Relative Closure and the Complexity of Pfaffian Elimination

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Abstract

We introduce the "relative closure" operation on one-parametric families of semi-Pfaffian sets. We show that finite unions of sets obtained with this operation ("limit sets") constitute a structure, i.e., a Boolean algebra closed under projections. Any Pfaffian expression, i.e., an expression with Boolean operations, quantifiers, equations and inequalities between Pfaffian functions, defines a limit set. The structure of limit sets is effectively o-minimal: there is an upper bound on the complexity of a limit set defined by a Pfaffian expression, in terms of the complexities of the expression and the Pfaffian functions in it.

1 Introduction

Pfaffian functions [14, 15] are solutions of a triangular system of first-order partial differential equations with polynomial coefficients (see Definition 2.1 below). A semi-Pfaffian set, defined by a Boolean formula with equations and inequalities between Pfaffian functions, is characterized by global finiteness properties. This means that the geometric and topological complexity of a semi-Pfaffian set admits an upper bound in terms of the complexity of its defining formula.

A sub-Pfaffian set Y is the image of a projection of a semi-Pfaffian set X into a subspace. Many finiteness properties of Y can be derived from the corresponding properties of X. These finiteness properties make semi- and sub-Pfaffian sets one of the favorite objects in the theory of o-minimal structures (see [3, 2]).

Upper bounds on the topological complexity of semi-Pfaffian sets were established in [15]. Different aspects of the geometric complexity of semi-Pfaffian and sub-Pfaffian sets, such as the order of tangency (Lojasiewicz inequality), stratification, frontier and closure, were addressed in [4, 5, 6, 7, 8].

For a restricted sub-Pfaffian set Y (projection of a restricted semi-Pfaffian set, see Definition 2.4) the complement of Y is sub-Pfaffian [5, 8, 22]). The algorithm in [8] provides an upper bound on the complexity of an existential expression for the complement of Y in terms of the complexity of an existential expression for Y.

For non-restricted semi-Pfaffian sets, Charbonnel [1] and Wilkie [23] introduced the "closure at infinity" operation. Charbonnel-Wilkie theorem ([23], see also [18, 13, 21]) implies that the sets constructed from non-restricted semi-Pfaffian sets by a finite sequence of projections and closures at infinity constitute an o-minimal structure.

In this paper, we introduce the "relative closure" operation (see Definition 3.5 below) on one-parametric families of semi-Pfaffian sets. A "limit set" is a finite union of the relative closures of semi-Pfaffian families. Every semi-Pfaffian set is a limit set. The main results of this paper (Theorems 3.10 and 6.1) state that limit sets constitute an effectively o-minimal structure, i.e., any expression with limit sets defines a limit set, with an upper bound on the complexity of the resulting limit set in terms of the complexity of the expression and of the limit sets in it. Since the number of connected components of a limit set admits an upper bound in terms of its complexity (Theorem 3.13) this provides an efficient version of the Charbonnel-Wilkie theorem for Pfaffian expressions.

2 Pfaffian functions and semi-Pfaffian sets

For a set $X \subset \mathbb{R}^n$, let $\overline{X}$ and $\partial X = \overline{X} \setminus X$ denote its closure and frontier. We assume that the closure points of X at infinity are included in $\overline{X}$ and ∂X. To avoid the separate treatment of infinity, we assume that $\mathbb{R}^n$ is embedded in the projective space, and all constructions are performed in an affine chart U such that X is relatively compact in U. To achieve this, it may be necessary to subdivide X into smaller pieces, each of them relatively compact in its own chart.

Definition 2.1 (See [15]). A *Pfaffian chain* of *order* $r \geq 0$ and *degree* $p \geq 1$ in $\mathbb{R}^n$ is a sequence of functions $y(x) = (y_1(x), \dots, y_r(x))$, each y_i defined and analytic in an open domain $G_i \subset \mathbb{R}^n$, satisfying a system of Pfaffian equations

$$dy_i(x) = \sum_{j=1}^{n} P_{ij}\big(x, y_1(x), \dots, y_i(x)\big) dx_j, \text{ for } x \in G_i,\ i = 1, \dots, r. \tag{1}$$

Here $P_{ij}(x, y_1, \dots, y_i)$ are polynomials of degree at most p. The system (1) is triangular: P_{ij} does not depend on y_k with $k > i$.

Each domain G_i should satisfy the following conditions:

(i) The graph $\Gamma_i = \{x \in G_i,\ t = y_i(x)\}$ of $y_i(x)$ belongs to an open domain

$$\Omega_i = \{x, t : x \in G_{i-1},\ S_{i\nu}(x, y_1(x), \dots, y_{i-1}(x), t) > 0, \text{ for } \nu = 1, \dots, N_i\}$$

with $S_{i\nu}$ polynomial in $x, y_1, \dots, y_{i-1}, t$, and $\partial\Gamma_i \subset \partial\Omega_i$.

(ii) Γ_i is a *separating submanifold* ("Rolle leaf") in Ω_i, i.e., Ω_i is a disjoint union of Γ_i and two open domains Ω_i^- and Ω_i^+. This is true, for example, when G_i is connected and Ω_i simply connected ([15], p.38).

A *Pfaffian function* of degree $d > 0$ with the Pfaffian chain $y(x)$ is a function $q(x) = Q(x, y(x))$, where $Q(x, y)$ is a polynomial of degree at most d. The function $q(x)$ is defined in a *semi-Pfaffian domain*

$$G = \bigcap G_i = \{S_{i\nu}(x, y_1(x), \dots, y_i(x)) > 0, \text{ for } i = 1, \dots, r,\ \nu = 1, \dots, N_i\}. \tag{2}$$

Remark 2.2. The above definition of a Pfaffian chain corresponds to the definition of a special Pfaffian chain in [4] (see also [7]). It is more restrictive than definitions in [15] and [4] where Pfaffian chains are defined as sequences of nested integral manifolds of polynomial 1-forms. Both definitions lead to (locally) the same class of Pfaffian functions.

More general definitions of Pfaffian functions, where the coefficients of equations (1) can be non-polynomial, are considered in [LR] and [MS]. Most of our constructions can be adjusted to this more general definition. However, efficient upper bounds on the complexity do not hold in this case.

Example 2.3 (Iterated exponential and logarithmic functions). For $r = 1, 2, \dots,$ let $e_r(t) = \exp(e_{r-1}(t))$, with $e_0(t) = t$. The functions $e_1, \dots, e_r$ constitute a Pfaffian chain of order r and degree r, since $de_r = e_r \cdots e_1\, dt$.

For $r = 1, 2, \dots$, let $l_r(t) = \ln(l_{r-1}(t))$ for $t > e_{r-1}(0)$, with $l_0(t) = t$. Define

$$\eta_r(\lambda) = 1/l_r(1/\lambda). \tag{3}$$

The function $\eta_r(\lambda)$ is defined in $G_r = \{0 < \lambda < 1/e_r(0)\}$. The functions $\eta_0, \dots, \eta_r$ constitute a Pfaffian chain of order $r + 1$ and degree $r + 2$, since

$$d\eta_0 = -\eta_0^2\, d\lambda,\ d\eta_1 = \eta_0\eta_1^2\, d\lambda, \dots, d\eta_r = (-1)^{r-1}\eta_0 \cdots \eta_{r-1}\eta_r^2\, d\lambda.$$

In the following, we fix a Pfaffian chain $y(x) = (y_1(x), \dots, y_r(x))$ and, if not explicitly stated otherwise, consider only Pfaffian functions with this particular Pfaffian chain, without explicit reference to the functions $y_i(x)$ and their domains of definition G_i.

Definition 2.4. A *basic semi-Pfaffian set* X of the *format* (I, J, n, r, p, d) in a semi-Pfaffian domain $G \subset \mathbb{R}^n$ is defined by a system of equations and inequalities

$$X = \{x \in G,\ \phi_i(x) = 0,\ \psi_j(x) > 0,\ \text{for } i = 1, \dots, I,\ j = 1, \dots, J\} \tag{4}$$

where ϕ_i and ψ_j are Pfaffian functions in G of degree not exceeding d, with a common Pfaffian chain of order r and degree p. We assume that G satisfies conditions (i) and (ii) of Definition 2.1, and the inequalities (2) for G are included in the definition of X.

The set X is *restricted* in G if $\overline{X} \subset G$.

A *semi-Pfaffian set* of the *format* (N, I, J, n, r, p, d) is a finite union of at most N basic semi-Pfaffian sets of the formats not exceeding (I, J, n, r, p, d) component-wise, all with the same Pfaffian chain. A semi-Pfaffian set X is *restricted* if it is a finite union of restricted basic semi-Pfaffian sets.

We need the following properties of semi-Pfaffian sets.

Proposition 2.5. *Semi-Pfaffian sets in G constitute a Boolean algebra. The format of a set defined by a Boolean formula with semi-Pfaffian sets admits an upper bound in terms of the formats of these sets and the complexity of the Boolean formula.*

Theorem 2.6 (Khovanskii [15], see also [24]). *The number of connected components of a semi-Pfaffian set X is finite, and admits an upper bound in terms of the format of X.*

Definition 2.7. A semi-Pfaffian set X is *nonsingular* of codimension k if, in a neighborhood of any point $x^0 \in X$, it coincides with a basic semi-Pfaffian set $\{\phi_1(x) = \cdots = \phi_k(x) = 0\}$ with the differentials of the functions $\phi_1, \dots, \phi_k$ independent at x^0.

Proposition 2.8 (See [7]). *Every semi-Pfaffian set X can be represented as a disjoint union of semi-Pfaffian subsets X^k, nonsingular of codimension k. For each k, $\bigcup_{l \geq k} X^l$ is relatively closed in X. The formats of X^k admit upper bounds in terms of the format of X.*

Definition 2.9. *Dimension* of a semi-Pfaffian set X is the maximum d such that X^{n-d} in Proposition 2.8 is nonempty.

Proposition 2.10. *Let X be a semi-Pfaffian set in a semi-Pfaffian domain G. Then $\overline{X} \cap G$ and $\partial X \cap G$ are semi-Pfaffian sets. The formats of these sets admit upper bounds in terms of the format of X.*

Proof. This follows from the algorithm [6] for the frontier and closure of a semi-Pfaffian set, and from the complexity estimates in [8]. □

Lemma 2.11 (Curve selection). *Let X be a semi-Pfaffian set in a semi-Pfaffian domain G such that $0 \in \overline{X \setminus \{0\}}$. There exists a one-dimensional nonsingular semi-Pfaffian subset γ of $X \setminus \{0\}$ such that $0 \in \overline{\gamma}$. The format of γ admits an upper bound in terms of the format of X.*

Proof. Due to Proposition 2.8, we can suppose X to be a nonsingular basic semi-Pfaffian set of codimension k such that the differentials of $\phi_1, \dots, \phi_k$ in (4) are independent at each point of X. Let ψ be the the product of all functions ψ_j in (4) multiplied by $1 + (c, x)$, with a generic vector c. If there are no inequalities in (4), we set $\psi = 1 + (c, x)$.

We assume (see Definition 2.4) that the functions ψ_j include the inequalities for G. In particular, ψ vanishes on ∂X.

Consider the set where $|\psi|$ is maximal over $X_\epsilon = \{x \in X : |x| = \epsilon\}$. This set is contained in the set γ_ϵ of critical points of $\psi|_{X_\epsilon}$. It follows from Lemma 2.15 below that, for a generic c, these critical points are non-degenerate, for small $\epsilon > 0$. Hence, for a small $\delta > 0$, the set $\gamma = \{(\epsilon, \gamma_\epsilon) : 0 < \epsilon < \delta\}$ is nonsingular one-dimensional. It is clear that γ is semi-Pfaffian and $0 \in \overline{\gamma}$. □

Proposition 2.12 (Exponential Lojasiewicz inequality, [12, 16, 17]). *Let X be a semi-Pfaffian set in $G \subset \mathbb{R}^n$ with a Pfaffian chain of order r, and let $q(x)$ be a Pfaffian function in $\mathbb{R}^n$. Suppose that 0 belongs to the closure of $X \cap \{q(x) > 0\}$. Then 0 belongs to the closure of*

$$x \in X : q(x) \geq 1/e_r(|x|^{-N}), \tag{5}$$

for some $N > 0$. Here $e_r(\lambda)$ is the iterated exponential function from Example 2.3.

Proof. Let $X_\epsilon = X \cap \{|x| = \epsilon\}$. Due to Lemma 2.11, we can suppose that $X \cap \{q > 0\}$ is a nonsingular curve. Let us choose a branch γ of this curve such that $0 \in \overline{\gamma}$. Let $y(x) = (y_1(x), \dots, y_r(x))$ be the Pfaffian chain for X. We have $\gamma \subset \{\phi_1(x) = \cdots = \phi_{n-1}(x) = 0\}$ where $\phi_j(x) = Q_j(x, y(x))$ are Pfaffian functions, with Q_j polynomial in (x, y), and the differentials of $\phi_j(x)$ are independent on γ. This implies that the differentials of $Q_1(x, y), \dots, Q_{n-1}(x, y)$ are independent on $\Gamma = \{x \in \gamma,\ y = y(x)\}$. In particular, there is a $(r+1)$-dimensional irreducible component Z of the algebraic set $\{Q_1(x, y) = \cdots = Q_{n-1}(x, y) = 0\}$ in $\mathbb{R}^{n+r}$ such that $\Gamma \subset Z$. After a linear change of variables in $\mathbb{R}^n$, we can suppose that $|x_n| = \max_i |x_i|$ on γ in the neighborhood of 0. Since $Z \not\subset \{x_n = 0\}$, there exist linear functions $l_1(x, y), \dots, l_r(x, y)$ in $\mathbb{R}^{n+r}$ such that $\mathbb{R}[x, y]/I(Z)$ is algebraic over $\mathbb{R}[x_n, l_1, \dots, l_r]$. In particular, functions $x_1, \dots, x_{n-1}$ and $y_1(x), \dots, y_r(x)$ restricted to γ are algebraic over the field generated by $r + 1$ functions $x_n, l_1(x, y(x)), \dots, l_r(x, y(x))$ restricted to γ.

Consider $t = 1/|x_n|$ as a parameter on γ in the neighborhood of $x = 0$. Restrictions of Pfaffian functions to γ can be considered as functions in t defined for large t. Due to the finiteness properties of Pfaffian functions [15], germs at $t = \infty$ of these functions generate a Hardy field H. The above arguments imply that H has transcendence degree at most r over $\mathbb{R}(t)$. Due to Proposition 5 of [20], rank of H does not exceed $r+1$. From Theorem 2 of [20], any function $h(t)$ in H is dominated by an iterated exponential function e_r (see Example 2.3 above): $|h(t)| < e_r(t^N)$ for some $N > 0$ as $t \to \infty$. Our statement follows from this inequality applied to $h = (1/q)|_\gamma$, since $|x_n| = \max_i |x_i| \geq |x|/\sqrt{n}$ on γ in the neighborhood of 0. □

Lemma 2.13. *Let X be a smooth manifold in $\mathbb{R}^n$. Let $f_c(x) = f(x) - \sum_\alpha c_\alpha g_\alpha(x)$ be a family of smooth functions on X depending on parameters $c \in \mathbb{R}^m$. Suppose that, for any $x \in X$, the differentials of g_α generate the cotangent space to X at x. Then, for a generic c, $f_c(x)$ has only non-degenerate critical points. More precisely, the values of c such that $f_c(x)$ has a degenerate critical point constitute a zero measure set $S \subset \mathbb{R}^m$.*

Proof. This is a variant of Thom's transversality theorem (See, e.g., [11], Ch. II). For convenience, we give a proof here. Let $d = \dim X$. Fix $x^0 \in X$. One can renumber g_α so that the differentials of $g_1, \dots, g_d$ generate the cotangent space to X at x^0. Let us change coordinates in a neighborhood U of x^0 so that $g_i(x) = x_i - a_i$, for $x \in U,\ i = 1, \dots, d$. Consider the mapping $df : U \to \mathbb{R}^d$ in these coordinates. The set of critical points of f_c in U coincides with $df^{-1}(c)$, and all these points are non-degenerate when c is not a critical value of df. From Sard's theorem, the set S_U of critical values of df has zero measure. Since the sets U selected for different points x^0 cover X, a countable covering of X by these sets can be found. Accordingly, the set S, a countable union of the sets S_U, has zero measure. □

Lemma 2.14. *Let X be a smooth manifold in $\mathbb{R}^n$, and $f(x)$ a smooth non-vanishing function on X. For a generic $c = (c_1, \dots, c_n)$, all critical points of a function $f(x)(1 + (c, x))$ are non-degenerate. More precisely, the values of c such that $f(x)(1+(c, x))$ has a degenerate critical point constitute a zero measure set $V \subset \mathbb{R}^n$.*

Proof. Consider the following family: $f_{a,c} = f(x) - af(x) + (c, x)f(x)$. It is easy to see that the differentials of $f(x)$ and $x_i f(x)$ generate the cotangent space to X at each point $x^0 \in X$. Lemma 2.13 implies that the set

$$S = \{(a, c) : f_{a,c} \text{ has a degenerate critical point}\}$$

has zero measure in $\mathbb{R}^{n+1}$. Since multiplication by a constant does not change critical points and their degeneracy, $S \cap \{a \neq 1\}$ is a cylinder over the set V. Hence V has zero measure in $\mathbb{R}^n$. □

Lemma 2.15. *Let X be a smooth manifold in $\mathbb{R}^n$, and $F(x,\lambda)$ a smooth non-vanishing function on $X\times\mathbb{R}^d$. For a fixed λ, consider $f_\lambda(x)=F(x,\lambda)$ as a function on X. For a generic c, the set*

$$W_c=\{\lambda : f_\lambda(x)(1+(c,x)) \text{ has a degenerate critical point}\}$$

has zero measure in $\mathbb{R}^d$.

Proof. Lemma 2.14 implies that, for each λ, the set

$$S_\lambda=\{c : f_\lambda(x)(1+(c,x)) \text{ has a degenerate critical point}\}$$

has zero measure in $\mathbb{R}^n$. Let $S=\cup_\lambda(S_\lambda,\lambda)\subset\mathbb{R}^n\times\mathbb{R}^d$. Due to Fubini theorem, S has measure zero in $\mathbb{R}^n\times\mathbb{R}^d$. This implies that, for a generic c, the set $W_c=S\cap\{c=\text{const}\}$ has zero measure in $\mathbb{R}^d$. □

3 Relative closure and limit sets

Let $\mathbb{R}^n\times\mathbb{R}$ be $(n+1)$-dimensional space, with coordinates $x=(x_1,\dots,x_n)$ and λ. For a set $X\subset\mathbb{R}^n\times\mathbb{R}$, we define $X_+=X\cap\{\lambda>0\}$, $X_\lambda=X\cap\{\lambda=\text{const}\}$, and $\check{X}=\overline{X_+}\cap\{\lambda=0\}$. Coordinate λ is considered as a *parameter*, and the set X is considered as a *family* of sets X_λ in $\mathbb{R}^n$.

Definition 3.1. Let G be a semi-Pfaffian domain (see Definition 2.1) in $\mathbb{R}^n\times\mathbb{R}$. A subset $X\subset G$ is a *semi-Pfaffian family* if X is a semi-Pfaffian set with a Pfaffian chain defined in G and, for any $\epsilon>0$, the set $X\cap\{\lambda>\epsilon\}$ is restricted in G. The *format* of X is defined as the format of a semi-Pfaffian set X_λ for a small $\lambda>0$.

Remark 3.2. In all constructions below, upper bounds on the complexity can be established for semi-Pfaffian families considered as semi-Pfaffian sets in $\mathbb{R}^n\times\mathbb{R}$. However, the upper bounds in terms of the format of a family (i.e., the complexity of the fibers X_λ) are more important in applications, since they provide better estimates for the geometric and topological complexity of limit sets.

Proposition 3.3. *Let X be a semi-Pfaffian family. Then $\overline{X}_+$ and $(\partial X)_+$ are semi-Pfaffian families. The formats of these families admit upper bounds in terms of the format of X.*

Proof. Since $X\cap\{\lambda>\epsilon\}$ is restricted in G, for any $\epsilon>0$, the set $\overline{X}_+$ is contained in G. Proposition 3.3 implies that $\overline{X}_+$ and $(\partial X)_+$ are semi-Pfaffian sets in G. The sets $\overline{X}_+\cap\{\lambda>\epsilon\}$ and $(\partial X)_+\cap\{\lambda>\epsilon\}$ are restricted in G, for any $\epsilon>0$, since this is true for X.

The statement on the formats follows from Proposition 2.10, since $(\overline{X})_\lambda=\overline{X_\lambda}$ and $(\partial X)_\lambda=\partial(X_\lambda)$ for a generic $\lambda>0$. These equalities can be derived from Proposition 2.8, Sard's theorem, and the finiteness properties of semi-Pfaffian sets. □

Definition 3.4. Two semi-Pfaffian families X and Y form a *semi-Pfaffian couple* (X,Y) if Y is relatively closed in $\{\lambda>0\}$ (i.e., $\overline{Y}_+=Y_+$) and contains $(\partial X)_+$. The format of the couple (X,Y) is defined as the component-wise maximum of the formats of X and Y.

Definition 3.5. Let (X,Y) be a semi-Pfaffian couple in $G\subset\mathbb{R}^n\times\mathbb{R}$. The *relative closure* of (X,Y) is defined as

$$(X,Y)_0=\check{X}\setminus\check{Y}\subset\check{G}\subset\mathbb{R}^n. \tag{6}$$

If $Y=(\overline{\partial X})_+$, we write X_0, the *relative closure* of X, instead of $(X,Y)_0$. The format of $(X,Y)_0$ is defined as the format of the couple (X,Y).

Definition 3.6. A *limit set* in $\Omega\subset\mathbb{R}^n$ is a finite union of the relative closures $(X_i,Y_i)_0$ of semi-Pfaffian couples (X_i,Y_i) in $G_i\subset\mathbb{R}^n\times\mathbb{R}$, such that $\check{G}_i=\Omega$ for all i. The format of a limit set is defined as (K,N,I,J,n,r,p,d) where (N,I,J,n,r,p,d) is the component-wise maximum of the formats of the couples (X_i,Y_i), and K is the number of these couples.

Proposition 3.7 (Complement of a limit set). *Let (X, Y) be a semi-Pfaffian couple in $G \subset \mathbb{R}^n \times \mathbb{R}$. Then the complement $\check{G} \setminus (X, Y)_0$ of $(X, Y)_0$ in $\check{G}$ is a limit set. The format of this limit set admits an upper bound in terms of the format of (X, Y).*

Proof. We assume that inequalities $s_{i\nu}(x, \lambda) = S_{i\nu}(x, y_1(x, \lambda), \dots, y_i(x, \lambda)) > 0$ (see (2)) defining G are included in the definition of X. Let $s(x, \lambda)$ be the product of all functions $s_\nu(x, \lambda)$ in these inequalities, so that $s > 0$ in G and $s = 0$ on ∂G. Let $G' = G \cap \{\lambda > 0,\ s(x, \lambda) \geq 1/e_r(\lambda^{-N})\}$. Here r is the order of the Pfaffian chain for X and N is a positive integer. Let $Z = G \setminus X$ and $Z' = Z \cap G'$. It is clear that Z' is a semi-Pfaffian family in G, and its format (as a family) does not depend on N (since $1/e_r(\lambda^{-N})$, for a fixed λ, is a constant). It follows from Proposition 2.12 that $\check{Z} = \check{Z}'$ for large N. We are going to prove that

$$\check{G} \setminus (X, Y)_0 = (Z', \overline{X}_+)_0 \cup (Y, \emptyset)_0. \tag{7}$$

By definition of the relative closure, the right side of (7) equals $(\check{Z}' \setminus \check{X}) \cup \check{Y} = (\check{Z} \setminus \check{X}) \cup \check{Y}$. Since $(X, Y)_0 \cap (\check{Z} \setminus \check{X}) = \emptyset$ and $(X, Y)_0 \cap \check{Y} = \emptyset$, the left side of (7) contains its right side. Let now $x \in \check{G} \setminus (X, Y)_0$. Note that x belongs either to $\check{X}$ or to $\check{Z}$ (or to both). If $x \in \check{X}$ then $x \in \check{Y}$. Otherwise, $x \in \check{Z} \setminus \check{X}$. This implies that the right side of (7) contains its left side. □

Proposition 3.8 (Product of limit sets). *Let (X, Y) and (X', Y') be two semi-Pfaffian couples in $G \subset \mathbb{R}^n \times \mathbb{R}$ and $G' \subset \mathbb{R}^m \times \mathbb{R}$, respectively. Then the product of $(X, Y)_0$ and $(X', Y')_0$ is a limit set in $\check{G} \times \check{G}' \subset \mathbb{R}^n \times \mathbb{R}^m$:*

$$(X, Y)_0 \times (X', Y')_0 = (X \times_{\mathbb{R}} X', Z)_0, \ \textit{where } Z = (\overline{X}_+ \times_{\mathbb{R}} Y') \cup (Y \times_{\mathbb{R}} \overline{X'}_+). \tag{8}$$

Here $X \times_{\mathbb{R}} X' = \{(x, x', \lambda) : (x, \lambda) \in X,\ (x', \lambda) \in X'\}$ is the fibered product over $\mathbb{R}$.

Proof. Let $z \in \check{X}$ and $z' \in \check{X}'$. From Lemma 2.11, one can find continuous functions $x = x(\lambda)$ and $x' = x'(\lambda)$ defined for small $\lambda > 0$ such that $(x(\lambda), \lambda) \in X$, $(x'(\lambda), \lambda) \in X'$, and $\lim_{\lambda \searrow 0}(x(\lambda), x'(\lambda)) = (z, z')$. Hence $(z, z') \in \check{(X \times_{\mathbb{R}} X')}$. This implies $\check{X} \times \check{X}' = \check{(X \times_{\mathbb{R}} X')}$. Similarly, $(\check{X} \times \check{Y}') \cup (\check{Y} \times (\check{X}') = \check{Z}$. The statement then follows from standard set-theoretic arguments. □

Proposition 3.9 (Intersection of limit sets). *Let (X, Y) and (X', Y') be two semi-Pfaffian couples. Then $(X, Y)_0 \cap (X', Y')_0$ is a limit set. The format of this limit set admits an upper bound in terms of the formats of the couples (X, Y) and (X', Y').*

Proof. We are going to prove that, for large integer N,

$$(X, Y)_0 \cap (X', Y')_0 = ((X \times_{\mathbb{R}} X') \cap W_N, Z)_0 \tag{9}$$

where Z is defined in (8) and

$$W_N = \{(x, x', \lambda) : |x - x'| \leq \eta_r(\lambda)^{1/N}\}. \tag{10}$$

Here r is the order of the Pfaffian chain for X, Y, X', Y', η_r is the iterated logarithmic function defined in (3), and $\check{G}$ is identified with its diagonal embedding in $\mathbb{R}^n \times \mathbb{R}^n$. The statement follows from Propositions 2.12 and 3.8, and the identity

$$(X, Y)_0 \cap (X', Y')_0 = [(X, Y)_0 \times (X', Y')_0] \cap \{x = x'\}.$$

We only have to show that

$$\check{((X \times_{\mathbb{R}} X') \cap W_N)} = \check{(X \times_{\mathbb{R}} X')} \cap \{x = x'\}.$$

Due to Lemma 2.11, a point (z, z) belongs to $\check{(X \times_{\mathbb{R}} X')}$ if and only if z belongs to $\check{X} \cap \check{X}'$. From (5) applied to $q \equiv \lambda$, the point $(z, 0)$ belongs to the closures of $X \cap \{(x, \lambda) : \eta_r(\lambda) \geq |x - z|^N\}$ and $X' \cap \{(x', \lambda) : \eta_r(\lambda) \geq |x' - z|^N\}$, for large enough N. Let (x, λ) and (x', λ) be two points in X and X', respectively, satisfying these two inequalities. Then $|x-x'| \le |x-z| + |x'-z| \le 2(\eta_r(\lambda))^{1/N}$. For small λ, this implies $|x-x'|^{N+1} \le \eta_r(\lambda)$, hence $(z,z,0)$ belongs to the closure of $(X_+ \times_{\mathbb{R}} X'_+) \cap W_{N+1}$, q.e.d.

To derive an upper bound for the format of $(X,Y)_0 \cap (X',Y')_0$, note that, for a fixed λ, $\eta_r(\lambda)^{1/N}$ is a constant, and $(W_N)_\lambda$ is a semialgebraic set of degree 2. □

Theorem 3.10. *Limit sets constitute a Boolean algebra. The format of a limit set defined by a Boolean formula with limit sets $X_1, \ldots, X_N$ admits an upper bound in terms of the complexity of the formula and the formats of $X_1, \ldots, X_N$.*

Proof. This follows from Propositions 3.7 and 3.9. □

Proposition 3.11. *Let (X,Y) be a semi-Pfaffian couple, and X' a semi-Pfaffian family such that X' is a relatively closed subset of X. Then $(X \setminus X', Y \cup X')$ and (X',Y) are semi-Pfaffian couples, and $(X,Y)_0$ is a disjoint union of $(X \setminus X', Y \cup X')_0$ and $(X',Y)_0$.*

Proof. Since X' is relatively closed in X, we have $(\partial X')_+ \subset (\partial X)_+ \subset Y$. In particular, (X',Y) is a semi-Pfaffian couple, and $Y \cup X'$ is relatively closed in $\{\lambda > 0\}$. Since a point in $\partial(X \setminus X')$ belongs either to ∂X or to X', we have $\big(\partial(X \setminus X')\big)_+ \subset Y \cup X'$, hence $(X \setminus X', Y \cup X')$ is a semi-Pfaffian couple.

It is clear that $(X \setminus X', Y \cup X')_0$ and $(X',Y)_0$ are disjoint subsets of $(X,Y)_0$. If a point $x^0 \in (X,Y)_0$ belongs to $\check{X}'$, then $x^0 \in (X',Y)_0$. Otherwise, x^0 belongs to $(X \setminus X', Y \cup X')_0$. □

Proposition 3.12. *Let (X,Y) be a semi-Pfaffian couple. Then $(X,Y)_0$ is a disjoint union of sets $(X^k, Y^k)_0$ with nonsingular k-dimensional sets X^k. Here $k = 0, \ldots, \dim X$. The formats of the semi-Pfaffian couples (X^k, Y^k) admit upper bounds in terms of the format of (X,Y).*

Proof. This follows from Propositions 2.8 and 3.11. □

Theorem 3.13 (See also [10]). *Let (X,Y) be a semi-Pfaffian couple. Then the number of connected components of $(X,Y)_0$ is finite, and admits an upper bound in terms of the format of (X,Y).*

Proof. Let $\Psi(x) = \min_{x' \in \check{Y}} (x-x')^2$ be the (squared) distance from x to $\check{Y}$ and, for $\lambda > 0$, let $\Psi_\lambda(x) = \min_{y \in Y_\lambda} (x-y)^2$ be the distance from x to Y_λ. Let Z_λ be the set of local maxima of $\Psi_\lambda|_{X_\lambda}$.

For every connected component C of $(X,Y)_0$, the function $\Psi(x)$ is positive on C and vanishes on ∂C, hence Ψ has a local maximum $x_0 \in C$. For small $\lambda > 0$, there exist $x_\lambda \in X_\lambda$ such that $|x_\lambda - x_0| \to 0$ as $\lambda \searrow 0$. This implies $\lim_{\lambda \searrow 0} \Psi_\lambda(x_\lambda) = \Psi(x_0) > 0$. In particular, there exists a positive constant ϵ such that $\Psi_\lambda(x_\lambda) > \epsilon$ for small $\lambda > 0$. Let $W_{\lambda,\epsilon} = \{x \in X_\lambda,\ \Psi_\lambda(x) > \epsilon$, and let C_λ be the connected component of x_λ in $W_{\lambda,\epsilon}$. Since $\Psi_\lambda(x) > \epsilon$ for any $x \in C_\lambda$, the sets C_λ are close to C for small positive λ, i.e., the closure of $\bigcup_{\lambda > 0} C_\lambda$ intersected with $\{\lambda = 0\}$ is a connected subset of $(X,Y)_0$ containing x_0, hence a subset of C. From the definition of C_λ, there exists a local maximum z_λ of $\Psi_\lambda|_{X_\lambda}$ in C_λ, and a connected component V_λ of Z_λ containing z_λ belongs to C_λ. Hence V_λ is close to C for small positive λ. This implies that the number of connected components of $(X,Y)_0$ does not exceed the number of connected components of Z_λ, for small positive λ.

Since Z_λ is a restricted sub-Pfaffian set, an upper bound on the number of its connected components in terms of the format of (X,Y) can be obtained either from [8] or from the bounds on the Betti numbers of restricted sub-Pfaffian sets in [9]. □

4 Regular families and dimension of limit sets

We consider $\mathbb{R}^n$ equipped with the standard Euclidean metric $|x|^2 = \sum x_i^2$. For a linear subspace $L \subset \mathbb{R}^n$, we define $L^\perp$ to be its orthogonal complement in $\mathbb{R}^n$. Let $\pi_L : \mathbb{R}^n \to L^\perp$ be a projection along L. For $x \in \mathbb{R}^n$ or $z = \pi_L x \in L^\perp$, let $L + x = L + z$ denote an affine subspace of $\mathbb{R}^n$ through x parallel to L.

For $I = \{i_1, \dots, i_d\} \subset \{1, \dots, n\}$, let $\mathbb{R}_I$ be the $(n-d)$-dimensional coordinate subspace of $\mathbb{R}^n$ defined by $x_I = (x_{i_1}, \dots, x_{i_d}) = 0$. Let π_I denote a projection along $\mathbb{R}_I$.

Definition 4.1. Let L and T be two linear subspaces in $\mathbb{R}^n$. We define *internal* distance from L to T as

$$\operatorname{dist}_i(L,T) = \sup_{x \in L,\, |x|=1} \ \inf_{y \in T,\, |y|=1} |x-y|. \tag{11}$$

Note that $\operatorname{dist}_i(L,T) \neq \operatorname{dist}_i(T,L)$, and $\operatorname{dist}_i(L,T) > 0$ if and only if $L \not\subset T$. *External* distance between L and T in $\mathbb{R}^n$ is defined as

$$\operatorname{dist}_e(L,T) = \inf_{x \in T^\perp,\, |x|=1} \ \inf_{y \in L^\perp,\, |y|=1} |x-y|. \tag{12}$$

Note that dist_e depends on the ambient space $\mathbb{R}^n$. When it is necessary to specify the ambient space, we write $\operatorname{dist}_e(L,T;\mathbb{R}^n)$ instead of $\operatorname{dist}_e(L,T)$. We have $\operatorname{dist}_e(L,T;\mathbb{R}^n) > 0$ if and only if L and T are transversal: $L + T = \mathbb{R}^n$.

Lemma 4.2. *For fixed dimensions, d and k, of L and T, both $\operatorname{dist}_i(L,T)$ and $\operatorname{dist}_e(L,T)$ are continuous nonnegative semialgebraic functions on $G_{d,n} \times G_{k,n}$, where $G_{d,n}$ denotes the Grassmannian of d-dimensional subspaces in $\mathbb{R}^n$.*

Proof. This follows from Definition 4.1 and the Tarski-Seidenberg principle. □

Lemma 4.3. *Let d and n be two positive integers, $d < n$. There exists a constant $C_{d,n} > 0$ such that, for any d-dimensional subspace L of $\mathbb{R}^n$, there is a subset $I = \{i_1, \dots, i_d\} \subset \{1, \dots, n\}$ with $\operatorname{dist}_e(L, \mathbb{R}_I) > C_{d,n}$.*

Proof. For any d-dimensional subspace L of $\mathbb{R}^n$, there exists $I = \{i_1, \dots, i_d\} \subset \{1, \dots, n\}$ such that L is transversal to $\mathbb{R}_I$. This implies that

$$\rho(L) = \max_{I : |I| = d} \operatorname{dist}_e(L, \mathbb{R}_I)$$

is positive. Since ρ is a continuous function on $G_{d,n}$, its minimum value $C_{d,n}$ is positive. □

Definition 4.4. Let X be a semi-Pfaffian family in $\mathbb{R}^n \times \mathbb{R}$, and L a linear subspace of $\mathbb{R}^n$. We say that X is *L-regular* at $x^0 \in \mathbb{R}^n$ if there exists a neighborhood Ω of x^0 and a constant $C > 0$ such that, for small $\lambda > 0$, the set $X_\lambda \cap \Omega$ is nonsingular and

$$\operatorname{dist}_e(L, T_x X_\lambda; \mathbb{R}^n) > C. \tag{13}$$

for all $x \in X_\lambda \cap \Omega$. In other words, for any sequence $(x^\nu, \lambda^\nu) \in X_+$ converging to $(x^0, 0)$, the limit of $T_{x^\nu} X_{\lambda^\nu}$, if exists, is transversal to L.

A couple (X,Y) is *L-regular* if X is L-regular at each point $x^0 \in (X,Y)_0$. For $L = \mathbb{R}_I$, an L-regular couple is called I-regular.

Proposition 4.5. *Let (X,Y) be a semi-Pfaffian couple in $G \subset \mathbb{R}^n \times \mathbb{R}$. Let L be a linear subspace in $\mathbb{R}^n$. Suppose that (X,Y) is L-regular at $x^0 \in (X,Y)_0$. Let $T = (L + x^0) \times \mathbb{R}$. Then $x^0 \in (X \cap T, Y)_0$.*

Proof. From the definition of L-regularity, there exists a neighborhood Ω of x^0 and a constant $C > 0$ such that (13) holds for small $\lambda > 0$ and $x \in X_\lambda \cap \Omega$. One can choose Ω a cylinder over a neighborhood U of $z^0 = \pi_L x^0$ in $L^\perp$.

Let (x^ν, λ^ν) be a sequence of points in X_+ converging to $(x^0, 0)$. We have $x^\nu \in X_{\lambda^\nu} \cap \Omega$ for large ν. Since $x^0 \in (X,Y)_0$, we have also $\partial X_{\lambda^\nu} \cap \Omega = \emptyset$ for large ν. Let $z^\nu = \pi_L x^\nu$. Let us connect (z^0, λ^ν) with (z^ν, λ^ν) by a line segment S_ν of the length $s_\nu = |z^\nu - z^0|$. We have $S_\nu \subset U$ for large ν. Let us parametrize S_ν by $t \in [0, s_\nu]$, with $t = 0$ corresponding to z^ν and $t = s_\nu$ to z^0. Let $\xi_\nu = \partial/\partial t$ be a unit tangent vector field to S_ν. For large ν the set $Z_\nu = X_{\lambda^\nu} \cap \pi_L^{-1} S_\nu \cap \Omega$ is nonsingular, and there is a unique smooth vector field ζ_ν on Z_ν orthogonal to $Z_\nu \cap (L + x^\nu)$ such that $\pi_L \zeta_\nu = \xi_\nu$. Due to (13), $sup_{Z_\nu}|\zeta_\nu|$ is bounded uniformly in ν.

Let γ_ν be a trajectory of ζ_ν starting at (x^ν, λ^ν). Since ζ_ν is uniformly bounded, we can assume, taking U small enough, that γ_ν cannot escape Ω at a point $x \in \partial\Omega$ such that $\pi_L x \in U$. Since $X \cap \{\lambda > \epsilon\}$ is restricted in G, for every $\epsilon > 0$, γ_ν cannot escape G other than through ∂X. Since $\partial X \subset Y$ and $Y_{\lambda^\nu} \cap \Omega = \emptyset$ for large ν, the only possibility for γ_ν is to end at a point $(u^\nu, \lambda^\nu) \in X_{\lambda^\nu} \cap \Omega$ such that $\pi_L u^\nu = z^0$, hence $(u^\nu, \lambda^\nu) \in X_+ \cap T$. Since $x^\nu \to x^0$ and ζ^ν is uniformly bounded, we have $u^\nu \to x^0$ as $\nu \to \infty$. This implies $x^0 \in \check{(}X \cap T)$. Since $x^0 \notin \check{Y}$, we have $x^0 \in (X \cap T, Y)_0$. ☐

Definition 4.6. Let L be a linear subspace in $\mathbb{R}^n$. A subset Z of $\mathbb{R}^n$ is *L-Lipschitz* if, in a neighborhood of each point $x^0 \in Z$, the set Z coincides with a finite union of graphs of Lipschitz functions $f_\nu : L^\perp \to L$. For $L = \mathbb{R}_I$, L-Lipschitz sets are called I-Lipschitz.

Proposition 4.7. *Let L be a linear subspace of $\mathbb{R}^n$ of codimension d. Let (X, Y) be a L-regular semi-Pfaffian couple in $\mathbb{R}^n \times \mathbb{R}$ with $\dim X = d + 1$. Then $(X, Y)_0$ is an L-Lipschitz set.*

Proof. Let $x^0 \in (X, Y)_0$. Due to Proposition 4.5, $(x^0, 0)$ belongs to the closure of $\Gamma = X_+ \cap T$ where $T = (L + x^0) \times \mathbb{R}$. The set Γ is nonsingular one-dimensional in the neighborhood of $(x^0, 0)$. Let Γ_k be distinct branches of Γ such that $(x^0, 0) \in \overline{\Gamma_k}$.

Let Ω be a neighborhood of x^0 in $\mathbb{R}^n$ such that, for small $\lambda > 0$, we have $Y_\lambda \cap \Omega = \emptyset$ and (13) holds at each point of $X_\lambda \cap \Omega$. We can choose Ω a cylinder over $U \subset L^\perp$ where U is a small neighborhood of $z^0 = \pi_L x^0$ in $L^\perp$. With the same arguments as in the proof of Proposition 4.5, one can show that, for small $\lambda > 0$, the set $X_\lambda \cap \Omega$ is a finite union of graphs of smooth functions $f_{k,\lambda}$ on U with values in L, with the graph of $f_{k,\lambda}$ passing Γ_k.

Since X is L-regular at x^0, the gradients of $f_{k,\lambda}$ are uniformly bounded, independent of λ. For a fixed $z \in U$ and a fixed k, the values $f_{k,\lambda}(z)$ are bounded and depend monotonously on λ as $\lambda \to 0$. Let X_k be the union over $\lambda > 0$ of the graphs of $f_{k,\lambda}$. Then $Z_k = \check{X}_k \subset \check{X} \cap \Omega$ is a graph of a Lipschitz function in U with values in L, and $(X, Y)_0 \cap \Omega = \cup_k Z_k$. ☐

Proposition 4.8. *Let (X, Y) be a semi-Pfaffian couple in $G \subset \mathbb{R}^n \times \mathbb{R}$ with $\dim X = d + 1$. Then*

$$(X, Y)_0 = \bigcup (X_I, Y_I)_0, \tag{14}$$

union over $I \subset \{1, \dots, n\}$ with $|I| \le d$, so that

(a) *(X_I, Y_I) is an I-regular semi-Pfaffian couple in G,*

(b) *$X_I \subset X$ is either empty or $(|I| + 1)$-dimensional, and $\dim Y_I \le \max(\dim Y, d)$. The formats of (X_I, Y_I) admit upper bounds in terms of the format of (X, Y).*

Proof. For $d = 0$, we can suppose X to be nonsingular 1-dimensional. Then (X, Y) is I-regular for $I = \emptyset$.

Due to Proposition 2.8, there exists a relatively closed subset $V \subset X$ such that $X \setminus V$ is nonsingular $(d + 1)$-dimensional, and $\dim V \le d$. For $I \subset \{1, \dots, n\}$ with $|I| = d$, let $X_I = \{(x, \lambda) \in X \setminus V : \operatorname{dist}_e(\mathbb{R}_I, T_x X_\lambda) > C_{d,n}\}$, where $C_{d,n}$ is defined in Lemma 3.3. Then $X \setminus V = \bigcup_{|I|=d} X_I$ and ∂X_I is relatively closed in $X \setminus V$. Due to Proposition 3.11,

$$(X, Y)_0 = \bigcup_{|I|=d} (X_I, Y_I)_0 \cup (W, Y)_0, \text{ where } Y_I = Y \cup V \cup \partial X_I \text{ and } W = V \bigcup_{|I|=d} (X \cap \partial X_I).$$

Note that each couple (X_I, Y_I) is I-regular, and $\dim W \le d$. The statement follows now from the induction hypothesis. ☐

Definition 4.9. For a semi-Pfaffian couple (X, Y) in $\mathbb{R}^n \times \mathbb{R}$, *dimension* $\dim(X, Y)_0$ is defined as maximum of $|I|$ over $I \subset \{1, \dots, n\}$ such that $(X_I, Y_I)_0 \neq \emptyset$ in (14).

Proposition 4.10. *Let $K \subset \{1, \dots, n\}$. Suppose that (X, Y) in Proposition 4.8 satisfies the following property: $X \subset Z$ where Z is a $(|K|+1)$-dimensional semi-Pfaffian family, K-regular at all $x \in (X, Y)_0$. Then the union in (14) can be taken over $I \subset K$.*

Proof. We repeat the arguments in the proof of Proposition 4.8, replacing the condition on $T_x X_\lambda$ in the definition of X_I by the corresponding condition on $\pi_K T_x X_\lambda$. Let $d = \dim X - 1$ and $k = |K|$. For $I \subset K$ with $|I| = d$, let

$$X_I = \{(x, \lambda) \in X \setminus V : \operatorname{dist}_e(\pi_K \mathbb{R}_I, \pi_K(T_x X_\lambda); \mathbb{R}_K^\perp) > C_{d,k}\},$$

where V is the singular set of X and $C_{d,k}$ is defined in Lemma 3.3. Then

$$X \setminus V = \bigcup_{I \subset K,\, |I|=d} X_I \text{ and } (X, Y)_0 = \bigcup_{I \subset K,\, |I|=d} (X_I, Y_I)_0 \bigcup (W, Y)_0,$$

where $Y_I = Y \cup V \cup \partial X_I$ and $W = V \bigcup_{I \subset K,\, |I|=d} (X \cap \partial X_I)$. □

5 L-tangent families and projections of limit sets

Definition 5.1. Let L be a linear subspace in $\mathbb{R}^n$. A nonsingular family X in $\mathbb{R}^n \times \mathbb{R}$ is *L-tangent* at $x^0 \in \mathbb{R}^n$ if, for any sequence (x^ν, λ^ν) of points in X_+ converging to $(x^0, 0)$, we have $\lim_{\nu \to \infty} \operatorname{dist}_i(L, T_{x^\nu} X_{\lambda^\nu}) = 0$. In other words, the limit of $T_{x^\nu} X_{\lambda^\nu}$, if exists, is contained in L. A couple (X, Y) is L-tangent if X is L-tangent at each point of $(X, Y)_0$. For $L = \mathbb{R}_I$, an L-tangent couple is called I-tangent.

Proposition 5.2. *Let (X, Y) be an L-tangent semi-Pfaffian couple in $\mathbb{R}^n \times \mathbb{R}$. Then $(X, Y)_0$ is contained in a finite number of affine subspaces parallel to L. The number of these planes admits an upper bound in terms of the format of (X, Y).*

Proof. One can assume $L = \mathbb{R}_K$ where $K = \{1, \dots, k\}$. Due to Proposition 4.8, $(X, Y)_0 = \cup_I (X_I, Y_I)_0$ with $X_I \subset X$ either empty or $(|I|+1)$-dimensional, and (X_I, Y_I) I-regular, for each $I \subset \{1, \dots, n\}$. Let $x^0 \in (X_I, Y_I)_0$, for some I. In particular, $(X_I, Y_I)_0 \neq \emptyset$. Since X is K-tangent at x^0 and $X_I \subset X$, X_I is K-tangent at x^0. This is only possible when $I \cap K = \emptyset$, i.e., $\mathbb{R}_K^\perp \subset \mathbb{R}_I$. According to Proposition 4.7, $(X_I, Y_I)_0$ is an I-Lipschitz set. In the neighborhood of x^0, it is a finite union of graphs of Lipschitz functions $f_\nu : \mathbb{R}_I^\perp \to \mathbb{R}_I$. Since X_I is K-tangent, the first k components of each f_ν are constants. This implies that $(X_I, Y_I)_0$ is contained in at most countable set of affine planes parallel to $\mathbb{R}_K$. The number of these planes does not exceed the number of connected components of $(X_I, Y_I)_0$, which admits an upper bound in terms of the format of (X, Y) (Theorem 3.13 and Proposition 4.8). □

Proposition 5.3. *Let (X, Y) be a semi-Pfaffian couple in $\mathbb{R}^n \times \mathbb{R}$ with $\dim X = d+1$, and $J \subset \{1, \dots, n\}$. Then*

$$(X, Y)_0 = \bigcup (X_I, Y_I)_0, \tag{15}$$

union over $I \subset \{1, \dots, n\}$ with $|I| \leq d$, so that

(a) *(X_I, Y_I) is an I-regular semi-Pfaffian couple in $\mathbb{R}^n \times \mathbb{R}$,*

(b) *$X_I \subset X$ is either empty or $(|I|+1)$-dimensional, and $\dim Y_I \leq \max(\dim Y, d)$.*

(c) *for any affine space $T \subset \mathbb{R}^n \times \mathbb{R}$ parallel to $\mathbb{R}_{I \cap J} \times \mathbb{R}$, $(X_I \cap T, Y_I)$ is J-tangent.*

Proof. We use induction on d, as in the proof of Proposition 4.8. For $d = 0$, the set X is 1-dimensional. Then (X, Y) is I-regular for $I = \emptyset$ and J-tangent for any J.

Let V be a relatively closed subset in X such that $X \setminus V$ is nonsingular $(d+1)$-dimensional, and $\dim V \leq d$. Let $I \subset \{1, \ldots, n\}$ with $|I| = d$. Let $K = I \cap J$, $k = |K|$, $m = |J|$. Define

$$X_I = \{(x, \lambda) \in X \setminus V : \quad \begin{array}{l} \mathrm{dist}_e(\mathbb{R}_K, T_x X_\lambda; \mathbb{R}^n) > C_{k,m}, \\ \mathrm{dist}_e(\mathbb{R}_I, \mathbb{R}_K \cap T_x X_\lambda; \mathbb{R}_K) > C_{d-k,n-m}, \\ \mathrm{dist}_i(\mathbb{R}_J, \mathbb{R}_K \cap T_x X_\lambda) < \eta_r(\lambda^N)\} \end{array}$$

where N is a large number, r is the order of the Pfaffian chain for X, and η_r is defined in (3). The constants $C_{k,m}$ and $C_{d-k,n-m}$ are defined in Lemma 3.3.

It can be shown, using Proposition 2.12, that

$$(X, Y)_0 = \bigcup_{|I|=d} (X_I, Y_I)_0 \bigcup (W, Y)_0 \text{ where } Y_I = Y \cup V \cup \partial X_I \text{ and } W = V \bigcup_{|I|=d} (X \cap \partial X_I).$$

The statement follows from the induction hypothesis, since (X_I, Y_I) satisfy conditions (a)–(c) and $\dim W \leq d$. $\square$

Definition 5.4. For $J \subset \{1, \ldots, n\}$, let $\pi_J : \mathbb{R}^n \to \mathbb{R}_J^\perp$ be a natural projection along $\mathbb{R}_J$. For a semi-Pfaffian couple (X, Y) in $\mathbb{R}^n \times \mathbb{R}$, *dimension* $\dim \pi_J(X, Y)_0$ is defined as maximum of $|I \cap J|$ over $I \subset \{1, \ldots, n\}$ such that $(X_I, Y_I)_0 \neq \emptyset$ in a decomposition (15) satisfying conditions (a)–(c) of Proposition 5.3.

Proposition 5.5. *Let $K, J \subset \{1, \ldots, n\}$. Suppose that (X, Y) in Proposition 5.3 satisfies the following property: $X \subset Z$ where Z is a semi-Pfaffian family in $\mathbb{R}^n \times \mathbb{R}$ such that*

(i) $\dim Z = |K| + 1$,

(ii) *Z is K-regular at all $x \in (X, Y)_0$,*

(iii) *for any affine space $T \subset \mathbb{R}^n \times \mathbb{R}$ parallel to $\mathbb{R}_{K\cap J} \times \mathbb{R}$, $Z \cap T$ is J-tangent at all $x \in (X, Y)_0$. Then the union in (15) can be taken over $I \subset K$.*

Proof. The proof is similar to the proof of Proposition 4.10. $\square$

Lemma 5.6 (Fiber cutting). *Let (X, Y) be a semi-Pfaffian couple in $\mathbb{R}^n \times \mathbb{R}$. Let $K, J \subset \{1, \ldots, n\}$ and $\pi = \pi_J$. Suppose that (X, Y) is K-regular and, for any affine subspace $T \subset \mathbb{R}^n \times \mathbb{R}$ parallel to $\mathbb{R}_{J\cap K} \times \mathbb{R}$, the couple $(X \cap T, Y)$ is J-tangent. In particular, $d = \dim \pi(X, Y)_0 = |J \cap K|$. Let $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, and $\rho : \mathbb{R}^{2n} \to \mathbb{R}^n$ a projection to the first factor.*

There exist semi-Pfaffian couples (V, W) and (V', W') in $\mathbb{R}^{2n} \times \mathbb{R}$ such that

(i)$\rho V, \rho V' \subset X$,

(ii)$\pi(X, Y)_0 = \pi\rho(V, W)_0 \cup \pi\rho(V', W')_0$,

(iii)$\dim \pi\rho(V', W')_0 < d$,

(iv)(V, W) *is* $(J \cap K)$*-regular,*

(v) V *is* $(d+1)$*-dimensional,*

(vi)*for any $\lambda > 0$ and any affine subspace L of $\mathbb{R}^{2n}$ parallel to $\mathbb{R}_{J\cap K} \times \mathbb{R}^n$, the set $V_\lambda \cap L$ is finite. The formats of (V, W) and (V', W') admit upper bounds in terms of the format of (X, Y).*

Proof. Due to Proposition 4.5, $(X, Y)_0$ is the union of $(X \cap T, Y)_0$ over all affine T parallel to $\mathbb{R}_{K\cap J} \times \mathbb{R}$. Due to Proposition 5.2, $\pi(X \cap T, Y)_0$ is finite, for any such T.

We want to apply the arguments in the proof of Theorem 3.13 to each couple $(X \cap T, Y)$. Let $Y = \cup_k Y^k$ be a weak stratification of Y (see Proposition 2.8). For a generic $2n$-vector (c, c'), consider a "distance" function $\Phi(x,x') = [1+(c,x)+(c',x')](x-x')^2$ on $\mathbb{R}^{2n}$. Suppose that (c,c') is chosen so that $1+(c,x)+(c',x')$ is positive on $X \times Y$. For $z \in \mathbb{R}^{\perp}_{K\cap J}$, let $T_z = \{x \in \mathbb{R}^n : \pi_{J\cap K} x = z\}$ be an affine subspace parallel to $\mathbb{R}_{J\cap K}$. Define semi-Pfaffian families $V^* = \bigcup_{z,\lambda} V_{z,\lambda}$ and $W^* = \bigcup_{z,\lambda} W_{z,\lambda}$ in $\mathbb{R}^{2n}\times\mathbb{R}$ as follows:

$$V^k_{z,\lambda} = \{x \in X_\lambda \cap T_z,\ x' \in Y^k_\lambda,\ (x,x') \text{ is a critical point of } \Phi|_{(X_\lambda\cap T_z)\times Y^k_\lambda}\};$$

$$V_{z,\lambda} = \bigcup_k V^k_{z,\lambda}; \quad W_{z,\lambda} = \bigcup_k \{(x,x') \in V^k_{z,\lambda} \text{ a degenerate critical point of } \Phi|_{(X_\lambda\cap T_z)\times Y^k_\lambda}\}.$$

Note that V^* and W^* are relatively closed in $X \times_{\mathbb{R}} Y$. The set $V_{z,\lambda}$ contains all points $x^0 \in (X_\lambda \cap T_z)\setminus Y$ where $\Psi_\lambda(x) = \min_{x'\in Y_\lambda} \Phi(x,x')$ has a local maximum on $X_\lambda\cap T_z$ at x^0. This implies $\pi\rho(V^*, Y\times_{\mathbb{R}} Y)_0 = \pi(X,Y)_0$.

Let S be the set of those (z,λ) for which $W_{z,\lambda}$ is non-empty. Due to Lemma 2.15, for a generic (c,c') the set S has zero measure in $\mathbb{R}_{J\cap K}\times\mathbb{R}$. Since $W^* \subset X \times_{\mathbb{R}} Y$ and (X,Y) is K-regular, Proposition 5.5 implies that $\dim \pi(W^*, Y\times_{\mathbb{R}} Y)_0 < d$.

For $(z,\lambda)\notin S$, the set $V_{z,\lambda}$ is discrete. Since $V^* \subset X\times_{\mathbb{R}} Y$, proposition 5.5 applied to $Z = X \times \mathbb{R}^n$ and $K' = K \cup \{n+1,\dots,2n\}$ implies that

$$(V^*, (Y\times_{\mathbb{R}} Y)\cup W^*)_0 = \bigcup (V_I, W_I)_0,$$

union over $I \subset K'$, where (V_I, W_I) satisfy conditions (a)–(c) of Proposition 5.3 and the sets $(V_I,W_I)_0$ are empty for all $I \supset J\cap K$ unless $I = J \cap K$.

Let $V = V_{J\cap K}$ and $W = W_{J\cap K}$. The set $W_{J\cap K}$ in the proof of Proposition 5.5 can be chosen so that $(V^*, (Y\times_{\mathbb{R}} Y)\cup W^*)_0 = (V,W)_0 \cup (W, (Y\times_{\mathbb{R}} Y)\cup W^*)_0$. and $\dim \pi\rho(W, (Y\times_{\mathbb{R}} Y)\cup W^*)_0 < d$ Let $V' = W^* \cup W$ and $W' = Y \times_{\mathbb{R}} Y$. Then the couples (V,W) and (V',W') satisfy conditions of Lemma 5.6. □

6 Projection theorem

In this section, we fix $J = \{1,\dots,m\} \subset \{1,\dots,n\}$ and denote $\pi = \pi_J : \mathbb{R}^n \to \mathbb{R}^m$. For $x \in \mathbb{R}^n$, let $x = (y,z)$ where $y = (x_1,\dots,x_m)$ and $z = (x_{m+1},\dots,x_n)$.

Theorem 6.1 (Projection of a limit set). *Let (X,Y) be a semi-Pfaffian couple in $G \subset \mathbb{R}^n\times\mathbb{R}$. Then $\pi(X,Y)_0$ is a limit set in $\pi\check{G} \subset \mathbb{R}^m$, and its format admits an upper bound in terms of the format of (X,Y).*

Proof of this theorem will be given at the end of this section. First, we prove it for X relatively closed in $\{\lambda > 0\}$, in two special cases: when Y is empty, and when each fiber of π restricted to $\check{X}$ contains at most one point. Next, we reduce the case of finite fibers to the case of one-point fibers. Finally, general case is reduced to the case of finite fibers by fiber-cutting.

Proposition 6.2. *Let X be a semi-Pfaffian family in $G \subset \mathbb{R}^n\times\mathbb{R}$. Suppose that X is relatively closed in $\{\lambda>0\}$, i.e., $X_+ = \overline{X}\cap\{\lambda>0\}$. Then $\pi\check{X}$ is a limit set in $\pi\check{G}\subset\mathbb{R}^m$.*

Proof. Let $f_1(x,\lambda),\dots,f_r(x,\lambda)$ be a Pfaffian chain for X. Define a "z-cone over X" as

$$CX = \{(y,z,\lambda) \in \mathbb{R}^n\times\mathbb{R} : \lambda > 0,\ (y, \frac{z}{\lambda}, \lambda) \in X\}.$$

This is a semi-Pfaffian family in the z-cone CG over G, with the Pfaffian chain

$$f_1(y,\frac{z}{\lambda},\lambda),\dots,f_r(y,\frac{z}{\lambda},\lambda).$$

Note that $\check{}(CG) = \pi\check{G}$. We have $\pi\check{X} = \check{}(CX) = (CX,\emptyset)_0$. □

Proposition 6.3. *Let (X,Y) be a semi-Pfaffian couple in $G\subset\mathbb{R}^n\times\mathbb{R}$ and Z a limit set in $\pi\check{G}\subset\mathbb{R}^m$. Suppose that X is relatively closed in $\{\lambda>0\}$ and, for each $y\in\pi(X,Y)_0\setminus Z$, the set $\check{X}\cap\pi^{-1}y$ contains at most one point. Then $\pi(X,Y)_0\setminus Z$ is a limit set in $\pi\check{G}$.*

Proof. Due to Proposition 6.2, $\pi\check{X}$ is a limit set in $\mathbb{R}^m$. Let $y=\pi x\in\pi\check{X}\setminus Z$, where $x=(y,z)\in\check{X}$. If $y\notin\pi(X,Y)_0$ then $x\in\check{Y}$, hence $y\in\pi(\check{X}\cap\check{Y})$. Conversely, if $y\in\pi(\check{X}\cap\check{Y})$ then $y\notin\pi(X,Y)_0$. Otherwise, x would be a unique point in $\check{X}\cap\pi^{-1}y$, hence $x\in\check{X}\cap\check{Y}$, and $y=\pi x\notin\pi(X,Y)_0$. This implies $\pi(X,Y)_0\setminus Z=\pi\check{X}\setminus(\pi(\check{X}\cap\check{Y})\cup Z)$. From (8) and (9) follows that $\check{X}\cap\check{Y}=\check{}((X\times_{\mathbb{R}}Y)\cap W)$, for a closed semi-Pfaffian family $W\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$. Due to Proposition 6.2, $\pi(\check{X}\cap\check{Y})$ is a limit set. Hence $\pi(X,Y)_0$ is a limit set. □

Proof of Theorem 6.1. We proceed by induction on $d=\dim\pi(X,Y)_0$. Due to Proposition 5.3, we can suppose that, for some $I\subset\{1,\dots,n\}$, the couple (X,Y) is I-regular, X is $(|I|+1)$-dimensional, and $(X\cap(T\times\mathbb{R}),Y)$ is J-tangent for any affine space T parallel to $\mathbb{R}_{I\cap J}$. Due to the induction hypothesis, we can consider only those I for which $|I\cap J|=d$.

Due to Lemma 5.6 (with $K=I$) we can replace $(X,Y)_0$ by $(V,W)_0\cup(V',W')_0$, where V is $(d+1)$-dimensional and projection of $(V',W')_0$ to $\mathbb{R}^m$ is less than d-dimensional. Due to the induction hypothesis, projection of $(V',W')_0$ to $\mathbb{R}^m$ is a limit set, hence it is enough to prove that projection of $(V,W)_0$ to $\mathbb{R}^m$ is a limit set. Accordingly, we can suppose from the very beginning that X is $(d+1)$-dimensional. Applying Proposition 5.3 to $(X,\partial X)$, we can suppose that, for a semi-Pfaffian family $S\supset\partial X$ with $\dim S\le d$, the semi-Pfaffian couple (X,S) is K-regular, for $K\subset J$ with $|K|=d$. Let Δ be projection of S_0 to $\mathbb{R}^d$. Due to the induction hypothesis, Δ is a limit set.

Let ρ denote projection from $\mathbb{R}^m$ to $\mathbb{R}^d$, and $y=(u,v)$ where $u=(x_1,\dots,x_d)$ and $v=(x_{d+1},\dots,x_m)$. For $u\in\mathbb{R}^d\setminus\Delta$ and $\lambda>0$, the sets $X_{u,\lambda}=X_\lambda\cap\{(x_1,\dots,x_d)=u\}$ are finite. Let $N_{\max}$ be the maximum, over $u\in\mathbb{R}^d\setminus\Delta$ and $\lambda>0$, number of points in $X_{u,\lambda}$. For $N=1,\dots,N_{\max}$, let

$$\begin{aligned}X_N=\{u,v_1,z_1,\dots,v_N,z_N,\lambda:\lambda>0,\\(u,v_1,z_1)\in X_\lambda,\dots,(u,v_N,z_N)\in X_\lambda,\\(v_1,z_1)<\dots<(v_N,z_N)\}.\end{aligned}$$

Here "$<$" is the lexicographic order. Each X_N is a semi-Pfaffian family in $\mathbb{R}^{d+N(n-d)}\times\mathbb{R}$, such that $X_{N,u,\lambda}=X_N\cap\{(u,\lambda)=\mathrm{const}\}$ contains exactly one point when $X_{u,\lambda}$ contains exactly N points, and $X_{N,u,\lambda}$ is empty when $X_{u,\lambda}$ contains less than N points. For $j=1,\dots,N$, let $\pi_{N,j}(u,v_1,z_1,\dots,u_N,z_N)=(u,v_j)$.

Let $Z=\rho^{-1}\Delta\cap\pi(X,Y)_0$. It is easy to show that Z is a projection of a limit set, and $\dim Z<d$. Due to the induction hypothesis, Z is a limit set.

For $y^0=(u^0,v^0)\in\pi(X,Y)_0\setminus Z$, let N be the maximum number such that $(y^0,0)\in\pi_{N,j}\overline{X_N}$, for some j. Let $x^0=(y^0,z^0)$ be a point in $(X,Y)_0\cap\pi^{-1}y^0$. Since X is K-regular at each point of $\pi^{-1}\rho^{-1}u^0$, the point $(x^0,0)$ belongs to the closure of $X\cap\{u=u^0\}$, due to Proposition 4.5. Since $(y^0,0)$ does not belong to $\pi_{N+1,j}\overline{X_{N+1}}$, for all j, the set $X_{u^0,\lambda}$ contains exactly N points, for small $\lambda>0$. Hence $X_{N,u^0,\lambda}$ contains exactly one point, for small $\lambda>0$. This implies that $\overline{X_{N,u^0}}\cap\{\lambda=0\}$ contains exactly one point. It is easy to see that X_N is K-regular at each point of $\pi^{-1}\rho^{-1}u^0$. Hence, $\overline{X_N}\cap\pi_{N,j}^{-1}(y^0,0)=\overline{X_{N,u^0}}\cap\pi_{N,j}^{-1}(y^0,0)$ contains exactly one point.

Let

$$Y_{N,j}=\partial X_N\bigcup\{u,v_1,z_1,\dots,u_N,z_N,\lambda:(u,v_j,z_j,\lambda)\in Y\},$$

and $Z_N=\cup_j\pi_{N,j}(X_N,Y_{N,j})_0$. Here we consider all sets $\pi_{N,j}(X_N,Y_{N,j})_0$ as subsets of the same space $\mathbb{R}^m$.

For $N=N_{\max}$, each set $\pi_{N,j}(X_N,Y_{N,j})_0\setminus Z$ is a limit set due to Proposition 6.3. In particular, $Z_{N_{\max}}\setminus Z$ is a limit set. Applying the same arguments to $N=N_{\max}-1$ and $Z\cup Z_{N_{\max}}$ instead of Z, we prove that each set $\pi N,j(X_N,Y_{N,j})_0\setminus(Z\cup Z_N)$ is a limit set, for $N=N_{\max}-1$, hence $Z_{N_{\max}-1}\setminus(Z\cup Z_{N_{\max}})$ is a limit set. Repeating these arguments for decreasing N, we prove that each set

$Z_N \setminus (Z \cup Z_{N+1})$ is a limit set. Finally, $\pi(X,Y)_0 = (\pi(X,Y)_0 \cap Z) \cup_N (Z_N \setminus (Z \cup Z_{N+1})$ is a limit set, since $\pi(X,Y)_0 \cap Z = \pi(X,Y)_0 \cap \rho^{-1}\Delta$ is a projection of a limit set and its dimension is less than d. $\square$

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