TORIC CUBES ARE CLOSED BALLS

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Abstract. We prove that toric cubes, which are images of $[0,1]^d$ under monomial maps, are the closures of graphs of monotone maps, and in particular semi-algebraically homeomorphic to closed balls.

1. Introduction

In [3] Engström, Hersh and Sturmfels introduced a class of compact semi-algebraic sets which they call *toric cubes*.

The following definition is adapted from [3].

Definition 1.1. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$, and $f_{\mathcal{A}} : [0,1]^d \to [0,1]^n$ be the map

$$\mathbf{t} = (t_1, \ldots, t_d) \mapsto (\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}),$$

where $\mathbf{t}^{\mathbf{a}_i} := t_1^{a_{i,1}} \cdots t_d^{a_{i,d}}$ for $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,d})$. The image of $f_{\mathcal{A}}$ is called a toric cube.

We call the image of the restriction of $f_{\mathcal{A}}$ to $(0,1)^d$ an *open toric cube*. The closure of an open toric cube is a toric cube. Note that an open toric cube is not necessarily an open subset of $\mathbb{R}^n$, and need not be contained in $(0,1)^n$ (if some $\mathbf{a}_i = \mathbf{0}$).

In [1, 2] the authors introduced a certain class of definable subsets of $\mathbb{R}^n$ (called *semi-monotone sets*) and definable maps $f : X \to \mathbb{R}^k$ (called *monotone maps*), where $X \subset \mathbb{R}^n$ is a semi-monotone set. Here "definable" means "definable in an o-minimal structure over $\mathbb{R}$", for example, real semi-algebraic.

These objects are meant to serve as building blocks for obtaining a conjectured cylindrical cell decomposition of definable sets into topologically regular cells, without changing the coordinate system in the ambient space $\mathbb{R}^n$ (see [1, 2] for a more detailed motivation behind these definitions).

The main result of this note is the following theorem.

Theorem 1.2. *An open toric cube $C \subset \mathbb{R}^n$ is the graph of a monotone map.*

As a result we obtain

Corollary 1.3. *An open toric cube $C \subset [0,1]^n$, with $\dim(C) = k$, is semi-algebraically homeomorphic to a standard open ball. The pair $(\overline{C}, C)$ is semi-algebraically homeomorphic to the pair $([0,1]^k, (0,1)^k)$, in particular, a toric cube is semi-algebraically homeomorphic to a standard closed ball.*

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Remark 1.4. Note that the first statement in Corollary 1.3 is also proved in [3, Proposition 1]. In conjunction with Theorem 2 in [3], Corollary 1.3 implies that any CW-complex in which the closures of each cell is a toric cube, must be a regular cell complex, and this answers in the affirmative the Conjecture 1 in [3].

2. Proof of Theorem 1.2 and Corollary 1.3

We begin with a few preliminary definitions.

Definition 2.1. Let $L_{j,\sigma,c} := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n |\ x_j \sigma c\}$ for $j = 1, \ldots, n$, $\sigma \in \{<, =, >\}$, and $c \in \mathbb{R}$. Each intersection of the kind

$$C := L_{j_1,\sigma_1,c_1} \cap \cdots \cap L_{j_m,\sigma_m,c_m} \subset \mathbb{R}^n,$$

where $m = 0, \ldots, n$, $1 \le j_1 < \cdots < j_m \le n$, $\sigma_1, \ldots, \sigma_m \in \{<, =, >\}$, and $c_1, \ldots, c_m \in \mathbb{R}$, is called a *coordinate cone* in $\mathbb{R}^n$.

Each intersection of the kind

$$S := L_{j_1,=,c_1} \cap \cdots \cap L_{j_m,=,c_m} \subset \mathbb{R}^n,$$

where $m = 0, \ldots, n$, $1 \le j_1 < \cdots < j_m \le n$, and $c_1, \ldots, c_m \in \mathbb{R}$, is called an *affine coordinate subspace* in $\mathbb{R}^n$.

In particular, the space $\mathbb{R}^n$ itself is both a coordinate cone and an affine coordinate subspace in $\mathbb{R}^n$.

Definition 2.2 ([1])**.** An open (possibly, empty) bounded set $X \subset \mathbb{R}^n$ is called *semi-monotone* if for each coordinate cone C the intersection $X \cap C$ is connected.

Remark 2.3. In fact, in Definition 2.2 above, it suffices to consider intersections with only affine coordinate subspaces (see [2, Theorem 4.3] or Theorem 2.5 below).

Notice that any convex open subset of $\mathbb{R}^n$ is semi-monotone.

The definition of *monotone maps* is given in [2] and is a bit more technical. We will not repeat it here but recall a few important properties of monotone maps that we will need. In particular, Theorem 2.5 below, which appears in [2], gives a complete characterization of monotone maps. For the purposes of the present paper this characterization can be taken as the definition of monotone maps.

Definition 2.4 ([2], Definition 1.4)**.** Let a bounded continuous map $\mathbf{f} = (f_1, \ldots, f_k)$ defined on an open bounded non-empty set $X \subset \mathbb{R}^n$ have the graph $\mathbf{F} \subset \mathbb{R}^{n+k}$. We say that $\mathbf{f}$ is *quasi-affine* if for any coordinate subspace T of $\mathbb{R}^{n+k}$, the projection $\rho_T : \mathbf{F} \to T$ is injective if and only if the image $\rho_T(\mathbf{F})$ is n-dimensional.

The following theorem is proved in [2].

Theorem 2.5 ([2], Theorem 4.3)**.** *Let a bounded continuous quasi-affine map* $\mathbf{f} = (f_1, \ldots, f_k)$ *defined on an open bounded non-empty set* $X \subset \mathbb{R}^n$ *have the graph* $\mathbf{F} \subset \mathbb{R}^{n+k}$*. The following three statements are equivalent.*

(i) *The map* $\mathbf{f}$ *is monotone.*
(ii) *For each affine coordinate subspace* S *in* $\mathbb{R}^{n+k}$ *the intersection* $\mathbf{F} \cap S$ *is connected.*
(iii) *For each coordinate cone* C *in* $\mathbb{R}^{n+k}$ *the intersection* $\mathbf{F} \cap C$ *is connected.*

Remark 2.6. In view of Theorem 2.5, it is natural to identify any semi-monotone set $X \subset \mathbb{R}^n$ with the graph of an identically constant function $f \equiv c$ on X, where c is an arbitrary real.

Definition 2.7. A definable bounded open set $U \subset \mathbb{R}^n$ is called (topologically) regular cell if $\overline{U}$ is definably homeomorphic to a closed ball, and the frontier $\overline{U} \setminus U$ is definably homeomorphic $(n-1)$-sphere. In other words, the pair $(\overline{U}, U)$ is definably homeomorphic to the pair $([0,1]^n, (0,1)^n)$.

Theorem 2.8 ([2], Theorem 5.1)**.** *The graph $\mathbf{F} \subset \mathbb{R}^{n+k}$ of a monotone map $\mathbf{f} : X \to \mathbb{R}^k$ on a semi-monotone set $X \subset \mathbb{R}^n$ is definably homeomorphic to a regular cell.*

Proof of Theorem 1.2. Let $C \subset [0,1]^n$ be an open toric cube and suppose that $C = f_{\mathcal{A}}((0,1)^d)$ for a monomial map $f_{\mathcal{A}}$ (see Definition 1.1).

Make the coordinate change $z_i = \log(t_i)$ for every $i = 1, \ldots, d$, and take the logarithm of every component of the map $f_{\mathcal{A}}$ expressed in coordinates z_i. Denote the resulting map by $\log f_{\mathcal{A}}$. Then $\log f_{\mathcal{A}}$ is the restriction of a linear map, namely

$$\log f_{\mathcal{A}} : (-\infty, 0)^d \to (-\infty, 0)^n,$$

defined by

$$\mathbf{z} = (z_1, \ldots, z_d) \mapsto (\mathbf{a}_1 \cdot \mathbf{z}, \ldots, \mathbf{a}_n \cdot \mathbf{z}).$$

Observe that log (the component-wise logarithm) maps the open cube, $(0,1)^d$ (resp. $(0,1)^n$) homeomorphically onto $(-\infty, 0)^d$ (resp. $(-\infty, 0)^n$). It follows that the fiber of the orthogonal projection of C to any k-dimensional coordinate subspace is the pre-image under the log map of an affine subset of $(-\infty, 0)^n$, and is a single point if it is zero-dimensional. Hence C is a graph of a quasi-affine map (choose any set of k coordinates such that the image of C under the orthogonal projection to the coordinate subspace of those coordinates is full dimensional).

Similarly, the intersection of C with any affine coordinate subspace is the pre-image under the log map, of an affine subset of $(-\infty, 0)^n$ and hence connected.

We proved that C satisfies the conditions of Theorem 2.5, hence C is the graph of a monotone map. □

Proof of Corollary 1.3. Immediate consequence of Theorem 1.2 and Theorem 2.8. □

References

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