Exceptional solutions to the Painlevé VI equation

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Abstract

We find all solutions of the Painlevé VI equations with the property that they have no zeros, no poles, no 1-points and no fixed points.

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Painlevé VI is the following second order ODE:

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}\\ &+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right],\end{aligned} \tag{1}$$

where $(\alpha,\beta,\gamma,\delta)$ are complex parameters.

It is known that each solution has a meromorphic continuation along every curve in $D=\mathbf{C}\backslash\{0,1\}$, see, for example, [10]. A solution $y(t)$ is called *exceptional* if $y(t)\notin\{0,1,\infty,t\}$ for all $t\in D$ (and for all branches of y). In [2] such solutions are called "smooth".

An interesting problem is to classify all exceptional solutions. As a motivation we mention a problem of Poincaré [17] of existence of a linear second order equation with 4 regular singularities and prescribed projective monodromy. It is known that generic monodromy representation with 4 generators can be realized by an equation with 5 regular singularities, 4 of them

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prescribed, say $(0, 1, \infty, t)$ and the fifth, $y(t)$ apparent [8]. To obtain an equation with 4 singularities one can perform an isomonodromic deformation: to move t until $y(t)$ collides with one of the singularities at $(0, 1, \infty, t)$. It is known that such an isomonodromic deformation is described by the Painlevé VI equation which $y(t)$ must satisfy [9]. So the desired collision of singularities can be achieved if and only if this solution $y(t)$ is not exceptional. See [13, 7] on the related problems.

When

$$(\alpha, \beta, \gamma, \delta) = (0, 0, 0, 1/2),$$

equation (1) was studied by Picard [16] 16 years before Painlevé, Gambier and R. Fuchs discovered it. Picard found all solutions for this case, and some of them are exceptional (see below). The following two results are known.

When

$$(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/8, 3/8),$$

there are exactly three exceptional solutions [2].

Local solutions are considered the same if they are obtained by an analytic continuation from each other.

When

$$(\alpha, \beta, \gamma, \delta) = (9/8, -1/8, 1/8, 3/8)$$

there is exactly one exceptional solution defined by the equation

$$3y^4 - 4ty^3 - 4y^3 + 6ty^2 - t^2 = 0. \tag{2}$$

This was recently found in [3].

We give a simple proof of these results. Moreover, we determine all values of parameters for which exceptional solutions exist, find their number for such values of parameters, and write down explicit representations of these solutions.

It will be convenient to work with the *elliptic form* of Painlevé VI discovered by Picard in a special case and by Painlevé in in the general case. Consider the lattice $\Lambda_\tau = \{m + n\tau : m, n \in \mathbf{Z}\}$, where τ is in the upper half-plane H. The Weierstrass function $\wp(z|\tau)$ is the solution of the differential equation

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

with the initial condition $\wp(0)=\infty$. Here the e_j are distinct and their sum is 0. We denote

$$\omega_0=0, \quad \omega_1=1/2, \quad \omega_2=\tau/2, \quad \omega_3=(1+\tau)/2; \tag{3}$$

then $e_k=\wp(\omega_k)$, $1\leq k\leq 3$, and $\wp$ is periodic with periods in Λ_τ.

Let us define the functions $t(\tau)$ and $p(\tau)$ by

$$t(\tau)=\frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}, \quad y(t)=\frac{\wp(p(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}. \tag{4}$$

The function $t(\tau)$ is the fundamental invariant of the group $\Gamma[2]$ of the linear fractional transformations represented by matrices $A\in SL(2,\mathbf{Z})$ satisfying $A\equiv I \pmod 2$.

Then $p(\tau)$ satisfies the elliptic form of Painlevé VI,

$$\frac{d^2p(\tau)}{d\tau^2}=-\frac{1}{4\pi^2}\sum_{k=0}^{3}\alpha_k\wp'(p(\tau)+\omega_k|\tau). \tag{5}$$

Here

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=(\alpha,-\beta,\gamma,1/2-\delta). \tag{6}$$

For the proof that (5) is equivalent to (1) we refer to [14].

Suppose that y is an exceptional solution. By (4) this means that

$$p(\tau)\not\equiv\omega_k \mod \Lambda_\tau, \quad \tau\in H, \quad k=0,\ldots,3. \tag{7}$$

Moreover, as the only critical points of $z\mapsto\wp(z,\tau)$ are those congruent to ω_k, we can locally solve the second equation in (4) with respect to p, and the implicit function theorem implies that p is holomorphic in H.

We use the following result of Earle [5, Thm. 4.13]:

Theorem A. *Let $p: H\to\mathbf{C}$ be a holomorphic function with the property that $p(\tau)\neq m+n\tau$ for all $\tau\in H$ and all integers m,n. Then*

$$p(\tau)=\mu+\nu\tau, \tag{8}$$

where μ and ν are real, and $(\mu,\nu)\notin\mathbf{Z}\times\mathbf{Z}$.

Applying this theorem, we obtain that a solution $y(t)$ of (1) described by (4) is exceptional if and only if p is of the form (8), with real $(\mu,\nu)\notin(\mathbf{Z}/2)\times(\mathbf{Z}/2)$. Substituting to (5), we obtain

$$\sum_{k=0}^{3}\alpha_k\wp'(\mu+\nu\tau+\omega_k|\tau)\equiv 0. \tag{9}$$

Such solutions are called *Picard's solutions*. They are exceptional when μ and ν are real. Picard [16] found that they exist in the case $\alpha_j = 0$, $0 \le j \le 3$. But of course they also exist whenever (9) holds. We mention the following

Corollary of Theorem A. *Let $y(t)$ be a multi-valued analytic function in $\mathbf{C}\backslash\{0,1\}$, which has an analytic continuation along every curve in $\mathbf{C}\backslash\{0,1\}$. Suppose that $y(t) \notin \{0,1,t\}$ for all $t \in \mathbf{C}\backslash\{0,1\}$ and for all branches of y. Then $y(t)$ is of the form (4) with p as in (8). In particular, y is a solution of (1) with parameters $(0,0,0,1/2)$.*

Now our problem of classification of exceptional solutions is reduced to a problem about elliptic functions:

For which α_k, μ, ν do we have the identity (9)?

To simplify (9) we use the formulas

$$\wp(z+\omega_k) = e_k + \frac{(e_k - e_i)(e_k - e_j)}{\wp(z) - e_k}, \quad \{i,j,k\} = \{1,2,3\}.$$

Differentiating these formulas with respect to z, we obtain

$$\wp'(z+\omega_k) = -\frac{(e_k - e_i)(e_k - e_j)}{(\wp(z) - e_k)^2}\wp'(z),$$

and substituting to (9) we obtain after simplification

$$\alpha_0(w-e_1)^2(w-e_2)^2(w-e_3)^2 = \sum_{k=1}^{3} \alpha_k (w-e_i)^2(w-e_j)^2(e_k-e_i)(e_k-e_j), \quad (10)$$

where $w(\tau) = \wp(\mu + \nu\tau|\tau)$.

Proposition 1. *If at least one $\alpha_k \neq 0$, the equation (10) can only hold when μ, ν are rational.*

Proof. The functions e_j are the roots of the equation

$$4x^3 - g_2 x - g_3 = 0, \quad (11)$$

whose coefficients are modular forms. In particular, if we set $T\tau = \tau + 1$, then the g_j are invariant with respect to T and thus the e_j are invariant with respect to T^3. Then it follows from (10) that $w(T^{18}\tau) = w(\tau)$. Now

$$w(T^n\tau) = \wp(\mu + n\nu + \nu\tau|\tau + n) = \wp(\mu + n\nu + \nu\tau|\tau) = \wp(\mu + \nu\tau + m + n\nu|\tau),$$

for all integers m, n. If ν is irrational we can arrange a sequence (m_k, n_k) such that n_k are divisible by 18, and $s_k = m_k + n_k\nu \to 0$. Then

$$w(\tau) = w(\tau + s_k), \quad s_k \to 0, \quad s_k \neq 0,$$

which cannot happen for a non-constant analytic function. This contradiction shows that ν is rational. A similar argument shows that μ is also rational.

Proposition 2. *If at least one $\alpha_k \neq 0$, then all exceptional solutions of (1) are algebraic.*

Proof. A direct computation shows that equation (10) is non-trivial if at least one $\alpha_j \neq 0$.

The function w satisfying (10) can take only finitely many values (at most 6) on any orbit of $\Gamma[2]$. Therefore y can take only finitely many values at each point. As y omits $0, 1, \infty$, Picard's Great Theorem implies that the singularities at $0, 1, \infty$ are algebraic.

Actually one can write an explicit algebraic equation which all exceptional solutions must satisfy. For this we express w in terms of y and the e_j in terms of t from (4) and substitute this expression to (10). We obtain:

$$\alpha_0 y^2(y-1)^2(y-t)^2 - \alpha_1 t(y-1)^2(y-t)^2 - \alpha_2(1-t)y^2(y-t)^2 \qquad (12)$$
$$-\alpha_3 t(t-1)y^2(y-1)^2 = 0.$$

To determine how many exceptional solutions are possible, one has to find for each $(\alpha_0, \ldots, \alpha_3)$ the number of irreducible factors of this equation, and to check which of these factors define algebraic solutions of (1). For example, in the case considered in [2], when all α_j are equal, we obtain three factors:

$$y^2(y-1)^2(y-t)^2 - t(y-1)^2(y-t)^2 - (1-t)y^2(y-t)^2 + t(t-1)y^2(y-1)^2$$
$$= (y^2 - t)(y^2 - 2y + t)(y^2 - 2yt + t). \qquad (13)$$

In this case, each of the three factors determines a solution of (1). So we obtained a simple proof of the main theorem of [2]. We can state the result as follows:

Proposition 3. *When not all $\alpha_j = 0$, exceptional solutions are algebraic functions given by the polynomial equation (12). Their number is the number* *of non-trivial irreducible factors of this polynomial that satisfy (1). A factor is called non-trivial if it depends on both y and t and is not a constant multiple of $y-t$.*

It is easy to see that an exceptional solution cannot be rational, see, for example, [4, Proposition 6], so the number of exceptional solutions is at most 3, and they are at most 6-valued.

Next we determine all cases when the polynomial in (12) is reducible.

Proposition 4. *When $\alpha_3 = 0$, the polynomial (12) factors as*

$$(y-t)^2 P_0(y,t),$$

where

$$P_0(y,t) = \alpha_0(y-1)^2y^2 - \alpha_2 y^2 - t\left(\alpha_1(y-1)^2 - \alpha_2 y^2\right)$$

has no non-trivial factors.

In this case we may have at most one exceptional solution defined by $P_0(y,t) = 0$.

Proposition 5. *If $\alpha_3 \neq 0$, then the polynomial (12) has a non-trivial factorization if*

$$\alpha_j = u_j^2, \quad \text{where} \quad \sum_{j=0}^{3} \pm u_j = 0,$$

for any choice of signs. An equivalent condition is

$$\left(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2-2(\alpha_0\alpha_1+\alpha_0\alpha_2+\alpha_0\alpha_3+\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\right)^2$$
$$= 64\alpha_0\alpha_1\alpha_2\alpha_3. \tag{14}$$

The surface defined by (14) contains three lines

$$\{\alpha_0 = \alpha_1,\ \alpha_2 = \alpha_3\}, \tag{15}$$

$$\{\alpha_0 = \alpha_2,\ \alpha_1 = \alpha_3\}, \tag{16}$$

$$\{\alpha_0 = \alpha_3,\ \alpha_1 = \alpha_2\}. \tag{17}$$

The polynomial (12) is a product of three non-trivial irreducible factors if $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ belongs to one of these lines, and $\alpha_0\alpha_1\alpha_2\alpha_3 \neq 0$.

Using Maple and Mathematica we determined that the cases listed in Proposition 5 exhaust all factorizations of our polynomial. But this fact will not be used in the proof of our main result.

When (14) does not hold, computation indicates that the polynomial (12) is irreducible, and can define at most one exceptional solution of (1). It remains to determine which algebraic functions defined by factors of (12) actually satisfy (1). Computation with Maple indicates that in the case when (12) is irreducible, the resulting algebraic function with 6 branches does not satisfy (1). In the case (14) when (12) splits into two irreducible factors, algebraic functions arising from these factors are of degrees 2 and 4. The function determined by the factor of degree 2 never satisfies (1), while the function determined by the factor of degree 4 satisfies (1) if and only if three of the α_j are equal and the fourth is equal to the sum of these three. If one of the equations (15), (16), or (17) is satisfied, then one of the factors satisfies the equation and the other two factors do not.

We will prove all these facts below without reliance on a computer. Our main result is the following.

Theorem 1. *The complete list of exceptional solutions of Painlevé VI is the following:*
If $\alpha_j = 0,\ 0 \leq j \leq 3$, *they are Picard's solutions with real* $(\mu, \nu) \in \mathbf{R}^2 \backslash \mathbf{Z}^2$.
If $\alpha_0 = \alpha_1, \quad \alpha_2 = \alpha_3$, *then there is a solution*

$$y(t) = \sqrt{t}. \tag{18}$$

If $\alpha_0 = \alpha_2, \quad \alpha_1 = \alpha_3$, *then there is a solution*

$$y(t) = 1 + \sqrt{1 - t}. \tag{19}$$

If $\alpha_0 = \alpha_3, \quad \alpha_1 = \alpha_2$, *then there is a solution*

$$y(t) = t + \sqrt{t^2 - t}. \tag{20}$$

If $\alpha_0 = 9\alpha_1 = 9\alpha_2 = 9\alpha_3 \neq 0$, *then there is a unique solution defined by*

$$3y^4 - 4ty^3 - 4y^3 + 6ty^2 - t^2 = 0. \tag{21}$$

If $9\alpha_0 = \alpha_1 = 9\alpha_2 = 9\alpha_3 \neq 0$, *then there is a unique solution defined by*

$$y^4 - 6ty^2 + 4t(t + 1)y - 3t^2. \tag{22}$$

If $9\alpha_0 = 9\alpha_1 = \alpha_2 = 9\alpha_3 \neq 0$, *then there is a unique solution defined by*

$$y^4 - 4y^3 + 6ty^2 - 4t^2y + t^2. \tag{23}$$

If $9\alpha_0 = 9\alpha_1 = 9\alpha_2 = \alpha_3 \neq 0$, *then there is a unique solution defined by*

$$y^4 - 4ty^3 + 6ty^2 - 4ty + t^2. \tag{24}$$

Equations (18), (19), (20) are permuted by the group generated by

$$(t, y) \mapsto (1 - t, 1 - y), \quad \text{and} \quad (t, y) \mapsto (1/t, y/t), \tag{25}$$

which is isomorphic to S_3. Equations (21), (22), (23), (24) are permuted by the group isomorphic to S_4 which is obtained by adding the transformation

$$(t, y) \mapsto (1/t, 1/y) \tag{26}$$

to (25).

All curves (21), (22), (23), (24) are of genus zero. A uniformization of (21) is

$$y = \frac{1}{1 - z^2}, \quad t = \frac{2z - 1}{(z - 1)^3(z + 1)}.$$

The rest are obtained by substitutions (26), (25):

$$y = 1 - z^2, \quad t = \frac{(z + 1)(z - 1)^3}{2z - 1} \qquad \text{for} \quad (22),$$

$$y = z^2, \quad t = -\frac{z^3(z - 2)}{2z - 1} \qquad \text{for} \quad (23),$$

$$y = -\frac{2z - 1}{z(z - 2)}, \quad t = -\frac{2z - 1}{z^3(z - 2)} \qquad \text{for} \quad (24).$$

Remark. If $\alpha_j \neq 0$ for at least one $j \in \{0, 1, 2, 3\}$, and we have an exceptional solution y as in (4), then (8) and (9) hold. So y also solves Picard's equation, and the whole one-parametric family of equations with parameters $k\alpha_j$, $k \in$ **C**. All such cases when a single solution satisfies a one-parametric family of Painlevé VI equations have been classified in [1]. Using this classification one can obtain an alternative proof of Theorem 1, as suggested by the referee. Our solutions correspond to the entries 2A, 2B, 2C, 3A, 3B, 3C, 3D in Table

2.1, [1, p. 3629]. We give an elementary proof of Theorem 1 independent of the results in [1].

Proof of Theorem 1. We have proved that all exceptional solutions are Picard solutions parametrized by

$$y(\tau) = \frac{\wp(\mu+\nu\tau|\tau) - e_1(\tau)}{e_2(\tau)-e_1(\tau)}, \quad t = \frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}.$$

Two such solutions are the same (obtained by an analytic continuation) if, and only if,

$$(\mu,\nu)A = \pm(\mu,\nu) \pmod{\mathbf{Z}\times\mathbf{Z}}, \quad \text{where} \quad A \in \Gamma[2].$$

Let us say that two rational vectors (μ,ν) and (μ',ν') are equivalent if

$$(\mu,\nu) = \pm(\mu',\nu') \pmod{\mathbf{Z}\times\mathbf{Z}}.$$

Then the group $\Gamma[2]$ acts on the equivalence classes, and we need the list of all classes whose orbit has length at most 6.

We have the following

Lemma 1. [15, Lemma 3] *Every $\Gamma[2]$ orbit contains a vector equivalent to one of the following:*

$$(0, M/N), \quad (M/N, 0), \quad (M/N, M/N), \tag{27}$$

where M, N are defined as follows. Let $\mu = \mu_1/\mu_0$, $\nu = \nu_1/\nu_0$ be the reduced representations. Then N is the least common multiple of μ_0, ν_0 and M is the greatest common divisor of $\mu_1 N/\mu_0$, $\nu_1 N/\nu_0$, so that M, N are coprime, and

$$\mu = mM/N, \quad \nu = nM/N,$$

where m, n are coprime.

Then the orbit of (μ,ν) contains a vector of the list (27) if, and only if, (m,n) is (even,odd), (odd,even) or (odd,odd), respectively.

The three solutions corresponding to the vectors (27) are permuted by the group generated by (25)

To understand the orbits completely, it remains to check which of the three vectors (27) are on the same orbit.

Lemma 2. *Let M, N be coprime integers. If N is odd, then the three points (27) are in one $\Gamma[2]$ orbit. If N is even, they are in three distinct orbits.*

Proof. The vectors $(0, M/N)$ and $(M/N, 0)$ are on the same orbit if, and only if, there exists a matrix in $\Gamma[2]$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} M/N \\ 0 \end{pmatrix} = \pm \begin{pmatrix} 0 \\ M/N \end{pmatrix} \pmod{\mathbf{Z} \times \mathbf{Z}}, \tag{28}$$

which is equivalent to

$$a \equiv 0 \pmod{N}, \tag{29}$$

and

$$cM \equiv \pm M \pmod{N}. \tag{30}$$

As a is odd, we conclude from (29) that N must be odd.

In the opposite direction, if N is odd, we can take

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -N & N+1 \\ -1-N^2 & 1+N(N+1) \end{pmatrix} \in \Gamma[2], \tag{31}$$

and (28) will be satisfied.

Now let us investigate when $(M/N, 0)$ and $(M/N, M/N)$ are on the same orbit. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} M/N \\ M/N \end{pmatrix} = \pm \begin{pmatrix} M/N \\ 0 \end{pmatrix} \pmod{\mathbf{Z} \times \mathbf{Z}}, \tag{32}$$

which is equivalent to

$$c + d \equiv 0 \pmod{N}, \tag{33}$$

and

$$(a+b)M = \pm M \pmod{N}. \tag{34}$$

As $c + d$ is always odd, we conclude from (33) that N must be odd. In the opposite direction, if N is odd, use the same matrix as in (31) and (32) will be satisfied.

Now it is easy to find the number of elements in an orbit. When N is odd, all vectors $(\mu_1/N, \nu_1/N)$ with the greatest common factor of μ_1, ν_1 coprime to N belong to one orbit. This orbit is of length greater than 6 when $N \geq 5$.

When N is even, such vectors lie on three orbits of equal length, corresponding to the three vectors (27), when $N = 4$ we have three orbits of length 2, and when $N = 6$ we have three orbits of length 4.

Thus the result is that exceptional solutions correspond to the following pairs (μ, ν):

a) $(1/4, 0)$, $(0, 1/4)$, $(1/4, 1/4)$, representing three distinct two-valued solutions.

b) $(1/3, 1/3)$, representing one four-valued solution.

c) $(1/6, 0)$, $(0, 1/6)$, $(1/6, 1/6)$, representing three four-valued solutions.

As one equation cannot have two different four-valued exceptional solutions, the three solutions in c) must belong to different equations. The three solutions in a) may belong to one equation, or to three different equations. The group (25) permutes solutions of the type a) and permutes solutions of the type b).

One can verify that vectors a) correspond to solutions (18), (19), (20) in Theorem 1, vector b) corresponds to (21), and vectors, c) correspond to the remaining three solutions (22), (23), (24).

For a) this is easy. To check b) and c), we write the tripling formula for the elliptic function

$$w(z) = \frac{\wp(z) - e_1}{e_2 - e_1},$$

which can be obtained from the well-known addition theorem for $\wp$. We have

$$w(3z) = y \left(\frac{y^4 + 4yt - 6y^2t - 3t^2 + 4yt^2}{4y^3t - 6y^2t + 4y^3 - 3y^4 + t^2} \right)^2 =: y \left(\frac{f(y,t)}{g(y,t)} \right)^2, \qquad (35)$$

where $y = w(z)$. At the points z of third order, we have $w(3z) = \infty$, while at the points of 6-th order, $w(3z) \in \{0, 1, t\}$. So we have to solve four equations

$$f(y,t) = 0, \quad g(y,t) = 0, \quad yf(y,t) - g(y,t) = 0, \quad yf(y,t) - tg(y,t) = 0.$$

The first two polynomials are irreducible. Factoring the other two we obtain:

$$f(y,t) - g(y,t) = (y-1)(4y^3 - y^4 - 6y^2t - t^2 + 4yt^2)^2$$

and

$$f(y,t) - tg(y,t) = (y-t)(y^4 - 4yt + 6y^2t - 4y^3t + t^2)^2,$$

which together with f and g gives the four polynomials in (21), (22), (23), (24).

Remarks.

1. The equation (14) defines a Kummer surface [11, p. 21, footnote].

2. All algebraic solutions of Painlevé VI have been classified in [12]. However this classification is only up to Bäcklund transformations, and Bäcklund transformations in general do not map exceptional solutions to exceptional solutions [2].

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