordinate subspace $S \subset \mathbb{R}^{n+1}$, the intersection $F \cap \{z < c\} \cap S$ is connected, since $\{z < c\} \cap S$ is a coordinate cone. Then part (iii) of this lemma implies that $F \cap \{z < c\}$ is either empty or the graph of a monotone map. The case of $F \cap \{z > c\}$ completely analogous. □

Lemma 5. *In the conditions of Lemma 3, the function $g(x_1, \dots, x_{n-1}, y)$ is monotone.*

Proof. The function g is defined on the semi-monotone set U. It has the same graph as the monotone function f, and hence, by Lemma 4, is itself monotone. □

Remark 3. The function g was constructed from f with respect to the variable x_n. An analogous function $g_j(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)$ can be constructed from f with respect to any variable x_j in which f is non-constant, and Lemma 5 implies that g_j is monotone. If g_j is non-constant in a variable x_ℓ, then the function constructed from g_j with respect to x_ℓ, coincides with the function

$$g_\ell(x_1, \ldots, x_{\ell-1}, y, x_{\ell+1}, \ldots, x_n).$$

The function constructed from g_j with respect to y coincides with f. All these functions have the same graph F as f.

Lemma 6. *Let X be an open, simply connected subset in $\mathbb{R}^n$, and let $\Sigma \subset X$ be a non-empty connected $(n-1)$-dimensional manifold closed in X. Then $X \setminus \Sigma$ has two connected components.*

Proof. There is a short exact sequence (a combination of a cohomological exact sequence of the pair (X, Σ) and the Poincare duality)

$$0 = H_1(X) \to H_0(\Sigma) \to H_0(X \setminus \Sigma) \to H_0(X) \to 0$$

which, given $H_0(\Sigma) = H_0(X) = \mathbb{Z}$, implies that $\mathrm{rank}(H_0(X \setminus \Sigma)) = 2$. □

Remark 4. Here is an alternative proof of Lemma 6, not using an exact sequence.

If Σ is not orientable, choose a normal at a point $x \in \Sigma$ and find a path in Σ that changes its orientation. Lift a path in the direction of the normal and connect its ends. The result is a loop in X intersecting Σ transversally at x. This loop cannot be contractible in X since its intersection index with Σ is ± 1. It follows that Σ is orientable.

If $X \setminus \Sigma$ is connected, take a segment transversal to Σ and connect its ends in $X \setminus \Sigma$. We get a loop in X which intersection index with Σ is ± 1. Thus, $X \setminus \Sigma$ is not connected.

Assume Σ is oriented. Every point $x \in X \setminus \Sigma$ can be connected in $X \setminus \Sigma$ to a point $v \in \Sigma$ by a path γ such that $\gamma \setminus \{v\} \subset X \setminus \Sigma$. If the path γ' for a point x' gets to Σ at the point v' from the same side of Σ as x, connect v and v' by a path ρ in Σ, then lift ρ along the normals to Σ. We get a path connecting x and x' in $X \setminus \Sigma$. It follows that $X \setminus \Sigma$ has exactly two connected components.

Lemma 7. *Let X be a semi-monotone set in $\mathbb{R}^n$ and $\Sigma \subset X$ a graph of a monotone function $x_n = h_n(x_1, \ldots, x_{n-1})$ on some semi-monotone $Y \subset \mathbb{R}^{n-1}$, such that $\partial\Sigma \subset \partial X$. Then $X \setminus \Sigma$ is a union of two semi-monotone sets.*

Proof. First notice that, by Lemma 6, $X \setminus \Sigma$ has two connected components, X_+ and X_-. For any variable x_j, $j = 1, \ldots, n$, and any $c \in \mathbb{R}$ the intersection $X \cap \{x_j = c\}$ is semi-monotone due to Corollary 1, while $\Sigma \cap \{x_j = c\}$ is either empty or the graph of a monotone function due to Corollary 2.

The rest of the proof is by induction on n, the base for $n = 1$ being trivial. If h_n is constant in each variable $x_1, \ldots, x_{n-1}$ then the statement of the theorem is trivially true. Let $X \cap \{x_j = c\}$ be non-empty for some variable x_j, $j = 1, \ldots, n$, and some $c \in \mathbb{R}$. Note that if h_n is non-constant in x_j, where $j < n$, then according to Remark 3, Σ is the graph of a monotone function $x_j = h_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ on some semi-monotone set Y_j. Now let $j = 1, \ldots, n$. If $\Sigma \cap \{x_j = c\} = \emptyset$, then either $X_+ \cap \{x_j = c\} = X \cap \{x_j = c\}$ or $X_- \cap \{x_j = c\} = X \cap \{x_j = c\}$, in any case the intersection is semi-monotone. Assume now that $\Sigma \cap \{x_j = c\} \neq \emptyset$. Observe that $\Sigma \not\subset \{x_j = c\}$, since h_n is not a constant function, and Σ is the graph of h_n. Hence, $(X_+ \cup X_-) \cap \{x_j = c\} \neq \emptyset$.

Both intersections, $X_+ \cap \{x_j = c\}$ and $X_- \cap \{x_j = c\}$ are non-empty. Indeed, if $j < n$ and $\{x_j = c\}$ contains a point $p \in \Sigma$, it also includes an open interval of a straight line parallel to x_n-axis containing p. The two parts into which p divides that interval belong one to X_+ and another to X_-. If $j = n$ then the restriction of the function h_n to a straight line parallel to x_i-axis, such that h_n is non-constant in x_i, has the graph which is a subset of Σ and has non-empty intersections with both $\{x_n < 0\}$ and $\{x_n > 0\}$.

By the inductive hypothesis, $(X \cap \{x_j = c\}) \setminus (\Sigma \cap \{x_j = c\})$ is a union of two semi-monotone sets. It follows that one of the connected components of $(X \cap \{x_j = c\}) \setminus (\Sigma \cap \{x_j = c\})$ lies in X_+ while another in X_-. Hence, the intersection of $\{x_j = c\}$ with each of X_+ and X_- is semi-monotone.

Finally, in the case when h_n is independent of x_j, $j < n$, the set Σ is a cylinder over the graph $\Sigma \cap \{x_j = c\}$ of a monotone function, for any $c \in \mathbb{R}$, and therefore one of the connected components of $(X \cap \{x_j = c\}) \setminus (\Sigma \cap \{x_j = c\})$ lies in X_+ while another X_-. It follows that the intersection of $\{x_j = c\}$ with each of X_+ and X_- is semi-monotone.

Corollary 1 now implies that each of X_+ and X_- is semi-monotone. □

Lemma 8. *Let $f : X \to \mathbb{R}$ be a continuous, bounded, non-constant function defined on a non-empty semi-monotone set X. The function f is monotone if and only if*

(i) *it is either strictly increasing in, or strictly decreasing in, or independent of x_j, for each $j = 1, \ldots, n$;*
(ii) *for every $b \in \mathbb{R}$ the set $\{\mathbf{x} \in X |\, f(\mathbf{x}) = b\}$ is either empty, or a graph of a monotone function in $n - 1$ variables.*

Proof. If f is monotone, then (i) follows from the definition of a monotone function, while (ii) is the statement of Corollary 2.

Conversely, suppose the conditions (i), (ii) are true.

Let $\Sigma := X \cap \{f(\mathbf{x}) = b\}$. Since Σ is a level set of a continuous function, $\partial\Sigma \subset \partial X$. By Lemma 7, $X \setminus \Sigma$ is a union of two semi-monotone sets. The condition (i) implies that one of these sets is $X \cap \{f(\mathbf{x}) < b\}$ and another is $X \cap \{f(\mathbf{x}) > b\}$. It follows that f is both sub- and supermonotone. □

Theorem 3. *A continuous function f, defined on a non-empty open bounded set $X \subset \mathbb{R}^n$, and not independent of x_n, is monotone if and only if it satisfies the following properties:*

(i) *f is either strictly increasing in or strictly decreasing in or independent of each of the variables x_j, where $j = 1, \ldots, n$;*
(ii) $\inf_{x_n} f$ *and* $\sup_{x_n} f$ *are sub- and supermonotone functions, respectively, in variables $x_1, \ldots, x_{n-1}$;*
(iii) *the intersection of X with any straight line parallel to the x_n-axis is either empty or an open interval;*
(iv) *the restriction of f to each non-empty set $X \cap \{x_n = a\}$, where $a \in \mathbb{R}$, is a monotone function.*

Proof. Assume first that f is not independent of x_n and satisfies the properties (i)–(iii). Let F be the graph of f. Then F can be represented as in Lemma 3,

$$F = \{(\mathbf{x}, y) |\, \mathbf{x} \in X,\ x_n = g(x_1, \ldots, x_{n-1}, y)\},$$

with the function g defined on the domain

$$U = \{(x_1, \ldots, x_{n-1}, y) |\, (x_1, \ldots, x_{n-1}) \in X',\ \inf_{x_n} f < y < \sup_{x_n} f\},$$

where X' is the projection of X to $\operatorname{span}\{x_1, \dots, x_{n-1}\}$. Observe that F is also the graph of g. By the property (ii), and by Proposition 2, the domain U is semi-monotone.

Now suppose that f satisfies also the property (iv). Applying Lemma 8 to g, we conclude that this function is monotone. Hence, by Lemma 4, f is also monotone.

Conversely, suppose that a function $f : X \to \mathbb{R}$, not independent of x_n, is monotone. Then properties (i) and (iii) follow from the definition of a monotone function. By Lemma 3 the set U is semi-monotone, hence, by Proposition 2, the property (ii) is satisfied. Property (iv) follows immediately from Lemma 4. $\Box$

3. Monotone maps

Definition 7. For a non-empty semi-monotone set $X \subset \mathbb{R}^n$ and $k \geq 1$, let

$$\mathbf{f} = (f_1, \dots, f_k) : X \to \mathbb{R}^k$$

be a continuous and bounded map. Let

$$H := \{x_{j_1}, \dots, x_{j_\alpha}, y_{i_1}, \dots, y_{i_\beta}\} \subset \{x_1, \dots, x_n, y_1, \dots, y_k\},$$

where $\alpha + \beta = n$. The set H is called a *basis* if the map

$$(x_{j_1}, \dots, x_{j_\alpha}, f_{i_1}, \dots, f_{i_\beta}) : X \to \mathbb{R}^n$$

is injective. Thus, a *system of basis sets* is associated with $\mathbf{f}$.

Example 3. Let $\mathbf{f} : X \to \mathbb{R}^k$ be a linear map on a non-empty semi-monotone set $X \subset \operatorname{span}\{x_1, \dots, x_n\}$, and $\mathbf{F}$ be its graph, which is an open set in n-dimensional linear subspace L in $\operatorname{span}\{\mathbf{x}_1, \dots, x_n, y_1, \dots, y_k\}$. Let $\mathbf{b} := \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis of the space L. Then a set $H := \{x_{j_1}, \dots, x_{j_\alpha}, y_{i_1}, \dots, y_{i_\beta}\}$, where $\alpha + \beta = n$, is a basis for the map $\mathbf{f}$ if and only if the projection of $\mathbf{b}$ to the space $\operatorname{span} H$ is a basis of $\operatorname{span} H$.

Lemma 9. *If $k = 1$, then the system of basis sets associated with $\mathbf{f} = (f_1) : X \to \mathbb{R}$ consists of $\{x_1, \dots, x_n\}$, and each set $\{x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n\}$ such that the function f_1 is either strictly increasing in x_j, or strictly decreasing in x_j (see Definition 4).*

Proof. Clearly, $\{x_1, \dots, x_n\}$ is a basis set.

If f_1 is either strictly increasing in x_j, or strictly decreasing in x_j, then the set $\{x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n\}$ is obviously a basis. Conversely, suppose that $\{x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n\}$ is a basis set, i.e., the restriction of f_1 to every non-empty interval

$$X \cap \{x_1 = c_1, \dots, x_{j-1} = c_{j-1}, x_{j+1} = c_{j+1}, \dots, x_n = c_n\},$$

where $c = (c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_n) \in \mathbb{R}^{n-1}$, is either strictly increasing or strictly decreasing. Let A (respectively, B) be the set of points c for which the restriction is strictly increasing (respectively, strictly decreasing), and X' the projection of X to $\operatorname{span}\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$. Thus, $X' = A \cup B$. Because f_1 is continuous, both sets, A and B, are open in X'. Since, by Proposition 1, X' is connected, we conclude that either $X' = A$ or $X' = B$. $\Box$

Definition 8. For a non-empty semi-monotone set $X \subset \mathbb{R}^n$ and $k \geq 1$, let

$$\mathbf{f} = (f_1, \dots, f_k) : X \to \mathbb{R}^k$$

be a continuous and bounded map and let $\mathbf{F} := \{(\mathbf{x}, \mathbf{y}) |\, \mathbf{x} \in X,\ \mathbf{y} = \mathbf{f}(\mathbf{x})\} \subset \mathbb{R}^{n+k}$ be its graph. Associate with $\mathbf{f}$ a system $\mathbf{m}$ of basis sets as in Definition 7. Define a map $\mathbf{f}$ to be *monotone*, by induction on $n \geq 1$.

If $n = 1$, the map $\mathbf{f}$ is monotone if for every i the function f_i is monotone.

Assume that monotone maps on non-empty semi-monotone subsets of $\mathbb{R}^{n-1}$ are defined.

A map $\mathbf{f}$ is *monotone* if for every $i = 1, \ldots, k$, and every $j = 1, \ldots, n$ such that f_i is not independent of x_j, the following holds.

(i) For every $b \in \mathbb{R}$, the intersection $\mathbf{F} \cap \{y_i = b\}$ (considered as a set in $\operatorname{span}\{x_1, \ldots, x_n, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k\}$), when non-empty, is the graph of a monotone map, denoted by $\mathbf{f}_{i,j,b}$, from a semi-monotone subset of $\operatorname{span}\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}$, into $\operatorname{span}\{y_1, \ldots, y_{i-1}, x_j, y_{i+1}, \ldots, y_k\}$.

(ii) The system of basis sets associated with $\mathbf{f}_{i,j,b}$ does not depend on $b \in \mathbb{R}$.

Remark 5. It follows from Theorem 6 that if the conditions in Definition 8 hold for any one j, then they hold for every j such that f_i is not independent of x_j.

Example 4. It is easy to check that an affine map $\mathbf{f} :\ X \to \mathbb{R}^k$ on a non-empty semi-monotone set $X \subset \operatorname{span}\{x_1, \ldots, x_n\}$ is a monotone map (cf. Example 3).

Lemma 10. *If the map* $\mathbf{f} := (f_1, \ldots, f_k) :\ X \to \mathbb{R}^k$ *is monotone, then the map* $(f_1, \ldots, f_{k-1}) :\ X \to \mathbb{R}^{k-1}$ *is also monotone.*

Proof. For any map $\mathbf{g} :\ X \to \mathbb{R}^k$, let $[\mathbf{g}] :\ X \to \mathbb{R}^{k-1}$ denote the map obtained from $\mathbf{g}$ by removing the k-th component.

The proof is by induction on n, with the base $n = 1$ being trivial. Choose any $b \in \mathbb{R}$. By the inductive hypothesis applied to the monotone map $\mathbf{f}_{i,j,b}$, where $i \neq k$, the map $[\mathbf{f}_{i,j,b}]$ is monotone. But $[\mathbf{f}_{i,j,b}]$ coincides with $[\mathbf{f}]_{i,j,b}$. Hence the requirements (i) and (ii) in Definition 8 are proved for $[\mathbf{f}]$. □

Theorem 4. *The map* $\mathbf{f} = (f_1) :\ X \to \mathbb{R}$ *is monotone if and only if the function* f_1 *is monotone.*

Proof. Suppose that $\mathbf{f}$ is a monotone map, let $\mathbf{F}$ be its graph. If f_1 is a constant function, then it is trivially monotone. Suppose that f_1 is non-constant. Then, by item (i) of Definition 8, the item (ii) of Lemma 8 is satisfied.

It remains to show that the condition (i) of Lemma 8 is also valid for the function f_1. We prove this by induction on n, the base for $n = 1$ being a requirement in Definition 8.

Suppose that for some $c_1, \ldots, c_{n-1} \in \mathbb{R}$ the set $X \cap \{x_1 = c_1, \ldots, x_{n-1} = c_{n-1}\}$ is non-empty, and the restriction of f_1 to this set is independent of x_n, i.e., identically equal to some $b \in \mathbb{R}$. We now prove that f_1 is independent of x_n, i.e., remains a constant *for all* fixed values of $x_1, \ldots, x_{n-1}$.

Since we assumed f_1 to be non-constant on X, there exists j such that f_1 is not independent of x_j. As $\mathbf{f}$ is a monotone map, by item (i) in Definition 8, $\mathbf{F} \cap \{y_1 = b\}$ is a graph of a monotone map $\mathbf{f}_{1,j,b} = (f_{1,j,b})$ defined on a semi-monotone subset of $\operatorname{span}\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}$. Then $j \neq n$, since $\mathbf{F} \cap \{x_1 = c_1, \ldots, x_{n-1} = c_{n-1}, y_1 = b\}$ contains a non-empty interval.

The function $f_{1,j,b}$ is monotone by the inductive hypothesis. The restriction of $f_{1,j,b}$ on $\{x_1 = c_1, \ldots, x_{j-1} = c_{j-1}, x_{j+1} = c_{j+1}, \ldots, x_{n-1} = c_{n-1}\}$ is constant (identically equal to c_j). Then, by item (i) of Lemma 8, $f_{1,j,b}$ is constant for all fixed values of $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1}$. Since, by item (ii) of Definition 8, the system of basis sets associated with the map $\mathbf{f}_{1,j,b}$ does not depend on $b \in \mathbb{R}$, the property of $f_{1,j,b}$ to be constant for all fixed values of $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}$ holds for all b. Since the graph $\{x_j = f_{1,j,b}\}$ of the function $f_{1,j,b}$ is the level set of f_1 at b, and the union of all level sets for all values b is X, it follows that for all fixed values of variables $x_1, \dots, x_{n-1}$ the function f_1 is independent of x_n.

It follows that if the restriction of f_1 to $X \cap \{x_1 = c_1, \dots, x_{n-1} = c_{n-1}\}$ is not independent of x_n for some $c_1, \dots, c_{n-1} \in \mathbb{R}$, then it is not independent for all fixed values of $x_1, \dots, x_{n-1}$. Repeating the argument from the proof of Lemma 9 (for $j = n$), we conclude that f_1 is either strictly increasing in, or strictly decreasing in, or independent of x_n.

Replacing in this argument n by each of $1, \dots, n-1$, we conclude that the item (i) of Lemma 8 is satisfied, and therefore the function f_1 is monotone.

Now suppose the function f_1 is monotone. Then, by item (ii) in Lemma 8, the map $\mathbf{f} = (f_1)$ satisfies (i) in Definition 8. To prove that $\mathbf{f}$ satisfies also (ii) in Definition 8, fix the numbers $b \in \mathbb{R}$ and $j \in \{1, \dots, n\}$. Suppose that $\{x_1, \dots, x_{n-1}\}$ is a basis set of $\mathbf{f}_{1,j,b}$, i.e., the fibre $\{x_1 = c_1, \dots, x_{n-1} = c_{n-1}, f_1 = b\}$, whenever non-empty, is a single point for each sequence $c_1, \dots c_{n-1} \in \mathbb{R}$. Then, according to (ii) in Definition 5, f_1 is non-constant on $\{x_1 = c_1, \dots, x_{n-1} = c_{n-1}\}$, in particular, the fibre $\{x_1 = c_1, \dots, x_{n-1} = c_{n-1}, f_1 = a\}$, whenever non-empty, is a single point for any other value $a \in \mathbb{R}$ of f_1. It follows that the set $\{x_1, \dots, x_{n-1}\}$ is a basis also for $\mathbf{f}_{1,j,a}$.

Replacing in this argument n by each of $1, \dots, n-1$, we conclude that the *system* of basis sets of $\mathbf{f}_{1,j,b}$ does not depend on b, hence $\mathbf{f}$ is monotone. □

Corollary 4. *If the map* $\mathbf{f} = (f_1, \dots, f_k) : X \to \mathbb{R}^k$ *is monotone, then every function* f_i *is monotone.*

Proof. Lemma 10 implies that the map $(f_i) : X \to \mathbb{R}$ is monotone for every $i = 1, \dots, k$. Then, by Theorem 4, the function f_i is monotone. □

Remark 6. The converse to Corollary 4 is false when $n > 1$ and $k > 1$. For example, consider the map $\mathbf{f} = (f_1, f_2) : (\frac{1}{2}, 1)^2 \to \mathbb{R}^2$, where

$$f_1 = x_2/x_1 \quad \text{and} \quad f_2 = x_1 - x_2.$$

Both functions, f_1 and f_2, are monotone on $(\frac{1}{2}, 1)^2$ but their level curves, $\{f_1 = 1\}$ and $\{f_2 = 0\}$, coincide while all other pairs of level curves are different. It follows that the map $\mathbf{f}_{1,1,1}$ has two basis sets, $\{x_1\}$ and $\{x_2\}$, while $\mathbf{f}_{1,1,2}$ has three basis sets, $\{x_1\}$, $\{x_2\}$ and $\{y_2\}$. Thus, the condition (ii) of Definition 8 is not satisfied for $\mathbf{f}$.

Lemma 11. *Let* $\mathbf{f} : X \to \mathbb{R}^k$ *be a monotone map,* $\mathbf{F}$ *the graph of* $\mathbf{f}$*. Then for any* $\{i_1, \dots, i_\beta\} \subset \{1, \dots, k\}$ *and* $b_1, \dots, b_\beta \in \mathbb{R}$*, where* $\beta \leq k$*, the intersection* $\mathbf{F}_\beta := \mathbf{F} \cap \{y_{i_1} = b_1, \dots, y_{i_\beta} = b_\beta\}$ *is either empty or the graph of a monotone map defined on a semi-monotone set in some space* $\operatorname{span}\{x_{j_1}, \dots, x_{j_\alpha}\}$*, where* $\alpha + \beta \geq n$*.*

Proof. The proof is by induction on β. If $\beta = 0$ (the base of the induction), then $\mathbf{F}_0 = \mathbf{F}$ and hence is the graph of a monotone map from $X \subset \operatorname{span}\{x_1, \dots, x_n\}$ to $\operatorname{span}\{y_1, \dots, y_k\}$. Let $I_0 = \emptyset$, and $J_0 = \{x_1, \dots, x_n\}$.

By the inductive hypothesis,

$$\mathbf{F}_{\beta-1} := \mathbf{F} \cap \{y_{i_1} = b_1, \dots, y_{i_{\beta-1}} = b_{\beta-1}\}$$

is a graph of a monotone map $\mathbf{h} = (h_1, \dots, h_k)$ from a semi-monotone subset of span $J_{\beta-1}$ to

$$\mathrm{span}((\{y_1, \dots, y_k\} \setminus I_{\beta-1}) \cup (\{x_1, \dots, x_n\} \setminus J_{\beta-1})).$$

If the function h_{i_β} is constant in each of the variables in $J_{\beta-1}$, then the graph $\mathbf{F}_{\beta-1}$ lies in $\{y_{i_\beta} = c\}$ for some $c \in \mathbb{R}$, hence the intersection $\mathbf{F}_\beta = \mathbf{F}_{\beta-1} \cap \{y_{i_\beta} = b_\beta\}$ is either empty (when $c \neq b_\beta$), or coincides with the graph $\mathbf{F}_{\beta-1}$ (when $c = b_\beta$). In this case we consider $\mathbf{F}_\beta$ as the graph of the same map $\mathbf{h}$, and assume $I_\beta = I_{\beta-1}$, $J_\beta = J_{\beta-1}$.

Suppose now that h_{i_β} is not constant in some of $J_{\beta-1}$, let it be, for definiteness, $x_{j_{\alpha+1}}$. Let $I_\beta := I_{\beta-1} \cup \{y_{i_\beta}\}$ and $J_\beta := J_{\beta-1} \setminus \{x_{j_{\alpha+1}}\}$. Then, by Definition 8, $\mathbf{F}_\beta = \mathbf{F}_{\beta-1} \cap \{y_{i_\beta} = b_\beta\}$ is the graph of the monotone map $\mathbf{h}_{i_\beta, x_{j_{\alpha+1}}, b_\beta}$ from a semi-monotone subset of $\mathrm{span}(J_\beta)$ to

$$\mathrm{span}((\{y_1, \dots, y_k\} \setminus I_\beta) \cup (\{x_1, \dots, x_n\} \setminus J_\beta)).$$

□

Notation 5. For a subset $H \subset \{x_1, \dots, x_n, y_1, \dots, y_k\}$, let $\mathbf{f}(H) : X \to \mathbb{R}^{|H|}$ denote a map defined by the functions corresponding to the elements of H.

Lemma 12. *If H is a basis set of a monotone map $\mathbf{f} : X \to \mathbb{R}^k$, then every component of $\mathbf{f}(H)$ is non-constant on X.*

Proof. If all components of $\mathbf{f}(H)$ are coordinate functions, then this is obvious. If non-coordinate functions exist and all are constants, then the dimension of each non-empty fibre of $\mathbf{f}(H)$ equals to the number of these functions, i.e., greater than zero, which contradicts to H being a basis set. Take a component f_i of $\mathbf{f}(H)$ which is not a constant on X, thus it is non-constant in some variable x_j. Then each non-empty fibre of f_i is the graph of a monotone map $\mathbf{f}_{i,j,b}$, according to (i) in the Definition 8. Observe that $H \setminus \{y_i\}$ is a basis set for the map $\mathbf{f}_{i,j,b}$, since the fibres of $\mathbf{f}_{i,j,b}(H \setminus \{y_i\})$ are exactly those fibres of $\mathbf{f}(H)$ on which $f_i = b$. If each component in $\mathbf{f}(H \setminus \{y_i\})$ is constant on all $(n-1)$-dimensional fibres of f_i, then we get a contradiction with $H \setminus \{y_i\}$ being a basis for $\mathbf{f}_{i,j,b}$. Continuing by induction, we conclude that all non-coordinate functions of $f(H)$ are non-constant on X. □

Theorem 6. *Let $\mathbf{f} : X \to \mathbb{R}^k$ be a monotone map on a non-empty semi-monotone $X \subset \mathbb{R}^n$, and $\mathbf{F}$ its graph. Then*

(i) *The system $\mathbf{m}$ of basis sets associated with $\mathbf{f}$ is a matroid of rank n.*
(ii) *For each independent set $I = \{x_{j_1}, \dots, x_{j_\alpha}, y_{i_1}, \dots, y_{i_\beta}\}$ of $\mathbf{m}$, and all sequences $c_1, \dots, c_\alpha, b_1, \dots, b_\beta \in \mathbb{R}$, the non-empty intersections*

$$\mathbf{F} \cap \{x_{j_1} = c_1, \dots, x_{j_\alpha} = c_\alpha, y_{i_1} = b_1, \dots, y_{i_\beta} = b_\beta\}$$

are graphs of monotone maps, having the same associated matroid $\mathbf{m}_I$ of rank $n - \alpha - \beta$ (the contraction of $\mathbf{m}$ by I). In particular, all such intersections have the same dimension $n - \alpha - \beta$.

Proof. By the definition of a matroid, to prove (i), we need to check the *basis axiom* ([9], p. 8), which states that for any two basis subsets $H, G \subset \{x_1, \dots, x_n, y_1, \dots, y_k\}$, if $h \in H \setminus G$ then there exists $g \in G \setminus H$ such that the set $\{g\} \cup (H \setminus \{h\})$ is a basis set. We prove this property by induction on n simultaneously with the property (ii). The base for $n = 1$ is obvious for both (i) and (ii).

First we prove the inductive step for (i). Fix any two basis subsets H, G, and an element $h \in (H \setminus G)$. Consider the set $Z = H \setminus \{h\}$. We prove that each non-empty fibre of $\mathbf{f}(Z)$ is a graph of a univariate monotone map. This is obvious if all components of $\mathbf{f}(Z)$ are coordinate functions. If some non-coordinate functions exist then, by Lemma 12, they are non-constant. Let f_i be one of them. In particular, f_i is not independent of some variable x_j. According to (i) in the Definition 8, each non-empty set $\mathbf{F} \cap \{y_i = b\}$ is the graph of a monotone map $\mathbf{f}_{i,j,b}$. Observe that $H \setminus \{y_i\}$ is a basis set for the map $\mathbf{f}_{i,j,b}$ since the fibres of $\mathbf{f}_{i,j,b}(H \setminus \{y_i\})$ are exactly those fibres of $\mathbf{f}(H)$ on which $f_i = b$. Note that the matroid $\mathbf{m}_i$ associated with $\mathbf{f}_{i,j,b}$ is the *contraction* of the matroid $\mathbf{m}$ by y_i. Since H is a basis set for $\mathbf{f}$, all these fibres are single points, hence $\mathbf{f}_{i,j,b}(H \setminus \{y_i\})$ is injective. Recall that $H \setminus \{y_i\} = Z \setminus \{h\}$.

By the inductive hypothesis of (ii), all non-empty fibres of $\mathbf{f}_{i,j,b}(H \setminus \{y_i\})$ are one-dimensional graphs of monotone functions. Since these fibres coincide with the fibres of $\mathbf{f}(Z)$ on which $f_i = b$, we conclude that all non-empty fibres of $\mathbf{f}(Z)$ are one-dimensional graphs of monotone maps.

Now let $z \in \{x_1, \dots, x_n, y_1, \dots, y_k\}$, and let ψ be the corresponding function. Suppose that $Z \cup \{z\}$ is not a basis set. Then the restriction of ψ to some fibre Ψ of the map $\mathbf{f}(Z)$ has fibres of dimension greater than zero. But, as we proved above, all fibres of $\mathbf{f}(Z)$ are one-dimensional graphs of monotone maps, hence ψ is constant on Ψ. Then item (ii) in the inductive hypothesis implies that ψ is constant on each fibre of $\mathbf{f}(Z)$.

Suppose that the basis axiom is violated, i.e., given $h \in H$, for every $g \in G \setminus H$ the set $Z \cup \{g\}$ is not a basis. It follows that the function corresponding to every $g \in G \setminus H$ is constant on fibres of $\mathbf{f}(Z)$ (which are curves). On the other hand, $G \cap H \subset Z$, so any function corresponding to $g \in G \cap H$ is constant on fibres of $\mathbf{f}(Z)$. Therefore functions corresponding to all $g \in G$ are constant on fibres of $\mathbf{f}(Z)$, hence fibres of $\mathbf{f}(G)$ are curves, thus G is not a basis set, which is a contradiction.

Now we prove the inductive step of (ii). It is sufficient to prove the statement for $|I| = 1$ since the case of the general I will follow by induction on $|I|$.

If $I = \{f_i\}$ for some $i = 1, \dots, k$, then according to Lemma 12, f_i is not a constant, and the statement follows immediately from Definition 8.

Now suppose that $I = \{x_\ell\}$.

By Definition 8 it is sufficient to prove that

(a) for all $i = 1, \dots k$, $j = 1, \dots, n$ and $b \in \mathbb{R}$, such that f_i is non-constant in x_j, the intersection $\mathbf{F} \cap \{x_\ell = c\} \cap \{y_i = b\}$ is the graph of a monotone map $\mathbf{g}_b$ on a semi-monotone set in $\mathrm{span}\{x_1, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$;

(b) The matroid associated with $\mathbf{g}_b$ is the same for every $b \in \mathbb{R}$.

By Definition 8, the set $\mathbf{F} \cap \{y_i = b\}$ is the graph of a monotone map $\mathbf{f}_{i,j,b}$. Applying the inductive hypothesis to this monotone map we conclude that the set $\mathbf{F} \cap \{y_i = b\} \cap \{x_\ell = c\}$ is the graph of a monotone map $\mathbf{g}_b$ on an $(n-2)$-dimensional semi-monotone set. Hence, (a) is established.

By Definition 8, the maps $\mathbf{f}_{i,j,b}$ have the same system of basis sets for all $b \in \mathbb{R}$. By the inductive hypothesis, this system is the matroid $\mathbf{m}_i$. The common matroid for the maps $\mathbf{g}_b$ is obtained from $\mathbf{m}_i$ as follows. Select all basis sets in $\mathbf{m}_i$ which contain the element x_ℓ, and remove this element from each of the selected sets. The resulting system of sets forms the matroid for $\mathbf{g}_b$. Note that this matroid is the *contraction* of $\mathbf{m}_i$ by x_ℓ. Since $\mathbf{m}_i$ is independent of b, so does this matroid, which proves (b). □

For any $I \subset \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$, and $\mathbf{F} \subset \mathbb{R}^{n+k}$, let $T := \operatorname{span}(I)$, and $\rho_T : \mathbf{F} \to T$ be the projection map.

Theorem 7. *Let $\mathbf{f} : X \to \mathbb{R}^k$ be a monotone map on a non-empty semi-monotone $X \subset \mathbb{R}^n$ having the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$. Let $\mathbf{m}$ be the matroid associated with $\mathbf{f}$, H a basis set of $\mathbf{m}$, and $T := \operatorname{span}(H)$. Then $\rho_T(\mathbf{F})$ is semi-monotone, and $\mathbf{F}$ is the graph of a monotone map on $\rho_T(\mathbf{F})$.*

Proof. Suppose that $H = \{x_1, \ldots, x_{j-1}, y_i, x_{j+1}, \ldots, x_n\}$ for some i and j.

Lemma 3 implies that $\rho_T(\mathbf{F})$ is a semi-monotone set. Observe that $\mathbf{F}$ is the graph of a continuous map $(f_1, \ldots, f_{i-1}, x_j, f_{i+1}, \ldots, f_k)$ on $\rho_T(\mathbf{F})$. This map is monotone by the definition, since according to Theorem 6, for each $h \in \{y_1, \ldots, y_{i-1}, x_j, y_{i+1}, \ldots, y_k\}$ and for each $b \in \mathbb{R}$, the intersection $\mathbf{F} \cap \{h = b\}$, if non-empty, is the graph of a monotone map with the matroid independent of b.

For an arbitrary basis H, by the matroid's basis axiom, there exists a sequence

$$H_0 := \{x_1, \ldots, x_n\}, H_1, \ldots, H_{t-1}, H_t = H$$

such that the basis $H_{\ell+1}$ is obtained from the basis H_ℓ by replacing a variable of the kind x_j by a variable of the kind y_i. Applying the argument, described above to each such replacement, we conclude the proof. □

Definition 9. Let a bounded continuous map $\mathbf{f} = (f_1, \ldots, f_k)$ defined on an open bounded non-empty set $X \subset \mathbb{R}^n$ have the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$. We say that $\mathbf{f}$ is *quasi-affine* if for any $T := \operatorname{span}\{x_{j_1}, \ldots, x_{j_\alpha}, y_{i_1}, \ldots, y_{i_\beta}\}$, where $\alpha + \beta = n$, the projection ρ_T is injective if and only if the image $\rho_T(\mathbf{F})$ is n-dimensional.

Remark 7. Observe that in Definition 9 the property of ρ_T to be injective is equivalent to $\{x_{j_1}, \ldots, x_{j_\alpha}, y_{i_1}, \ldots, y_{i_\beta}\}$ being a basis set associated with $\mathbf{f}$.

In the case of a function, the property to be quasi-affine is equivalent to the condition on the function to be either strictly increasing in, strictly decreasing in, or independent of any variable.

Lemma 13. *The system of basis sets associated with a quasi-affine map $\mathbf{f} : X \to \mathbb{R}^k$, having the graph $\mathbf{F}$, coincides with the system of bases of the matroid associated with the affine map whose graph is the tangent space of $\mathbf{F}$ at a generic smooth point.*

Proof. Follows immediately from Definitions 7 and 9.

Observe that for some smooth points of $\mathbf{F}$ the matroids associated with the affine maps, whose graphs are tangent spaces of $\mathbf{F}$ at these points, may be different from the matroid at a generic smooth point (e.g., the origin for $\mathbf{f}(x) = x^3$ on $X = (-1, 1)$). □

Theorem 8. *Every monotone map $\mathbf{f} : X \to \mathbb{R}^k$ is quasi-affine.*

Proof. Let $\mathbf{f} : X \to \mathbb{R}^k$ be a monotone map having the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$, and $T := \operatorname{span}\{x_{j_1}, \ldots, x_{j_\alpha}, y_{i_1}, \ldots, y_{i_\beta}\}$, where $\alpha + \beta = n$.

If the projection ρ_T is injective, then obviously $\rho_T(\mathbf{F})$ is n-dimensional.

Conversely, observe that, by Lemma 10, the map $\mathbf{g} := (f_{i_1}, \ldots, f_{i_\beta})$ is monotone. Suppose that, contrary to the claim, ρ_T is not injective. Lemma 11 implies that if the fiber of a monotone map over a value is 0-dimensional then it is a single point. Hence, there are two points

$$(a_{j_1},\dots,a_{j_\alpha},a_{i_1},\dots,a_{i_\beta}) \text{ and } (b_{j_1},\dots,b_{j_\alpha},b_{i_1},\dots,b_{i_\beta})$$

in $\operatorname{span}\{x_{j_1},\dots,x_{j_\alpha},y_{i_1},\dots,y_{i_\beta}\}$ such that the fiber of $\mathbf{g}' := \mathbf{g}|_{x_{j_1}=a_{j_1},\dots,x_{j_\alpha}=a_{j_\alpha}}$ over $(a_{i_1},\dots,a_{i_\beta})$ is a single point, while the fiber of $\mathbf{g}'' := \mathbf{g}|_{x_{j_1}=b_{j_1},\dots,x_{j_\alpha}=b_{j_\alpha}}$ over $(b_{i_1},\dots,b_{i_\beta})$ has the positive dimension. By the part (ii) of Theorem 6, applied to the independent set $\{x_{j_1},\dots,x_{j_\alpha}\}$ as I, the matroids, associated with $\mathbf{g}'$ and $\mathbf{g}''$ coincide. In particular, there exists a point $(b^{(0)}_{i_1},\dots,b^{(0)}_{i_\beta})$ such that the fiber of $\mathbf{g}'$ over this point has the positive dimension. Let $\mathbf{G}_0$ be the graph of $\mathbf{g}'$.

We proceed by induction on $\nu=0,1,\dots,\beta$. According to Lemma 11, for every $\nu \le \beta$ the intersection $\mathbf{G}_\nu := \mathbf{G}_0 \cap \{y_{i_1}=a_{i_1}\dots,y_{i_\nu}=a_{i_\nu}\}$ is the graph of a monotone map $\mathbf{g}^{(\nu)}$ (since the fiber of $\mathbf{g}'$ over $a_{i_1},\dots,a_{i_\beta}$ is 0-dimensional, the set $\mathbf{G}_\nu$ is non-empty). Also, since the fiber of $\mathbf{g}'$ is 0-dimensional, the map $\mathbf{g}^{(\nu)}$ is of the form $\mathbf{g}^{(\nu-1)}_{i_\nu,j,a_{i_\nu}}$ for an appropriate j. Because $\mathbf{g}^{(\nu-1)}$ is monotone, the map $\mathbf{g}^{(\nu-1)}_{i_\nu,j,b^{(\nu-1)}_{i_\nu}}$ has the same matroid as $\mathbf{g}^{(\nu-1)}_{i_\nu,j,a_{i_\nu}}$. In particular, there exists a point $(b^{(\nu)}_{i_{\nu+1}},\dots,b^{(\nu)}_{i_\beta})$ such that the fiber of $\mathbf{g}^{(\nu-1)}_{i_\nu,j,a_{i_\nu}}$ over this point has the positive dimension.

On the last step, for $\nu=\beta-1$ and an appropriate j, the two maps, $\mathbf{g}^{(\beta-1)} = \mathbf{g}^{(\beta-2)}_{i_{\beta-1},j,a_{i_{\beta-1}}}$ and $\mathbf{g}^{(\beta-2)}_{i_{\beta-1},j,b^{(\beta-2)}_{i_{\beta-1}}}$, defined on an interval, have the same matroids. On the other hand, the component of $\mathbf{g}^{(\beta-2)}_{i_{\beta-1},j,a_{i_{\beta-1}}}$, corresponding to y_{i_β}, is a non-constant monotone function on that interval, while the component of $\mathbf{g}^{(\beta-2)}_{i_{\beta-1},j,b^{(\beta-2)}_{i_{\beta-1}}}$, corresponding to y_{i_β}, is a constant function. We get a contradiction. $\square$

Corollary 5. *Let a monotone map $\mathbf{f}: X \to \mathbb{R}^k$ have the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$ and the associated matroid $\mathbf{m}$. A subset $H \subset \{x_1,\dots,x_n,y_1,\dots,y_k\}$ is an independent set of $\mathbf{m}$ if and only if $\dim(\rho_L(\mathbf{F})) = |H|$, where $L := \operatorname{span} H$.*

Proof. Let $|H|=m$. If H is an independent set of $\mathbf{m}$ then, by the matroid theory's Augmentation Theorem ([9], Ch. 1, Section 5), there is a basis set I of $\mathbf{m}$ such that $H \subset I$. By Theorem 8, $\dim(\rho_T(\mathbf{F}))=n$, where $T := \operatorname{span} I$, and therefore $\dim(\rho_L(\mathbf{F}))=m$, since $\rho_L(\mathbf{F})$ is the image of the projection of $\rho_T(\mathbf{F})$ to L.

Conversely, suppose that $\dim(\rho_L(\mathbf{F}))=m$. Clearly, $m \le n$. Observe that for any definable pure-dimensional set U in $\mathbb{R}^{n+k}$, with $\dim U = n$, if $\rho_L(U)=m$ then there is a subset $I \subset \{x_1,\dots,x_n,y_1,\dots,y_k\}$ such that $H \subset I$ and $\dim(\rho_T(U))=n$, where $T=\operatorname{span} I$. Since $\mathbf{F}$, being the graph of continuous map on an open set in $\mathbb{R}^n$, satisfies all the properties of U, there exists a subset I such that $\dim(\rho_T(\mathbf{F}))=n$. Then H is an independent set of $\mathbf{m}$, being a subset of its basis. $\square$

Remark 8. Lemma 13 immediately implies that the matroid associated with a monotone map, having the graph $\mathbf{F}$, coincides with the matroid associated with the affine map whose graph is the tangent space of $\mathbf{F}$ at a generic point.

4. Equivalent definitions of a monotone map, and their corollaries

Lemma 14. *Let $\mathbf{f}: X \to \mathbb{R}^k$ be a monotone map on a semi-monotone set $X \subset \mathbb{R}^n$, and C a coordinate cone in $\mathbb{R}^n$. Then the restriction $\mathbf{f}|_C : X \cap C \to \mathbb{R}^k$ is a monotone map.*

Proof. It is sufficient to consider just the cases of $C = \{x_\ell = c\}$ and $C = \{x_\ell > c\}$. The first case follows directly from Theorem 6, (ii). The proof for the case $C = \{x_\ell > c\}$ can be conducted by induction on n, completely analogous to the last part of the proof of Theorem 6. $\square$

Corollary 6. *Let $\mathbf{F} \subset \mathbb{R}^{n+k}$ be the graph of a monotone map $\mathbf{f} : X \to \mathbb{R}^k$, and let*

$$P := \bigcap_{1 \le j \le n+k} \{(z_1, \dots, z_{n+k}) |\, a_j < z_j < b_j\} \subset \mathbb{R}^{n+k}$$

for some $a_j,\ b_j \in \mathbb{R},\ j = 1, \dots, n+k$. Then $\mathbf{F} \cap P$ is either empty or the graph of a monotone function.

Proof. Let P' be the image of the projection of P to X. By Lemma 14, the restriction $\mathbf{f}_{P'} : P' \to \mathbb{R}^k$ is a monotone map. Theorem 7 allows to apply the same argument to projections of P to other subspaces of $\mathbb{R}^{n+k}$. $\square$

The following theorem is a generalization of Lemma 4 from monotone functions to monotone maps.

Theorem 9. *Let a bounded continuous quasi-affine map $\mathbf{f} = (f_1, \dots, f_k)$ defined on an open bounded non-empty set $X \subset \mathbb{R}^n$ have the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$. The following three statements are equivalent.*

(i) *The map $\mathbf{f}$ is monotone.*
(ii) *For each affine coordinate subspace S in $\mathbb{R}^{n+k}$ the intersection $\mathbf{F} \cap S$ is connected.*
(iii) *For each coordinate cone C in $\mathbb{R}^{n+k}$ the intersection $\mathbf{F} \cap C$ is connected.*

Proof. We first prove that (i) implies (ii). Suppose that $\mathbf{f}$ is monotone. Consider an affine coordinate subspace $S = \{x_{j_1} = c_1, \dots, x_{j_\alpha} = c_\alpha, y_{i_1} = b_1, \dots, y_{i_\beta} = b_\beta\}$ in $\mathbb{R}^{n+k}$. Lemma 14 implies that the intersection $\mathbf{F} \cap \{x_{j_1} = c_1, \dots, x_{j_\alpha} = c_\alpha\}$ is the graph of a monotone map $\mathbf{g}$, and hence connected. Applying Lemma 11 to $\mathbf{g}$, we conclude that $\mathbf{F} \cap S$ is also the graph of a monotone map, and therefore connected.

The part (ii) implies the part (iii) by Lemma 1 and a straightforward induction on the number of strict inequalities defining the coordinate cone.

Now we prove that (iii) implies (i). Suppose the condition (iii) is satisfied. Let C' be a coordinate cone in $\operatorname{span}\{x_1, \dots, x_n\}$. Then the intersection $\mathbf{F} \cap (C' \times \mathbb{R}^k)$ is connected, hence the image of its projection, $C' \cap X$, is connected. It follows that X is semi-monotone.

We prove that $\mathbf{f}$ is a monotone map by induction on n, the base for $n = 1$ being trivial. Choose any i and j such that f_i is non-constant in x_j, and consider the map $\mathbf{f}_{i,j,b}$ for some $b \in \mathbb{R}$. This map and its graph $\mathbf{F} \cap \{y_i = b\}$ inherit from $\mathbf{f}$ and $\mathbf{F}$ properties (i) and (ii) in the conditions of the theorem. Thus, by the inductive hypothesis, $\mathbf{f}_{i,j,b}$ is a monotone map. It remains to prove that the system of basis sets associated with $\mathbf{f}_{i,j,b}$ does not depend on $b \in \mathbb{R}$. Let $T = \operatorname{span}\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$. Any basis set of $\mathbf{f}_{i,j,b}$ is a subset

$$\{x_{j_1}, \dots, x_{j_\alpha}, y_{i_1}, \dots, y_{i_\beta}\} \subset \{x_1, \dots, x_n, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_k\},$$

where $\alpha + \beta = n - 1$, such that the map

$$(x_{j_1}, \dots, x_{j_\alpha}, f_{i_1}, \dots, f_{i_\beta}) :\ \rho_T(\mathbf{F} \cap \{y_i = b\}) \to \mathbb{R}^{n-1}$$

is injective. Since $\mathbf{f}$ is quasi-affine, the injectivity does not depend on the choice of b. $\square$

Corollary 7. *Let* $\mathbf{f} : X \to \mathbb{R}^k$ *be a monotone map having the graph* $\mathbf{F} \subset \mathbb{R}^{n+k}$. *Then for every* $z \in \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$ *and every* $c \in \mathbb{R}$ *each of the intersections* $\mathbf{F} \cap \{z\, \sigma\, c\}$, *where* $\sigma \in \{<, >, =\}$, *is either empty or the graph of a monotone map.*

Proof. Let $\mathbf{F} \cap \{z\, \sigma\, c\} \neq \emptyset$. According to part (iii) of Theorem 9, for any affine coordinate subspace $S \subset \mathbb{R}^{n+k}$, the intersection $\mathbf{F} \cap \{z\, \sigma\, c\} \cap S$ is connected, since $\{z \sigma c\} \cap S$ is a coordinate cone. Then part (ii) of this theorem implies that $\mathbf{F} \cap \{z \sigma c\}$ is the graph of a monotone map. □

The following corollary is a generalization of Definition 5 from monotone functions to monotone maps.

Corollary 8. *Let a bounded continuous map* $\mathbf{f} = (f_1, \ldots, f_k)$ *on a non-empty semi-monotone set* $X \subset \mathbb{R}^n$ *have the graph* $\mathbf{F} \subset \mathbb{R}^{n+k}$. *The map* $\mathbf{f}$ *is monotone if and only if*

(i) *it is quasi-affine;*
(ii) *for every subset* $\{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$ *the intersection*

$$S_{i_1,\ldots,i_m} := \bigcap_{1 \le \ell \le m} \{f_{i_\ell} \sigma_\ell 0\} \subset X,$$

where $\sigma_\ell \in \{<, >\}$, *is semi-monotone.*

Proof. Suppose that the map $\mathbf{f}$ is monotone. Then, by Theorem 9, the condition (i) is satisfied. Let C be a coordinate cone in $\operatorname{span}\{x_1, \ldots, x_n\}$. The intersection $S_{i_1,\ldots,i_m} \cap C$ is the projection on $\operatorname{span}\{x_1, \ldots, x_n\}$ of the intersection of $\mathbf{F}$ with the coordinate cone

$$K := (C \times \mathbb{R}^k) \cap \bigcap_{1 \le \ell \le m} \{y_{i_\ell} \sigma_\ell 0\}$$

in $\operatorname{span}\{x_1, \ldots, x_n, y_1, \ldots, y_k\}$. By item (ii) in Theorem 9, this intersection is connected, hence the projection is also connected. It follows that $S_{i_1,\ldots,i_m}$ is semi-monotone.

Conversely, suppose that the conditions (i) and (ii) of the theorem are satisfied, and let K be a coordinate cone in $\operatorname{span}\{x_1, \ldots, x_n, y_1, \ldots, y_k\}$. The projection of $\mathbf{F} \cap K$ on $\operatorname{span}\{x_1, \ldots, x_n\}$ is of the kind $S_{i_1,\ldots,i_m} \cap C$ for some $\{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$ and a coordinate cone C in $\operatorname{span}\{x_1, \ldots, x_n\}$. Since $S_{i_1,\ldots,i_m}$ is semi-monotone, its intersection with C is connected. Because $\mathbf{F}$ is the graph of a continuous map, the intersection $\mathbf{F} \cap K$ is also connected, hence, by Theorem 9, the map $\mathbf{f}$ is monotone. □

The following theorem is another corollary to Theorem 9

Theorem 10. *Let* $\mathbf{f} : X \to \mathbb{R}^k$ *be a monotone map defined on a semi-monotone set* $X \subset \mathbb{R}^n$, *with the associated matroid* $\mathbf{m}$, *and graph* $\mathbf{F} \subset \mathbb{R}^{n+k}$. *Then for any subset* $I \subset \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$, *with* $T := \operatorname{span} I$, *the image* $\rho_T(\mathbf{F})$ *under the projection map* $\rho_T : \mathbf{F} \to T$ *is either a semi-monotone set or the graph of a monotone map, whose matroid is a minor of* $\mathbf{m}$.

Proof. By Theorem 9, it is sufficient to prove that $\mathbf{G} := \rho_T(\mathbf{F})$ is either a semi-monotone set or the graph of a quasi-affine map, and that for each affine coordinate subspace S in $\operatorname{span} I$ the intersection $\rho_T(\mathbf{F}) \cap S$ is connected.

Let $\dim \mathbf{G} = m$ and assume first that $m < \dim T$.

Let H be a subset of I such that $|H| = m$ and $\dim(\lambda_L(\mathbf{G})) = m$, where $L := \operatorname{span} H$, and $\lambda_L : \mathbf{G} \to L$ is the projection map (obviously, there is such a subspace). By Corollary 5, there exists a subset $J \subset \{x_1, \dots, x_n, y_1, \dots, y_k\}$ such that $J \cap I = H$, $|J| = n$, and $\dim(\rho_N(\mathbf{F})) = n$, where $N := \operatorname{span} J$.

Since $\mathbf{f}$ is quasi-affine, the projection map $\rho_N : \mathbf{F} \to N$ is injective. We now prove that the projection map λ_L is also injective. Because ρ_N is injective, the set J is a basis of $\mathbf{m}$, and therefore its subset H is an independent set of $\mathbf{m}$. According to Theorem 6, all non-empty fibers of $\rho_L : \mathbf{F} \to L$ are graphs of monotone maps, having the same matroids of rank $n - m$. Since $\dim \mathbf{G} = m$, the image of the projection to T of a generic fiber of ρ_L (hence every fiber of ρ_L, since all matroids are the same) to T is 0-dimensional, thus it is a single point. We conclude that λ_L is injective, hence $\mathbf{G}$ is the graph of a map, defined on $\lambda_L(\mathbf{G})$, and that this map is quasi-affine. Denote this map by $\mathbf{g}$.

Let S be an affine coordinate subspace in T. Since $\mathbf{f}$ is monotone, the intersection $\mathbf{F} \cap (S \times (\operatorname{span}(J \setminus H)))$ is connected. It follows that $\mathbf{G} \cap S = \rho_L(\mathbf{F} \cap (S \times (\operatorname{span}(J \setminus H)))$ is connected, hence, by Theorem 9, $\mathbf{g}$ is a monotone map.

Let $\mathbf{m}_I$ be the matroid associated with $\mathbf{g}$. Observe that if the projection map ρ_T is injective, then the family of all independent sets of $\mathbf{m}_I$ consists of subsets $H \subset I$ which are independent sets of $\mathbf{m}$. It follows that $\mathbf{m}_I$ is a *restriction* of the matroid $\mathbf{m}$ to I ([9], Ch. 4, Section 2). In fact, $\mathbf{m}_I$ includes some basis sets of $\mathbf{m}$. In the case of a general projection map, the family of all independent sets of $\mathbf{m}$ consists of subsets $H \subset I$ which can be appended by maximal independent subsets of $J \setminus H$ so that to become independent sets of $\mathbf{m}$. Thus, $\mathbf{m}_I$ is a *contraction* of $\mathbf{m}$ ([9], Ch. 4 Section 3). In fact, for each basis set of $\mathbf{m}_I$ there is a subset of $J \setminus H$ such that their union is a basis of $\mathbf{m}$. It follows that $\mathbf{m}_I$ is a *minor* of $\mathbf{m}$.

Now let $\dim \mathbf{G} = m = \dim T$. By Corollary 5, I is an independent set of the matroid $\mathbf{m}$, hence, by the matroid theory's Augmentation Theorem ([9], Ch. 1, Section 5), there is a basis J of $\mathbf{m}$, containing I. The image of the projection of $\mathbf{F}$ to $\operatorname{span} J$ is a semi-monotone set, according to Theorems 8 and 7. Then, by Proposition 1, $\mathbf{G}$ is also a semi-monotone set. $\square$

Theorem 11. *Let $\mathbf{F}$ be the graph of a monotone map $\mathbf{f} : X \to \mathbb{R}^k$ on a semi-monotone set $X \subset \mathbb{R}^n$. Let $\mathbf{G} \subset \mathbf{F}$ be the graph of a monotone map $\mathbf{g}$ such that $\dim \mathbf{G} = n - 1$, and $\partial \mathbf{G} \subset \partial \mathbf{F}$. Then $\mathbf{F} \setminus \mathbf{G}$ is a disjoint union of two graphs of some monotone maps.*

Proof. Let $T := \operatorname{span}\{x_1, \dots, x_n\}$. According to Theorem 10, $\rho_T(\mathbf{G})$ is a graph of a monotone function, hence, by Lemma 7, $X \setminus \rho_T(\mathbf{G})$ has two connected components, each of which is a semi-monotone set. Their pre-images in $\mathbf{F}$ are the two connected components of $\mathbf{F} \setminus \mathbf{G}$, we denote them by $\mathbf{F}_+$ and $\mathbf{F}_-$, which are graphs of some continuous maps, $\mathbf{f}_+$ and $\mathbf{f}_-$. We prove that $\mathbf{f}_+$ and $\mathbf{f}_-$ are monotone maps by checking that they satisfy Definition 8.

Suppose that there exist $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n\}$ such that the component f_i of $\mathbf{f}$ is not independent of a coordinate x_j, otherwise $\mathbf{f}$ is identically constant on X, and the theorem becomes trivially true. Fix one such pair i, j.

The intersection $\mathbf{F} \cap \{y_i = b\}$ is either empty or the graph of a monotone map, due to Definition 8. If $\mathbf{G} \subset \mathbf{F} \cap \{y_i = b\}$, then $\mathbf{G} = \mathbf{F} \cap \{y_i = b\}$, and hence $(\mathbf{F}_+ \cup \mathbf{F}_-) \cap \{y_i = b\} = \emptyset$. We now prove, by induction on n, that if $\mathbf{F} \cap \{y_i = b\} \neq \emptyset$ and $\mathbf{G} \not\subset \mathbf{F} \cap \{y_i = b\}$, then each of intersections $\mathbf{F}_+ \cap \{y_i = b\}$ and $\mathbf{F}_- \cap \{y_i = b\}$

is either empty or the graph of a monotone map. The base of the induction, for $n = 1$, is trivial.

If $\mathbf{G} \cap \{y_i = b\} = \emptyset$, then either $\mathbf{F}_+ \cap \{y_i = b\} = \mathbf{F} \cap \{y_i = b\}$ or $\mathbf{F}_- \cap \{y_i = b\} = \mathbf{F} \cap \{y_i = b\}$. In any case, one of the two intersections is the graph of a monotone map and the other one is empty. Assume now that $\mathbf{G} \cap \{y_i = b\} \neq \emptyset$. Then both intersections, $\mathbf{F}_+ \cap \{y_i = b\}$ and $\mathbf{F}_- \cap \{y_i = b\}$ are non-empty. Indeed, if one of them is empty, then the component g_i of $\mathbf{g}$ attains the global extremum b at some point in its domain, which contradicts the monotonicity of g_i.

By the inductive hypothesis, $(\mathbf{F} \cap \{y_i = b\}) \setminus (\mathbf{G} \cap \{y_i = b\})$ is a disjoint union of two graphs of some monotone maps. One of its connected components lies in $\mathbf{F}_+$ while another in $\mathbf{F}_-$. Hence, both intersections $\mathbf{F}_+ \cap \{y_i = b\}$ and $\mathbf{F}_- \cap \{y_i = b\}$ are graphs of monotone maps, for example, of the maps $\mathbf{f}_{+\ i,j,b}$ and $\mathbf{f}_{-\ i,j,b}$. Thus, the part (i) of Definition 8 is proved for $\mathbf{f}_+$ and $\mathbf{f}_-$.

Observe that the matroid associated with the monotone map $\mathbf{f}_{i,j,b}$ does not depend on b (by the part (ii) of Definition 8), and, since $\mathbf{f}_{i,j,b}$ is quasi-affine (by Theorem 8), coincides with systems of basis sets of each of the maps $\mathbf{f}_{+\ i,j,b}$ and $\mathbf{f}_{-\ i,j,b}$. It follows that the systems of basis sets $\mathbf{f}_{+\ i,j,b}$ and $\mathbf{f}_{-\ i,j,b}$ do not depend on b, which proves the part (ii) of Definition 8 for $\mathbf{f}_+$ and $\mathbf{f}_-$.

We conclude that the maps $\mathbf{f}_+$ and $\mathbf{f}_-$ are monotone. $\square$

Definition 10. Let $g_1, \ldots, g_m$ be continuous functions on a set $Y \subset \mathbb{R}^n$. A *sign condition in* $g_1, \ldots, g_m$ *on* Y is any intersection of the kind $Y \cap \bigcap_{1 \le i \le m} \{g_i \sigma_i 0\}$, where $\sigma_i \in \{<, =, >\}$.

Corollary 9. *Let* $\mathbf{F}$ *be the graph of a monotone map* $\mathbf{f} : X \to \mathbb{R}^k$ *on a semi-monotone set* $X \subset \mathbb{R}^n$. *Let* $g_1, \ldots, g_m$ *be continuous functions* $\mathbf{F} : \to \mathbb{R}$ *such that for any subset* $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, m\}$ *and any* $j > 1$, *the intersection* $\mathbf{F} \cap \{g_{i_1} = \cdots = g_{i_j} = 0\}$ *either coincides with the intersection* $\mathbf{F} \cap \{g_{i_1} = \cdots = g_{i_{j-1}} = 0\}$ *or is the graph of a monotone map having codimension 1 in* $\mathbf{F}$. *Then every sign condition in* $g_1, \ldots, g_m$ *on* $\mathbf{F}$ *is the graph of a monotone function.*

Proof. Given a sign condition in $g_1, \ldots, g_m$ on $\mathbf{F}$, assume, without loss of generality, that all equalities among σ_i correspond to $i = 1, \ldots, r$, where $r \le m$. Then, by the conditions of the lemma, $\mathbf{F} \cap \{g_1 = \cdots = g_r = 0\}$ is the graph of a monotone map. Adding the strict inequalities $g_j \sigma_j 0$ for $j > r$ one by one we conclude, by Theorem 11, that the set in $\mathbf{F}$, defined by equations and inequalities on each step, is the graph of a monotone map. $\square$

The following theorem is a version for monotone maps of Theorem 3.

Theorem 12. *Let a bounded continuous quasi-affine map* $\mathbf{f} = (f_1, \ldots, f_k)$ *on a non-empty semi-monotone set* $X \subset \mathbb{R}^n$ *have the graph* $\mathbf{F} \subset \mathbb{R}^{n+k}$. *Let*

$$\{x_{j_1}, \ldots, x_{j_\alpha}, y_{i_1}, \ldots, y_{i_\beta}\} \subset \{x_1, \ldots, x_n, y_1, \ldots, y_k\},$$

where $\alpha + \beta = k$, *and*

$$T := \operatorname{span}(\{x_1, \ldots, x_n, y_1, \ldots, y_k\} \setminus \{x_{j_1}, \ldots, x_{j_\alpha}, y_{i_1}, \ldots, y_{i_\beta}\}).$$

Assume that $\rho_T(\mathbf{F})$ *is* n*-dimensional. The map* $\mathbf{f}$ *is monotone if and only if*

(i) *The image* $\rho_T(\mathbf{F})$ *is a semi-monotone set.*
(ii) *For every* $j \in \{j_1, \ldots, j_\alpha\}$ *and every* $i \in \{i_1, \ldots, i_\beta\}$, *such that the function* f_i *is not independent of a variable form* $\{x_1, \ldots, x_n\} \setminus \{x_{j_1}, \ldots, x_{j_\alpha}\}$, *all*

non-empty intersections $\mathbf{F} \cap \{x_j = a\}$ *and* $\mathbf{F} \cap \{y_i = a\}$, *where* $a \in \mathbb{R}$, *are either empty or graphs of monotone maps.*

Proof. Suppose that $\mathbf{f}$ is a monotone map. Then (i) and (ii) are satisfied by Theorem 7, because $\mathbf{f}$ is quasi-affine.

Conversely, assume that a bounded continuous quasi-affine map $\mathbf{f}$ satisfies the properties (i) and (ii). Because $\mathbf{f}$ is quasi-affine and (i) is satisfied, $\mathbf{F}$ is the graph of a map $\mathbf{g} : \rho_T(\mathbf{F}) \to \operatorname{span}\{x_{j_1}, \dots, x_{j_\alpha}, y_{i_1}, \dots, y_{i_\beta}\}$. Again, since $\mathbf{f}$ is quasi-affine, all monotone (according to (ii)) maps, with graphs $\mathbf{F} \cap \{x_j = a\}$ and $\mathbf{F} \cap \{y_i = a\}$, have associated matroids that don't depend on a. It follows from Definition 8 that $\mathbf{g}$ is monotone. Since $\mathbf{f}$ and $\mathbf{g}$ have the same graph, Theorem 9 implies that that $\mathbf{f}$ is also monotone. □

5. Graphs of monotone maps are regular cells

It is known (see Proposition 3) that any compact definable set $X \subset \mathbb{R}^n$ is definably homeomorphic to a finite simplicial complex $\widetilde{X}$, which is a *polyhedron* [6]. In this section we will use Lemma 23 and some known results from PL topology (formulated, for the reader's convenience, in Section 7) to introduce or to prove some definable homeomorphisms of definable sets. Thus, the relation $X \sim Y$ ("X is definably homeomorphic to Y") we will understand as $\widetilde{X} \sim_{PL} \widetilde{Y}$ ("$\widetilde{X}$ is PL homeomorphic to $\widetilde{Y}$").

Throughout this section the term "regular cell" means "topologically regular cell". In line with the convention above, we will actually use a slightly stronger version of this notion than the one in Definition 12. Namely, we say that a definable set V is a closed n-ball if $\widetilde{V} \sim_{PL} [-1,1]^n$, and is an $(n-1)$-sphere if $\widetilde{V} \sim_{PL} ([-1,1]^n \setminus (-1,1)^n)$. A definable bounded open set $U \subset \mathbb{R}^n$ is called (topologically) regular cell if $\overline{U}$ is a closed ball, and the frontier $\overline{U} \setminus U$ is an $(n-1)$-sphere. By Proposition 4, such U is also a regular cell in the sense of Definition 12.

Theorem 13. *The graph* $\mathbf{F} \subset \mathbb{R}^{n+k}$ *of a monotone map* $\mathbf{f} : X \to \mathbb{R}^k$ *on a semi-monotone set* $X \subset \operatorname{span}\{x_1, \dots, x_n\}$ *is a regular* n*-cell.*

Remark 9. Theorem 2.2 in [1] states that every semi-monotone set is a regular cell. Example 4.3 in [1] shows that the graph a bounded submonotone (but not supermonotone) function, satisfying the condition (ii) in Definition 5, may not be a regular cell.

We are going to prove Theorem 13 by induction on the dimension n of a regular cell. For $n = 1$ the statement is obvious. Assume it to be true for $n-1$, we will refer to this statement as to the *global inductive hypothesis.*

Lemma 15. *Let* $\mathbf{F} \subset \mathbb{R}^{n+k}$ *be a graph of a monotone map. Let*

$$\mathbf{F}_0 := \mathbf{F} \cap X_{j,=,c}, \quad \mathbf{F}_+ := \mathbf{F} \cap X_{j,>,c}, \quad \text{and} \quad \mathbf{F}_- := \mathbf{F} \cap X_{j,<,c}$$

for some $1 \le j \le n+k$ *and* $c \in \mathbb{R}$. *Then* $\overline{\mathbf{F}}_+ \cap \overline{\mathbf{F}}_- = \overline{\mathbf{F}}_0$.

Proof. Let a point $\mathbf{x} = (x_1, \dots, x_n) \in X_{j,=,c} \setminus \overline{\mathbf{F}}_0$ belong to $\overline{\mathbf{F}}_+ \cap \overline{\mathbf{F}}_-$. Then there is an $\varepsilon > 0$ such that an open cube centered at $\mathbf{x}$,

$$P_\varepsilon := \bigcap_{1 \le j \le n+k} \{(y_1, \dots, y_{n+k}) | \, |x_j - y_j| < \varepsilon\} \subset \mathbb{R}^{n+k},$$

has non-empty intersections with both $\mathbf{F}_+$ and $\mathbf{F}_-$ and the empty intersection with $\mathbf{F}_0$. Thus, $P_\varepsilon \cap \mathbf{F}$ is not connected, which is not possible since, according to Corollary 6, $P_\varepsilon \cap \mathbf{F}$ is the graph of a monotone map. □

Corollary 10. *Let* $\mathbf{F} \subset \mathbb{R}^{n+k}$ *be a graph of a monotone map. If* $\mathbf{F}_+$ *and* $\mathbf{F}_-$ *in Lemma 15 are regular cells, then* $\mathbf{F}$ *is a regular cell.*

Proof. We need to prove that $\overline{\mathbf{F}}$ is a closed n-ball, and that the frontier $\overline{\mathbf{F}} \setminus \mathbf{F}$ is an $(n-1)$-sphere. The only non-trivial case is when $\mathbf{F}_0$ is non-empty.

Since $\mathbf{F}_0$ is the graph of a monotone function due to Theorem 6 (ii), $\mathbf{F}_0$ is a regular $(n-1)$-cell by the inductive hypothesis. Thus, $\overline{\mathbf{F}}_0$, $\overline{\mathbf{F}}_+$, and $\overline{\mathbf{F}}_-$ are closed balls, while $\overline{\mathbf{F}}_0 \setminus \mathbf{F}$ is an $(n-2)$-sphere. Hence $\overline{\mathbf{F}}$ is obtained by gluing together two closed n-balls, $\overline{\mathbf{F}}_+$ and $\overline{\mathbf{F}}_-$ along closed $(n-1)$-ball $\overline{\mathbf{F}}_0$ (see Definition 13). Proposition 6 implies that $\overline{\mathbf{F}}$ is a closed n-ball.

According to Proposition 5, the sets $\overline{\mathbf{F}}_+ \setminus \mathbf{F} = \partial\overline{\mathbf{F}}_+ \setminus \mathbf{F}_0$ and $\overline{\mathbf{F}}_- \setminus \mathbf{F} = \partial\overline{\mathbf{F}}_- \setminus \mathbf{F}_0$ are closed $(n-1)$-balls. The frontier $\overline{\mathbf{F}} \setminus \mathbf{F}$ of $\mathbf{F}$ is obtained by gluing $\overline{\mathbf{F}}_+ \setminus \mathbf{F}$ and $\overline{\mathbf{F}}_- \setminus \mathbf{F}$ along the set $(\overline{\mathbf{F}}_+ \cap \overline{\mathbf{F}}_-) \setminus \mathbf{F}$ which, by Lemma 15, is equal to $\overline{\mathbf{F}}_0 \setminus \mathbf{F}$ and thus, is an $(n-2)$-sphere, the common boundary of $\overline{\mathbf{F}}_+ \setminus \mathbf{F}$ and $\overline{\mathbf{F}}_- \setminus \mathbf{F}$. It follows from Proposition 4 that $\overline{\mathbf{F}} \setminus \mathbf{F}$ is an $(n-1)$-sphere. □

Lemma 16. *If* $\mathbf{F}$ *and* $\mathbf{F}_-$ *in Lemma 15 are regular cells, then* $\mathbf{F}_+$ *is also a regular cell.*

Proof. Proposition 7 implies that $\overline{\mathbf{F}}_+$ is a closed n-ball. By the global inductive hypothesis of the Theorem 13, $\mathbf{F}_0$ is a regular cell. By Proposition 5, $\overline{\mathbf{F}}_+ \setminus \mathbf{F} = \partial\overline{\mathbf{F}}_+ \setminus \mathbf{F}_0$ is a closed $(n-1)$-ball. Then the frontier $\overline{\mathbf{F}}_+ \setminus \mathbf{F}_+$ of $\mathbf{F}_+$ is obtained by gluing two closed $(n-1)$-balls, $\overline{\mathbf{F}}_+ \setminus \mathbf{F}$ and $\overline{\mathbf{F}}_0$ along the $(n-2)$-sphere $\overline{\mathbf{F}}_0 \setminus \mathbf{F}$. Therefore, by Proposition 4, the frontier of $\mathbf{F}_+$ is an $(n-1)$-sphere. □

The following lemma is used on the inductive step of the proof of Theorem 13 and assumes the global inductive hypothesis (that a graph of a monotone map in less than n variables is a regular cell).

Introduce the notation $\mathbb{R}_+ := \{x \in \mathbb{R} |\ x > 0\}$.

Lemma 17. *Let* $\mathbf{F} \subset \mathbb{R}_+^{n+k}$ *be a graph of a monotone map* $\mathbf{f}$ *on a semi-monotone set* $X \subset \operatorname{span}\{x_1, \ldots, x_n\}$*, such that the origin is in* $\overline{\mathbf{F}}$*. Let* $c(t) = (c_1(t), \ldots, c_{n+k}(t))$ *be a definable generic curve inside the smooth locus of* $\mathbf{F}$ *converging to the origin as* $t \to 0$*. Then, for all small positive* t*, the set*

$$\mathbf{F}_t := \mathbf{F} \cap \{x_1 < c_1(t), \ldots, x_{n+k} < c_{n+k}(t)\}$$

is a cone with the vertex at the origin and a regular cell as the base, i.e., $\mathbf{F}_t$ *is a regular* n*-cell.*

Proof. For a non-empty subset $J := \{j_1, \ldots, j_i\} \subset \{1, \ldots, n+k\}$, let

$$C_{J,t} := \mathbf{F} \cap \{x_{j_1} = c_{j_1}(t), \ldots, x_{j_i} = c_{j_i}(t),\ x_\ell < c_\ell(t) \text{ for all } \ell \neq j_1, \ldots, j_i\}.$$

Due to the theorem on triangulation of definable functions ([3], Th. 4.5), for all small positive t, $\overline{\mathbf{F}}_t$ is definably homeomorphic to a closed cone with the vertex at the origin and the base definably homeomorphic to $\overline{C}_t$, where C_t is the union of non-empty $C_{J,t}$ for all non-empty J. To complete the proof of the lemma, it is enough to show that C_t is a regular cell.

According to Theorem 6 (ii), for every non-empty J the $(n-i)$-dimensional set $C_{J,t}$ is either empty or a graph of a monotone map. Hence, by the global inductive hypothesis, $C_{J,t}$ is a regular $(n-i)$-cell. Its closure, $\overline{C}_{J,t}$, is a closed cell (i.e., is definably homeomorphic to the closed cube $[0,1]^{n-i}$). Note that if $J \subset K \subset \{1,\dots,n+k\}$ then the closed cell $\overline{C}_{K,t}$ is a face of $\overline{C}_{J,t}$.

We prove by induction on n the following claim. If $\mathbf{F}$ is a graph in $\mathbb{R}^{n+k}_+$ of a monotone map on a semi-monotone set $X \subset \mathbb{R}^n_+$, and $c(t)$ is a smooth point in $\mathbf{F}$ (i.e., we don't assume that that the origin is necessarily in $\overline{\mathbf{F}}$), then C_t is a regular cell. The base for $n=1$ is obvious. The case of an arbitrary n we prove according to the following plan.

(a) Prove that for all J the difference $\overline{C}_{J,t} \setminus \widehat{C}_{J,t}$, where $\widehat{C}_{J,t} := \bigcup_{K \supset J} C_{K,t}$, is a closed cell. Then $\overline{C}_t$ is an $(n-1)$-dimensional cell complex (we will use the same notation for a complex and its underlying polyhedron), consisting of the closed cells $\overline{C}_{J,t}$ and the closed cells $\overline{C}_{J,t} \setminus \widehat{C}_{J,t}$ for all J.

(b) Construct a linear cell complex, $\overline{D}_t$, similar to $\overline{C}_t$, replacing $\mathbf{F}$ by the tangent space to $\mathbf{F}$ at $c(t)$, and prove that D_t is a regular cell.

(c) Prove that $\overline{C}_t$ and $\overline{D}_t$ are abstractly isomorphic, which implies that the pairs $(\overline{C}_t, C_t)$ and $(\overline{D}_t, D_t)$ are homeomorphic.

To prove (a) observe that, since $C_{J,t}$ is a regular $(n-i)$-cell, its boundary $\partial C_{J,t}$ is the PL $(n-i-1)$-sphere, while by the inductive hypothesis the difference $\widehat{C}_{J,t} \setminus C_{J,t}$ is a regular $(n-i-1)$-cell. Since

$$(\overline{C}_{J,t} \setminus \widehat{C}_{J,t}) \cup (\widehat{C}_{J,t} \setminus C_{J,t}) = \partial C_{J,t},$$

the difference $\overline{C}_{J,t} \setminus \widehat{C}_{J,t}$ is a closed cell by Newman's theorem (Corollary 3.13 in [6]). Note that if $J \subset K \subset \{1,\dots,n+k\}$ then the closed cell $\overline{C}_{K,t} \setminus \widehat{C}_{K,t}$ is a face of $\overline{C}_{J,t} \setminus \widehat{C}_{J,t}$. It is clear that for any J the closed cell $\overline{C}_{J,t}$ is a cell complex of the required type.

Now we construct the cell complex to satisfy (b). Recall that $c(t)$ is a smooth point of $\mathbf{F}$. Let $L(t)$ be the tangent space to $\mathbf{F}$ at $c(t)$. For every non-empty subset

$$J := \{j_1,\dots,j_i\} \subset \{1,\dots,n+k\},$$

introduce the $(n-i)$-dimensional convex polyhedron

$$D_{J,t} := L(t) \cap \{x_{j_1} = c_{j_1}(t),\dots,x_{j_i} = c_{j_i}(t),\ x_\ell < c_\ell(t) \text{ for all } \ell \neq j_1,\dots,j_i\},$$

and let D_t be the union of sets $D_{J,t}$ for all non-empty J. The same argument as in the case of $\overline{C}_{J,t}$, shows that the difference $\overline{D}_{J,t} \setminus \widehat{D}_{J,t}$, where $\widehat{D}_{J,t} := \bigcup_{K \supset J} D_{K,t}$, is a closed cell. Then $\overline{D}_{J,t}$ is a cell complex with closed cells of the kind $\overline{D}_{K,t}$, for all $K \supset J$, and the unique closed cell $\overline{D}_{J,t} \setminus \widehat{D}_{J,t}$. It follows that $\overline{D}_t$ is a cell complex with closed cells of the kind $\overline{D}_{J,t}$ and $\overline{D}_{J,t} \setminus \widehat{D}_{J,t}$ for all non-empty J.

Observe that $\overline{D}_t$ is a cone with the vertex $c(t)$ and the base B obtained by intersecting D_t with a hyperplane in $\mathbb{R}^{n+k}$ which separates $c(t)$ from all other vertices of $\overline{D}_t$. The base B is the boundary of a convex polyhedron, and therefore a PL sphere. It follows, using Lemma 1.10 in [6], that D_t is a regular cell.

To prove (c) we claim that for each $J = \{j_1,\dots,j_i\}$, if the cell $C_{J,t}$ is non-empty then the cell $D_{J,t}$ is non-empty, the converse implication being obvious. Assume the opposite, i.e., that $C_{J,t} \neq \emptyset$ while $D_{J,t} = \emptyset$. Then there exists $\ell \in \{n+1,\dots,n+k\} \setminus J$ such that the tangent space to $\overline{C}_{J,t}$ at $c(t)$ lies in $\{x_\ell = c_\ell\}$.

Since the map $\mathbf{f}$ is monotone, the component function f_ℓ of $\mathbf{f}$ is independent of each variable x_r, where $r \in \{j_1, \ldots, j_i\} \cap \{1, \ldots n\}$. It follows that the graph F_ℓ of f_ℓ lies in $\{x_\ell = c_\ell\}$ (cf. Remark 8), therefore so does $C_{J,t}$. This is a contradiction since, by the definition, $C_{J,t} \subset \{x_\ell < c_\ell\}$.

Thus, we have a bijective correspondence between the regular cells in C_t and D_t. Relating, in addition, for each J, the cell $\overline{C}_{J,t} \setminus \widehat{C}_{J,t}$ to the cell $\overline{D}_{J,t} \setminus \widehat{D}_{J,t}$ we obtain a bijective correspondence between the closed cells in the cell complexes $\overline{C}_t$ and $\overline{D}_t$. Note that the adjacency relations in both complexes are determined by the same simplicial subcomplex of the simplex with vertices $\{1, \ldots, n+k\}$, i.e., the complexes have the common nerve. It follows that the complexes $\overline{C}_t$ and $\overline{D}_t$ are abstractly isomorphic (see [6]), and their boundaries are abstractly isomorphic. Lemma 2.18 in [6] implies that the sets $\overline{C}_t$ and $\overline{D}_t$ are homeomorphic, and the boundaries $\partial \overline{C}_t$ and $\partial \overline{D}_t$ are homeomorphic. Then, by Lemma 1.10 in [6], the pairs $(\overline{C}_t, C_t)$ and $(\overline{D}_t, D_t)$ are PL homeomorphic. Therefore C_t is a regular cell, since D_t is a regular cell. $\square$

We now generalize Lemma 17, by removing the assumption that the curve $c(t)$ lies necessarily inside $\mathbf{F}$.

Lemma 18. *Let $\mathbf{F}$ be a graph in $\mathbb{R}_+^{n+k}$ of a monotone map $\mathbf{f}$ on a semi-monotone set $X \subset \operatorname{span}\{x_1, \ldots, x_n\}$, such that the origin is in $\overline{\mathbf{F}}$. Let $c(t) = (c_1(t), \ldots, c_{n+k}(t))$ be a definable curve inside $\mathbb{R}_+^{n+k}$ (not necessarily inside $\mathbf{F}$) converging to the origin as $t \to 0$. Then, for all small positive t,*

$$\mathbf{F}_t := \mathbf{F} \cap \{x_1 < c_1(t), \ldots, x_n < c_{n+k}(t)\}$$

is a cone with the vertex at the origin and a regular cell C_t as the base, i.e., $\mathbf{F}_t$ is a regular n-cell.

Proof. We use the notations from the proof of Lemma 17. As in the proof of Lemma 17, it is sufficient to show that C_t is a regular cell.

Observe that for a small enough t, the set $\overline{C}_t$ is a link at the origin in $\overline{\mathbf{F}}$.

Choose another definable curve, $s(t)$, converging the origin as $t \to 0$, so that $s(t)$ lies inside the smooth locus of $\mathbf{F}$ and is generic. For a non-empty $J = \{j_1, \ldots, j_i\}$, let

$$S_{J,t} := \mathbf{F} \cap \{x_{j_1} = s_{j_1}(t), \ldots, x_{j_i} = s_{j_i}(t),\ x_\ell < s_\ell(t) \text{ for all } \ell \neq j_1, \ldots, j_i\},$$

and S_t be the union of non-empty sets $S_{J,t}$ for all non-empty J. According to Lemma 17, S_t is a regular cell. For a small enough t, the closed cell $\overline{S}_t$ is also a link at the origin in $\overline{\mathbf{F}}$. By the theorem on the PL invariance of a link ([6], Lemma 2.19), the two links $\overline{C}_t$ and $\overline{S}_t$ are PL homeomorphic.

The same argument shows that the two links, ∂C_t and ∂S_t, at the origin in $\partial \mathbf{F}$ are PL homeomorphic. Then, by Lemma 1.10 in [6], the pairs $(\overline{C}_t, C_t)$ and $(\overline{S}_t, S_t)$ are PL homeomorphic. Therefore C_t is a regular cell, since S_t is a regular cell. $\square$

Lemma 19. *Let $\mathbf{F}$ be a graph in $\mathbb{R}_+^{n+k}$ of a monotone map $\mathbf{f}$ on a semi-monotone set $X \subset \operatorname{span}\{x_1, \ldots, x_n\}$, such that the origin is in $\overline{\mathbf{F}}$, and let $c = (c_1, \ldots, c_{n+k}) \in \mathbb{R}_+^{n+k}$. Then $\mathbf{F}_c := \mathbf{F} \cap \{x_1 < c_1, \ldots, x_n < c_{n+k}\}$ is a regular cell for a generic c with a sufficiently small $\|c\|$.*

Proof. Consider a definable set $\mathbf{F_y} := \mathbf{F} \cap \{x_1 < y_1, \dots, x_{n+k} < y_{n+k}\} \subset \mathbb{R}_+^{2(n+k)}$ with coordinates $x_1, \dots, x_{n+k}, y_1, \dots, y_{n+k}$ and $\mathbf{y} = (y_1, \dots, y_{n+k})$. By Corollary 12, there is a partition of $\mathbb{R}_+^{n+k}$ (having coordinates $y_1, \dots, y_{n+k}$) into definable sets T such that if any T is fixed, then for all $\mathbf{y} \in T$ the closures $\overline{\mathbf{F}}_\mathbf{y}$ are definably homeomorphic to the same polyhedron, and the frontiers $\overline{\mathbf{F}}_\mathbf{y} \setminus \mathbf{F_y}$ are definably homeomorphic to the same polyhedron.

For every n-dimensional T, such that the origin is in $\overline{T}$, there is, by the curve selection lemma ([3], Th. 3.2) a definable curve $c(t)$ converging to 0 as $t \to 0$. Hence, by Lemma 18, for each $c \in T$ the set $\overline{\mathbf{F}}_c$ is a closed n-ball, while $\overline{\mathbf{F}}_c \setminus \mathbf{F}_c$ is an $(n-1)$-sphere. Therefore, $\mathbf{F}_c$ is a regular cell. □

Lemma 20. *Using the notation from Lemma 19, for a generic $c \in \mathbb{R}_+^{n+k}$ with a sufficiently small $\|c\|$, the intersection*

$$\mathbf{F}_c \cap \bigcap_{1 \le \nu \le \ell} \{x_{j_\nu} \sigma_\nu a_\nu\},$$

for any $\ell \le n+k$, $j_\nu \in \{1, \dots, n+k\}$, $\sigma_\nu \in \{<, >\}$, and for any generic sequence $a_1 > \dots > a_\ell$, is either empty or a regular cell.

Proof. It is sufficient to assume that $a_\nu < c_{j_\nu}$ for all ν. Induction on ℓ. For $\ell = 1$, the set $\mathbf{F}_c \cap \{x_{j_1} < a_1\}$ is itself a set of the kind $\mathbf{F}_c$, and therefore is a regular cell, by Lemma 19. Then the set $\mathbf{F}_c \cap \{x_{j_1} > a_1\}$ is a regular cell due to Lemma 16.

By the inductive hypothesis, every non-empty set of the kind

$$\mathbf{F}_c^{(\ell-1)} := \mathbf{F}_c \cap \bigcap_{1 \le \nu \le \ell-1} \{x_{j_\nu} \sigma_\nu a_\nu\}$$

is a regular cell. Also by the inductive hypothesis, replacing c_{j_ℓ} by a_ℓ if $a_\ell < c_{j_\ell}$, every set $\mathbf{F}_c^{(\ell-1)} \cap \{x_{j_\ell} < a_\ell\}$ is a regular cell. Since both $\mathbf{F}_c^{(\ell-1)}$ and $\mathbf{F}_c^{(\ell-1)} \cap \{x_{j_\ell} < a_\ell\}$ are regular cells, so is $\mathbf{F}_c^{(\ell-1)} \cap \{x_{j_\ell} > a_\ell\}$, by Lemma 16, which completes the induction. □

Lemma 21. *Let $\mathbf{F}$ be a graph in $\mathbb{R}^{n+k}$ of a monotone map, and let a point $\mathbf{y} = (y_1, \dots, y_{n+k})$ belong to $\mathbf{F}$. Then for two generic points $a = (a_1, \dots, a_{n+k})$, $b = (b_1, \dots, b_{n+k}) \in \mathbb{R}_+^{n+k}$, with sufficiently small $\|a\|$ and $\|b\|$, the intersection*

$$\mathbf{F}_{a,b} := \mathbf{F} \cap \bigcap_{1 \le j \le n+k} \{-a_j < x_j - y_j < b_j\}$$

is a regular cell.

Proof. Induction on $m := n+k$ with the base $m = 1$ $(n = 0, k = 1)$ being obvious.

Translate the point $\mathbf{y}$ to the origin. Let $c = (c_1, \dots, c_m) \in \mathbb{R}^m$ be a generic point, and $\mathbb{P}_{m,c}$ the open octant of $\mathbb{R}^m$ containing c. By Lemma 19, if $\|c\|$ is sufficiently small, the set

$$\mathbf{F}_c := \mathbf{F} \cap \mathbb{P}_{m,c} \cap \{|x_1| < |c_1|, \dots, |x_m| < |c_m|\}$$

is either empty or a regular cell. Choose such a point c in every octant of $\mathbb{R}^m$.

Choose $(-a_i)$ (respectively, b_i) as the maximum (respectively, minimum) among the negative (respectively, positive) c_i over all octants $\mathbb{P}_{m,c}$. We now prove that, with so chosen a and b, the set $\mathbf{F}_{a,b}$ is a regular cell. Induction on a parameter

$r = 0, \dots, m-1$. For the base of the induction, with $r = 0$, if $d = (d_1, \dots, d_m)$ is a vertex of

$$\bigcap_{1 \le j \le m} \{-a_j < x_j < b_j\}$$

belonging to one of the $2^m = 2^{m-r}$ octants of $\mathbb{R}^m$, then $\mathbf{F}_d$ is either empty or a regular cell, by Lemma 20. Partition the family of all sets of the kind $\mathbf{F}_d$ into pairs $(\mathbf{F}_{d'}, \mathbf{F}_{d''})$ so that $d'_1 = a_1$, $d''_1 = b_1$ and $d'_i = d''_i$ for all $i = 2, \dots, m$. Whenever the cells $\mathbf{F}_{d'}$, $\mathbf{F}_{d''}$ are both non-empty, they have the common $(n-1)$-face

$$\mathbf{F} \cap (\{0\} \times \mathbb{P}_{m-1,(d'_2,\dots,d'_m)}) \cap \{x_1 = 0, |x_2| < |d'_2|, \dots, |x_m| < |d'_m|\},$$

which, by the inductive hypothesis of the induction on m, is a regular cell. Then, according to Corollary 10, the union of the common face and $\mathbf{F}_{d'} \cup \mathbf{F}_{d''}$ is a regular cell. Gluing in this way all pairs $(\mathbf{F}_{d'}, \mathbf{F}_{d''})$, we get a family of 2^{m-1} either empty or regular cells. This family is partitioned into pairs of regular cells each of which has the common regular cell face in the hyperplane $\{x_2 = 0\}$. On the last step of the induction, for $r = m-1$, we are left with at most two regular cells having, in the case of the exactly two cells, the common regular cell face in the hyperplane $\{x_m = 0\}$. Gluing these sets along the common face, we get, by Corollary 10, the regular cell $\mathbf{F}_{a,b}$. □

Lemma 22. *Using the notations from Lemma 21, the intersection*

$$V_{a,b} := \mathbf{F}_{a,b} \cap \bigcap_{1 \le \nu \le \ell} \{x_{j_\nu} \sigma_\nu d_\nu\}, \tag{5.1}$$

for any $\ell \le n+k$, $j_\nu \in \{1, \dots, n+k\}$, $\sigma_\nu \in \{<, >\}$, and for any generic $d_1 > \dots > d_\ell$, is either empty or a regular cell.

Proof. Analogous to the proof of Lemmas 20. □

Proof of Theorem 13. For each point $\mathbf{y} \in \overline{\mathbf{F}}$ choose generic points a, $b \in \mathbb{R}^{n+k}$ as in Lemma 21, so that the set $\mathbf{F}_{a,b}$ becomes a regular cell. We get an open covering of the compact set $\overline{\mathbf{F}}$ by the sets of the kind

$$A_{a,b} := \bigcap_{1 \le j \le n+k} \{-a_j < x_j - y_j < b_j\},$$

choose any finite subcovering $\mathcal{C}$. For every $j = 1, \dots, n+k$ consider the finite set D_j of j-coordinates a_j, b_j for all sets $A_{a,b}$ in $\mathcal{C}$. Let

$$\bigcup_{1 \le j \le n+k} D_j = \{d_1, \dots, d_L\}$$

with $d_1 > \dots > d_L$. Every set $V_{a,b}$, corresponding to a subset of a cardinality at most ℓ of $\{d_1, \dots, d_L\}$ (see (5.1)), is a regular cell, by Lemma 22. The graph $\mathbf{F}$ is the union of those $V_{a,b}$ and their common faces, for which $A_{a,b} \in \mathcal{C}$.

The rest of the proof is similar to the final part of the proof of Lemma 21. Use induction on $r = 1, \dots, n+k$, within the current induction step of the induction on $m = n+k$. The base of the induction is for $r = 1$. Let $D_1 = \{d_{1,1}, \dots, d_{1,k_1}\}$ with $d_{1,1} > \dots > d_{1,k_1}$. Partition the finite family of all regular cells $V_{a,b}$, for all $A_{a,b} \in \mathcal{C}$, into $(|D_1| - 1)$-tuples so that the projections of cells in a tuple on the x_1-coordinate are exactly the intervals

$$(d_{1,k_1}, d_{1,k_1-1}), (d_{1,k_1-1}, d_{1,k_1-2}), \dots, (d_{1,2}, d_{1,1}), \tag{5.2}$$

and any two cells in a tuple having as projections two consecutive intervals in (5.2) have the common $(n-1)$-dimensional face in a hyperplane $\{x_1 = \text{const}\}$. This face, by the external inductive hypothesis (of the induction on m), is a regular cell. According to Corollary 10, the union of any two consecutive cells and their common face is a regular cell. Gluing in this way all consecutive pairs in every $(|D_1|-1)$-tuple, we get a smaller family of regular cells. This family, on the next induction step $r = 2$, is partitioned into $(|D_2|-1)$-tuples of cells such that in each of these tuples two consecutive cells have the common regular cell face in a hyperplane $\{x_2 = \text{const}\}$. On the last step, $r = m$, of the induction we are left with one $(|D_n|-1)$-tuple of regular cells such that two consecutive cells have the common regular cell face in a hyperplane $\{x_n = \text{const}\}$. Gluing all pairs of consecutive cells along their common faces, we get, by Corollary 10, the regular cell $\mathbf{F}$. □

Graphs of monotone maps over real closed fields. Fix an arbitrary real closed field R. In [1] semi-algebraic semi-monotone sets in R^n were considered, and in particular it was proved that every such set X is a regular cell. The latter means that there exists a *semi-algebraic* homeomorphism $h : (\overline{X}, X) \to ([-1,1]^n, (-1,1)^n)$ (cf. Definition 12).

One can expand these results to semi-algebraic functions and maps over R (the graphs of such functions and maps are semialgebraic sets). In particular, the following statement is true.

Theorem 14. *The graph $\mathbf{F} \subset \mathrm{R}^{n+k}$ of a semi-algebraic monotone map $\mathbf{f} : X \to \mathrm{R}^k$ on a semi-algebraic semi-monotone set $X \subset \mathrm{R}^n$ is a regular n-cell.*

The proof of this theorem is based on applying the Tarski-Seidenberg transfer principle (Proposition 5.2.3 in [2]) to a first-order formalization of the statement of Theorem 13, and is completely analogous to the proof of Theorem 3.3 in [1].

6. Example: toric cubes

In [4] Engström, Hersh and Sturmfels introduced a class of compact semi-algebraic sets which they call *toric cubes.*

The following definition is adapted from [4].

Definition 11. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$, and $f_{\mathcal{A}} : [0,1]^d \to [0,1]^n$ be the map

$$\mathbf{t} = (t_1, \ldots, t_d) \mapsto (\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}),$$

where $\mathbf{t}^{\mathbf{a}_i} := t_1^{a_{i,1}} \cdots t_d^{a_{i,d}}$ for $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,d})$. The image of $f_{\mathcal{A}}$ is called a toric cube.

We call the image of the restriction of $f_{\mathcal{A}}$ to $(0,1)^d$ an *open toric cube.* The closure of an open toric cube is a toric cube. Note that an open toric cube is not necessarily an open subset of $\mathbb{R}^n$, and need not be contained in $(0,1)^n$ (if some $\mathbf{a}_i = \mathbf{0}$).

In this section we prove the following theorem.

Theorem 15. *An open toric cube $C \subset \mathbb{R}^n$ is the graph of a monotone map.*

As a result we obtain

Corollary 11. *An open toric cube $C \subset [0,1]^n$, with $\dim(C) = k$, is semi-algebraically homeomorphic to a standard open ball. The pair $(\overline{C}, C)$ is semi-algebraically homeomorphic to the pair $([0,1]^k, (0,1)^k)$, in particular, a toric cube is semi-algebraically homeomorphic to a standard closed ball.*

Remark 10. Note that the first statement in Corollary 11 is also proved in [4], Proposition 1. In conjunction with Theorem 2 in [4], Corollary 11 implies that any CW-complex in which the closures of each cell is a toric cube, must be a regular cell complex, and this answers in the affirmative the Conjecture 1 in [4].

Proof of Theorem 15. Let $C \subset [0,1]^n$ be an open toric cube and suppose that $C = f_{\mathcal{A}}((0,1)^d)$ for a monomial map $f_{\mathcal{A}}$ (see Definition 11).

Make the coordinate change $z_i = \log(t_i)$ for every $i = 1, \dots, d$, and take the logarithm of every component of the map $f_{\mathcal{A}}$ expressed in coordinates z_i. Denote the resulting map by $\log f_{\mathcal{A}}$. Then $\log f_{\mathcal{A}}$ is the restriction of a linear map, namely

$$\log f_{\mathcal{A}} : (-\infty, 0)^d \to (-\infty, 0)^n,$$

defined by

$$\mathbf{z} = (z_1, \dots, z_d) \mapsto (\mathbf{a}_1 \cdot \mathbf{z}, \dots, \mathbf{a}_n \cdot \mathbf{z}).$$

Observe that log (the component-wise logarithm) maps the open cube, $(0,1)^d$ (resp. $(0,1)^n$) homeomorphically onto $(-\infty,0)^d$ (resp. $(-\infty,0)^n$). It follows that the fiber of the orthogonal projection of C to any k-dimensional coordinate subspace is the pre-image under the log map of an affine subset of $(-\infty,0)^n$, and is a single point if it is zero-dimensional. Hence C is a graph of a quasi-affine map (choose any set of k coordinates such that the image of C under the orthogonal projection to the coordinate subspace of those coordinates is full dimensional).

Similarly, the intersection of C with any affine coordinate subspace is the pre-image under the log map, of an affine subset of $(-\infty,0)^n$ and hence connected.

We proved that C satisfies the conditions of Theorem 9, hence C is the graph of a monotone map. □

Proof of Corollary 11. Immediate consequence of Theorem 15 and Theorem 13. □

7. Appendix

Here we formulate some propositions, mostly from PL topology, which are used in the proofs above.

Proposition 3 ([3], Theorem 4.4)**.** *Let $X \subset \mathbb{R}^n$ be a compact definable set and let Y_i, $i = 1, \dots, k$ be definable subsets of X. Then there exists a finite simplicial complex K and a definable homeomorphism (triangulation) $\varphi : K \to X$ such that each Y_i is a union of images by φ of open simplices of K.*

Proposition 3 implies, in particular, every compact definable set in $\mathbb{R}^n$ is a *polyhedron* [6].

Let $\sim$ (respectively, $\sim_{PL}$) denote the relation of definable (respectively, PL) homeomorphism.

Lemma 23. *Let $X, Y \subset \mathbb{R}^n$ be two definable compact sets, and $\widetilde{X}, \widetilde{Y}$ two polyhedra, such that $X \sim \widetilde{X}$, $Y \sim \widetilde{Y}$, and $\widetilde{X} \sim_{PL} \widetilde{Y}$. Then $X \sim Y$.*

Proof. Straightforward, since, by Theorems 2.11, 2.14 in [6], any PL homeomorphism of compact polyhedra is definable. □

Definition 12. A definable set X is called a (topologically) regular m-cell if the pair $(\overline{X}, X)$ is definably homeomorphic to the pair $([-1,1]^m, (-1,1)^m)$.

Definition 13. Let Z be a closed (open) PL $(n-1)$-ball, X, Y be closed (respectively, open) PL n-balls, and

$$\overline{Z} = \overline{X} \cap \overline{Y} = \partial X \cap \partial Y.$$

We say that $X \cup Y \cup Z$ is obtained by *gluing X and Y along Z*.

Proposition 4 ([6], Lemma 1.10)**.** *Let X and Y be closed PL n-balls and $h : \partial X \to \partial Y$ a PL homeomorphism. Then h extends to a PL homeomorphism $h_1 : X \to Y$.*

Proposition 5 ([6], Corollary 3.13_n)**.** *Let X be a closed PL n-ball, Y be a closed $(n+1)$-ball, ∂Y be its boundary (the PL n-sphere), and let $X \subset \partial Y$. Then $\overline{\partial Y \setminus X}$ is a PL n-ball.*

Proposition 6 ([6], Corollary 3.16)**.** *Let X, Y, Z be closed PL balls, as in Definition 13, and $X \cup Y$ be obtained by gluing X and Y along Z. Then $X \cup Y$ is a closed PL n-ball.*

Proposition 7 ([7], Lemma I.3.8)**.** *Let $X, Y \subset \mathbb{R}^n$ be compact polyhedra such that X and $X \cup Y$ are closed PL n-balls. Let $X \cap Y$ be a closed PL $(n-1)$-ball contained in ∂X, and let the interior of $X \cap Y$ be contained in the interior of $X \cup Y$. Then Y is a closed PL n-ball.*

Proposition 8 ([8], Ch. 8, (2.14))**.** *Let $X \subset \mathbb{R}^{m+n}$ be a definable set, and let $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be the projection map. Then there exist an integer $N > 0$ and a definable (not necessarily continuous) map $f : X \to \Delta$, where Δ is an $(N-1)$-simplex, such that for every $\mathbf{x} \in \mathbb{R}^m$ the restriction $f_{\mathbf{x}} : (X \cap \pi^{-1}(\mathbf{x})) \to \Delta$ of f to $X \cap \pi^{-1}(\mathbf{x})$ is a definable homeomorphism onto a union of faces of Δ.*

Corollary 12. *Using the notations from Proposition 8, let all fibres $X \cap \pi^{-1}(\mathbf{x})$ be definable compact sets. Then there is a partition of $\pi(X)$ into a finite number of definable sets $T \subset \mathbb{R}^m$ such that all fibres $X \cap \pi^{-1}(\mathbf{x})$ with $\mathbf{x} \in T$ are definably homeomorphic, moreover each of these fibres is definably homeomorphic to the same simplicial complex.*

Proof. There is a finite number of different unions of faces in Δ. Since f is definable, the pre-image of any such union under the map $f \circ \pi^{-1}$ is a definable set. □

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