LIPSCHITZ GEOMETRY OF PAIRS OF NORMALLY EMBEDDED HÖLDER TRIANGLES

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1. Introduction

The question of bi-Lipschitz classification of semialgebraic surfaces has become in recent years one of the central questions of Metric Geometry of Singularities. There are two natural structures of a metric space on a connected semialgebraic set $X \subset \mathbb{R}^n$. The first one is the inner distance, the length of a minimal path in X connecting two points. The second one is the outer distance, defined as the distance in $\mathbb{R}^n$ between two points of X. A germ X is called *normally embedded* (see [2]) if its inner and outer metrics are equivalent. There are three natural equivalence relations associated with these distances. Two sets X and Y are inner (resp., outer) equivalent if there is a inner (resp., outer) bi-Lipschitz homeomorphism $h : X \to Y$. The sets X and Y are ambient bi-Lipschitz equivalent if the homeomorphism $h : X \to Y$ can be extended to a bi-Lipschitz homeomorphism H of the ambient space. The ambient equivalence is stronger than the outer equivalence, and the outer equivalence is stronger then the inner equivalence. Finiteness theorems of Mostowski and Valette (see [9] and [10]) show that there are finitely many ambient bi-Lipschitz equivalence classes in any semialgebraic family of semialgebraic sets.

The paper [1] of the first author presents a complete bi-Lipschitz classification of semialgebraic surface germs at the origin with respect to the inner metric. It is based on a canonical partition of a surface germ into Hölder triangles and isolated arcs. The exponents of these triangles, and the combinatorics of the graph defined by their links, constitute a complete inner Lipschitz invariant.

The outer Lipschitz geometry of semialgebraic surface germs is considerably more complicated, and their outer bi-Lipschitz classification is still work in progress. An important step towards such classification was made in [3], where classification of the germs at the origin of $\mathbb{R}^2$ of semialgebraic (or, more generally, definable in a polynomially bounded o-minimal structure) Lipschitz functions with respect to contact Lipschitz equivalence relation was suggested. It was based on a complete combinatorial invariant of contact Lipschitz equivalence, called *pizza*.

Another important step was made in [7], where an "abnormal" semialgebraic surface germ was canonically partitioned into normally embedded Hölder triangles. Several constructions and results from [7] are used in the present paper.

Normally embedded Hölder triangles are the simplest "building blocks" of semialgebraic surface germs: the only Lipschitz invariant of a normally embedded Hölder triangle is its exponent. In the present paper we consider the next, a little bit more complicated, case

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of a *pair* of normally embedded Hölder triangles: a surface germ $X = T \cup T'$ which is the union of two normally embedded Hölder triangles T and T'. Let $f : T \to \mathbb{R}$ and $g : T' \to \mathbb{R}$ be the distances from the points in one of these two triangles to the other one. The pizzas of f and g, being contact Lipschitz invariants of these two functions, are outer Lipschitz invariants of X. The first question is whether X is outer bi-Lipschitz equivalent to the union of T and the graph of the distance function f. Simple examples (see Fig. 4) show that the answer may be negative. Another natural question is whether the pizzas of f and g are equivalent. The answer, in general, is again negative. We show (see Theorem 3.20) that the answers to both questions are positive if the pair (T, T') is *elementary* (see Definition 2.10) and satisfies boundary conditions (5). The conditions (5) appear naturally in the paper [7], where some standard building blocks (*clusters*) are defined in the link of a singular surface. Any two Hölder triangles in a cluster satisfy (5). Although a pair X satisfying (5) is simpler than the general pair of normally embedded Hölder triangles, its outer Lipschitz geometry is still rather complicated. If one considers a pair $X = T \cup T'$ of two normally embedded Hölder triangles, such that T' is a graph of a Lipschitz function defined on T, then X automatically satisfies the condition (5). A natural question is whether the opposite is true. Suppose that a pair $X = T \cup T'$ of normally embedded Hölder triangles satisfies (5). Is it true that X is outer Lipschitz equivalent to the union of T and the graph of a function f defined on T? The answer is negative, and we present several examples when this is not true (see Section 4). In this paper we define an outer Lipschitz invariant of a pair of normally embedded Hölder triangles satisfying (5), called *$\sigma\tau$-pizza*, and conjecture that it is a complete invariant: all pairs with the same $\sigma\tau$-pizza should be outer bi-Lipschitz equivalent.

In Section 2 we give basic definitions and reformulate the pizza invariant in the language of *zones* (see Definition 2.20).

In Section 3 we establish properties of elementary pairs of Hölder triangles and give examples of non-elementary pairs for which these properties fail. We also discuss conditions satisfied by a surface germ $X = T \cup T'$ equivalent to the union of a Hölder triangle T and the graph of the distance function f defined on T.

In Section 4 the $\sigma\tau$-pizza is defined. The main result of the section, Theorem 4.13, states that it is an outer Lipschitz invariant of a pair of normally embedded Hölder triangles satisfying (5): the $\sigma\tau$-pizzas of outer bi-Lipschitz equivalent pairs are combinatorially equivalent. We conjecture that the converse of Theorem 4.13 is also true, but the proof needs some additional work.

Some remarks about the figures. Since it is practically impossible to adequately show outer Lipschitz geometry of a surface germ in a plot, we draw instead its link (intersection with a small sphere centered at the singular point) indicating higher tangency orders by smaller Euclidean distances. We hope these plots will help to create geometric intuition.

2. Preliminaries

All sets, functions and maps in this paper are germs at the origin of $\mathbb{R}^n$ definable in a polynomially bounded o-minimal structure over $\mathbb{R}$ with the field of exponents $\mathbb{F}$. The simplest (and most important in applications) examples of such structures are real semialgebraic and (global) subanalytic sets, with $\mathbb{F} = \mathbb{Q}$.

Definition 2.1. A germ X at the origin inherits two metrics from the ambient space: the *inner metric* where the distance between two points of X is the length of the shortest path connecting them inside X, and the *outer metric* with the distance between two points of X being their distance in the ambient space. A germ X is *normally embedded* if its inner and outer metrics are equivalent.

For a point $x \in X$ and a subset $Y \subset X$ we define the *outer distance* $dist(x,Y) = \inf_{y\in Y}|x-y|$, and the *inner distance* $idist(x,Y)$ as the infimum of the lengths of paths connecting x with points $y \in Y$.

A *surface germ* is a closed germ X such that $\dim_{\mathbb{R}} X = 2$, and it is pure dimensional.

Definition 2.2. An *arc* in $\mathbb{R}^n$ is (a germ at the origin of) a mapping $\gamma : [0,\epsilon) \to \mathbb{R}^n$ such that $\gamma(0) = 0$. Unless otherwise specified, we suppose that arcs are parameterized by the distance to the origin, i.e., $|\gamma(t)| = t$. We usually identify an arc γ with its image in $\mathbb{R}^n$. The *Valette link* of X is the set $V(X)$ of all arcs $\gamma \subset X$.

Definition 2.3. Let $f \not\equiv 0$ be (a germ at the origin of) a function defined on an arc γ. The *order* of f on γ is the value $q = ord_\gamma f \in \mathbb{F}$ such that $f(\gamma(t)) = ct^q + o(t^q)$ as $t \to 0$, where $c \neq 0$. If $f \equiv 0$ on γ, we set $ord_\gamma f = \infty$.

Definition 2.4. The *tangency order* of two arcs γ and γ' is defined as $tord(\gamma,\gamma') = ord_\gamma|\gamma(t)-\gamma'(t)|$. The tangency order of an arc γ and a set of arcs $Z \subset V(X)$ is defined as $tord(\gamma,Z) = \sup_{\lambda\in Z} tord(\gamma,\lambda)$. The tangency order of two subsets Z and Z' of $V(X)$ is defined as $tord(Z,Z') = \sup_{\gamma\in Z} tord(\gamma,Z')$. Similarly, $itord(\gamma,\gamma')$, $itord(\gamma,Z)$ and $itord(Z,Z')$ denote the tangency orders with respect to the inner metric. If T is a Hölder triangle and γ is an arc we are going to use the notation $tord(\gamma,T)$ instead of $tord(\gamma,V(T))$ and $itord(\gamma,T)$ instead of $itord(\gamma,V(T))$.

The tangency order defines a non-Archimedean metric on the set of arcs: if $tord(\gamma,\gamma') > tord(\gamma,\gamma'')$ then $tord(\gamma',\gamma'') = tord(\gamma,\gamma'')$.

Remark 2.5. The inner metric on a semialgebraic set is bi-Lipschitz equivalent to a semialgebraic metric (so-called pancake metric, see the theorem of Kurdyka and Orro [8] and also [2]). The inner order of tangency of two arks $itord(\gamma_1,\gamma_2))$ is also defined in [6].

Definition 2.6. For $\beta \in \mathbb{F}$, $\beta \geq 1$, the *standard β-Hölder triangle* is (a germ at the origin of) the set

$$T_\beta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 \leq y \leq x^\beta\}. \tag{1}$$

The curves $\{x \geq 0,\ y = 0\}$ and $\{x \geq 0,\ y = x^\beta\}$ are the *boundary arcs* of T_β.

Definition 2.7. A *β-Hölder triangle* is (a germ at the origin of) a set $T \subset \mathbb{R}^n$ that is inner bi-Lipschitz homeomorphic to the standard β-Hölder triangle (1). The number $\beta = \mu(T) \in \mathbb{F}$ is called the *exponent* of T. The arcs γ_1 and γ_2 of T mapped to the boundary arcs of T_β by an inner bi-Lipschitz homeomorphism are the *boundary arcs* of T (notation $T = T(\gamma_1,\gamma_2)$). All other arcs of T are its *interior arcs*. The set of interior arcs of T is denoted by $I(T)$. An arc $\gamma \subset T$ is *generic* if $itord(\gamma,\gamma_1) = itord(\gamma,\gamma_2)$. The set of generic arcs of T is denoted by $G(T)$.

Definition 2.8. Let X be a surface germ. An arc $\gamma \in V(X)$ is *Lipschitz non-singular* if there exists a normally embedded Hölder triangle $T \subset X$ such that $\gamma \in I(T)$ and $\gamma \not\subset \overline{X \setminus T}$. Otherwise, γ is *Lipschitz singular*. A Hölder triangle T is *non-singular* if any arc $\gamma \in I(T)$ is Lipschitz non-singular.

Definition 2.9. For a Lipschitz function f defined on a Hölder triangle T, let

$$Q_f(T) = \bigcup_{\gamma\in V(T)} ord_\gamma f. \tag{2}$$

It was shown in [3] that $Q_f(T)$ is either a point or a closed interval in $\mathbb{F}\cup\{\infty\}$.

Definition 2.10. A Hölder triangle T is *elementary* with respect to a Lipschitz function f if, for any $q\in Q_f(T)$ and any two arcs γ and γ' in T such that $ord_\gamma f = ord_{\gamma'} f = q$, the order of f is q on any arc in the Hölder triangle $T(\gamma,\gamma')\subset T$.

Remark 2.11. Examples 4.4, 4.5, 4.6 in [3] make the definition 2.10 more clear.

Definition 2.12. Let T be a Hölder triangle and f a Lipschitz function defined on T. For each arc $\gamma\subset T$, the *width* $\mu_T(\gamma,f)$ of γ with respect to f is the infimum of exponents of Hölder triangles $T'\subset T$ containing γ such that $Q_f(T')$ is a point. For $q\in Q_f(T)$ let $\mu_{T,f}(q)$ be the set of exponents $\mu_T(\gamma,f)$, where γ is any arc in T such that $ord_\gamma f = q$. It was shown in [3] that, for each $q\in Q_f(T)$, the set $\mu_{T,f}(q)$ is finite. This defines a multivalued *width function* $\mu_{T,f} : Q_f(T)\to\mathbb{F}\cup\{\infty\}$. If T is an elementary Hölder triangle with respect to f then the function $\mu_{T,f}$ is single valued. When f is fixed, we write $\mu_T(\gamma)$ and μ_T instead of $\mu_T(\gamma,f)$ and $\mu_{T,f}$.

The *depth* $\nu_T(\gamma,f)$ of an arc γ with respect to f is the infimum of exponents of Hölder triangles $T'\subset T$ such that $\gamma\in G(T')$ and $Q_f(T')$ is a point. By definition, $\nu_T(\gamma,f)=\infty$ when there are no such triangles T'.

Definition 2.13. Let T be a non-singular Hölder triangle and f a Lipschitz function defined on T. We say that T is a *pizza slice* associated with f if it is elementary with respect to f and, unless $Q_f(T)$ is a point, $\mu_{T,f}(q)=aq+b$ is an affine function on $Q_f(T)$. If T is a pizza slice such that $Q_f(T)$ is not a point, then the *supporting arc* $\tilde\gamma$ of T with respect to f is the boundary arc of T such that $\mu_T(\tilde\gamma,f)=\max_{q\in Q_f(T)}\mu_{T,f}(q)$.

Proposition 2.14. (See [3].) *Let T be a β-Hölder triangle which is a pizza slice associated with a non-negative Lipschitz function f, such that $Q=Q_f(T)$ is not a point. Then* $\mu_T\not\equiv \mathrm{const}$ *and the following holds:*
(1) $\beta\le\mu_T(q)\le\max(q,\beta)$ *for all* $q\in Q$,
(2) $\mu_T(\gamma)=\beta$ *for* $\gamma\in G(T)$,
(3) *If $\tilde\gamma$ is the supporting arc of T with respect to f, then $\mu_T(\gamma)=itord(\tilde\gamma,\gamma)$ for all arcs $\gamma\subset T$ such that $\mu_T(\gamma)<\mu_T(\tilde\gamma)$.*

Definition 2.15. (See [3].) Let f be a non-negative Lipschitz function defined on an oriented β-Hölder triangle T. A *pizza decomposition* of T (or just a *pizza* on T) associated with f is a decomposition $\{T_i\}_{i=1}^p$ of T into β_i-Hölder triangles $T_i=T(\lambda_{i-1},\lambda_i)$ ordered according to the orientation of T, such that
(1) λ_0 and λ_p are the boundary arcs of T,
(2) $T_i\cap T_{i+1}=\lambda_i$ for $1\le i<p$,
(3) $T_i\cap T_j=\{0\}$ when $|i-j|>1$,
(4) each Hölder triangle T_i is a pizza slice associated with f.
We write $q_i=ord_{\lambda_i}f$, $Q_i=Q_f(T_i)$, $\mu_i(q)=\mu_{T_i,f}(q)$. If Q_i is not a point, then $\tilde\gamma_i$ denotes the supporting arc of T_i with respect to f.

Definition 2.16. A pizza decomposition $\{T_i\}$ of T associated with f is *minimal* if $T_{i-1}\cup T_i$ is not a pizza slice associated with f for any $i>1$.

Definition 2.17. For two non-negative Lipschitz functions f on T and g on T', a pizza decomposition $\{T_i = T(\lambda_{i-1}, \lambda_i)\}$ of T associated with f is *equivalent* to a pizza decomposition $\{T'_i = T(\lambda'_{i-1}, \lambda'_i)\}$ of T' associated with g if there is an orientation preserving inner bi-Lipschitz homeomorphism $h : T \to T'$ such that $h(\lambda_i) = \lambda'_i$, $ord_{\lambda_i} f = ord_{\lambda'_i} g$, $Q_f(T_i) = Q_g(T'_i)$ and $\mu_{T_i,f} \equiv \mu_{T'_i,g}$, for all i, and moreover, $h(\tilde\gamma_i) = \tilde\gamma'_i$ if $Q_f(T_i) = Q_g(T'_i)$ is not a point, where $\tilde\gamma_i$ and $\tilde\gamma'_i$ are the supporting arcs for T_i and T'_i with respect to f and g.

Definition 2.18. Let T and T' be two β-Hölder triangles. Two Lipschitz function germs $f : (T, 0) \longrightarrow (\mathbb{R}, 0)$ and $g : (T', 0) \longrightarrow (\mathbb{R}, 0)$ are *Lipschitz contact equivalent* if there exist two germs of inner bi-Lipschitz homeomorphisms $h : (T, 0) \longrightarrow (T', 0)$ and $H : (T \times \mathbb{R}, 0) \longrightarrow (T' \times \mathbb{R}, 0)$ such that $H(T \times \{0\}) = T' \times \{0\}$ and the following diagram is commutative:

$$\begin{array}{ccccc} (T,0) & \xrightarrow{(id,f)} & (T\times\mathbb{R},0) & \xrightarrow{\pi} & (T,0) \\ h\downarrow & & H\downarrow & & h\downarrow \\ (T',0) & \xrightarrow{(id,g)} & (T'\times\mathbb{R},0) & \xrightarrow{\pi'} & (T',0) \end{array} \tag{3}$$

Here $\pi : T \times \mathbb{R} \to T$ and $\pi' : T' \times \mathbb{R} \to T'$ are natural projections.

The main result of [3], reformulated for non-negative Lipschitz functions defined on Hölder triangles, is the following theorem.

Theorem 2.19. *Let T and T' be oriented Hölder triangles. Non-negative Lipschitz functions $f : T \to \mathbb{R}$ and $g : T' \to \mathbb{R}$ are Lipschitz contact equivalent if and only if a minimal pizza decomposition of T associated with f and a minimal pizza decomposition of T' associated with g are equivalent. In particular, any two minimal pizza decompositions associated with the same function $f : T \to \mathbb{R}$ are equivalent.*

Definition 2.20. (See [7, Definition 2.34].) Let X be a surface germ. A non-empty set of arcs $Z \subset V(X)$ is called a *zone* if, for any two arcs $\gamma_1 \neq \gamma_2$ in Z, there exists a non-singular Hölder triangle $T = T(\gamma_1, \gamma_2)$ such that $V(T) \subset Z$. A *singular zone* is a zone $Z = \{\gamma\}$ consisting of a single arc γ. A zone Z is *normally embedded* if, for any two arcs $\gamma_1 \neq \gamma_2$ in Z, there exists a normally embedded Hölder triangle $T = T(\gamma_1, \gamma_2)$ such that $V(T) \subset Z$.

Definition 2.21. (See [7, Definition 2.37].) The *order* of a zone Z is defined as $\mu(Z) = \inf_{\gamma,\gamma' \in Z} tord(\gamma, \gamma')$. If Z is a singular zone then $\mu(Z) = \infty$. If $\mu(Z) = \beta$ then Z is called a β-zone.

Definition 2.22. (See [7, Definition 2.40].) A β-zone Z is *closed* if there are two arcs γ and γ' in Z such that $tord(\gamma, \gamma') = \beta$. Otherwise, Z is an *open* zone. By definition, any singular zone is closed.

Definition 2.23. A zone $Z \subset V(X)$ is *perfect* if, for any two arcs γ and γ' in Z, there exists a Hölder triangle $T \subset X$ such that $V(T) \subset Z$ and both γ and γ' are generic arcs of T. By definition, any singular zone is perfect.

Definition 2.24. Let $f : T \to \mathbb{R}$ be a Lipschitz function defined on a non-singular Hölder triangle T. A zone $Z \subset V(T)$ is a *q-order zone* for f if $ord_\gamma f = q$ for any arc $\gamma \in Z$. A

q-order zone for f is *maximal* if it is not a proper subset of any other q-order zone for f. The *width zone* $W_T(\gamma, f)$ of an arc $\gamma \subset T$ with respect to f is the maximal q-order zone for f containing γ, where $q = ord_\gamma f$. The order of $W_T(\gamma, f)$ is $\mu_T(\gamma, f)$. The *depth zone* $D_T(\gamma, f)$ of an arc $\gamma \subset T$ with respect to f is the union of zones $G(T')$ for all triangles $T' \subset T$ such that $\gamma \in G(T')$ and $Q_f(T')$ is a point. By definition, $D_T(\gamma, f) = \{\gamma\}$ when there are no such triangles T'. The order of $D_T(\gamma, f)$ is $\nu_T(\gamma, f)$.

Lemma 2.25. *Let $f : T \to \mathbb{R}$ be a Lipschitz function defined on a non-singular Hölder triangle T. For any arc $\gamma \subset T$, the width zone $W_T(\gamma, f)$ is closed.*

Proof. If $f|_\gamma \equiv 0$, then either γ is an isolated arc in the closed subset $T_0 = \{f(x) = 0\}$ of T and a singular zone $W_T(\gamma, f) = \{\gamma\}$ is closed by definition, or there is a maximal Hölder triangle $\tilde{T}_0 \subset T_0$ containing γ. Then $\mu = \mu_T(\gamma, f)$ is the exponent of $\tilde{T}_0$, and $W_T(\gamma, f) = V(\tilde{T}_0)$ is a closed μ-zone. Otherwise, let $f(\gamma(t)) = c_0 t^q + o(t^q)$ where $c_0 \neq 0$, and let $\tilde{T}_c$ be the maximal Hölder triangle containing γ in the subset $T_c = \{|f(x)| \leq ct^q\}$ of T, where $c \geq |c_0|$. Then the family $\{\tilde{T}_c\}$ is definable, Hölder triangles $\tilde{T}_c$ have the same exponent $\mu = \mu_T(\gamma, f)$ for large enough c, and $W_T(\gamma, f) = \bigcup_{c \geq |c_0|} V(\tilde{T}_c)$. Thus $W_T(\gamma, f)$ is a closed μ-zone. □

Definition 2.26. Let T be a non-singular Hölder triangle and f a Lipschitz function defined on T. If $Z \subset V(T)$ is a zone, we define $Q_f(Z)$ as the set of all exponents $ord_\gamma f$ for $\gamma \in Z$. The zone Z is *elementary* with respect to f if the set of arcs $\gamma \in Z$ such that $ord_\gamma f = q$ is a zone for each $q \in Q_f(Z)$.

For $\gamma \in Z$ and $q = ord_\gamma f$, the *width* $\mu_Z(\gamma, f)$ of γ with respect to f is the infimum of exponents of Hölder triangles T' containing γ such that $V(T') \subset Z$ and $Q_f(T')$ is a point. The *width zone* $W_Z(\gamma, f)$ of γ with respect to f is the maximal subzone of Z containing γ such that $q = ord_\lambda f$ for all arcs $\lambda \subset W_Z(\gamma, f)$. The order of $W_Z(\gamma, f)$ is $\mu_Z(\gamma, f)$. For $q \in Q_f(Z)$ let $\mu_{Z,f}(q)$ be the set of exponents $\mu_Z(\gamma, f)$, where $\gamma \in Z$ is any arc such that $ord_\gamma f = q$. It follows from [3] that, for each $q \in Q_f(Z)$, the set $\mu_{Z,f}(q)$ is finite. This defines a multivalued *width function* $\mu_{Z,f} : Q_f(Z) \to \mathbb{F} \cup \{\infty\}$. If Z is an elementary zone with respect to f then the function $\mu_{Z,f}$ is single valued.

We say that Z is a *pizza slice zone* associated with f if it is elementary with respect to f, $Q_f(Z)$ is a closed interval in $\mathbb{F} \cup \{\infty\}$ and, unless $Q_f(Z)$ is a point, $\mu_{Z,f}(q) = aq + b$ is an affine function on $Q_f(Z)$. If Z is a pizza slice zone such that $Q_f(Z)$ is not a point, then the *supporting subzone* $\tilde{Z}$ of Z with respect to f is the set of arcs $\lambda \in Z$ such that $\mu_Z(\lambda, f) = \max_{q \in Q_f(Z)} \mu_{Z,f}(q)$.

Lemma 2.27. *Let f be a Lipschitz function defined on a non-singular Hölder triangle T. Let γ be an interior arc of T, so that $T = T' \cup T''$ and $T' \cap T'' = \{\gamma\}$. Then either $\mu_{T'}(\gamma, f) = \mu_{T''}(\gamma, f)$ and $\nu_T(\gamma, f) = \mu_T(\gamma, f)$, or $\nu_T(\gamma, f) = \max(\mu_{T'}(\gamma, f), \mu_{T''}(\gamma, f)) > \mu_T(\gamma, f)$. In both cases, $D_T(\gamma, f)$ is a closed perfect zone.*

Proof. Let $\mu = \mu_T(\gamma, f)$, $\mu' = \mu_{T'}(\gamma, f)$ and $\mu'' = \mu_{T''}(\gamma, f)$. By definition of the width, $\mu = \min(\mu', \mu'')$. By definition of the depth, $\nu_T(\gamma, f) \geq \max(\mu', \mu'')$. According to Lemma 2.25, the width zones $W_{T'}(\gamma, f)$ and $W_{T''}(\gamma, f)$ are closed zones of orders μ' and μ''. If $\mu' = \mu'' = \mu$ then there are two arcs $\gamma' \subset W_{T'}(\gamma, f)$ and $\gamma'' \subset W_{T''}(\gamma, f)$ such that $tord(\gamma, \gamma') = tord(\gamma, \gamma'') = \mu$ and $ord_\lambda f = ord_\gamma f$ for all arcs $\lambda \subset T(\gamma', \gamma'')$. Then γ is a generic arc of a μ-Hölder triangle $T(\gamma', \gamma'')$, thus $\nu_T(\gamma, f) \leq \mu$. Since $\nu_T(\gamma, f) \geq \mu$, we have $\nu_T(\gamma, f) = \mu$ in this case. Otherwise, if $\mu' > \mu''$ then, according to Lemma 2.25, there are two arcs $\gamma' \subset W_{T'}(\gamma, f)$ and $\gamma'' \subset W_{T''}(\gamma, f)$ such that $tord(\gamma, \gamma') = tord(\gamma, \gamma'') = \mu'$ and $ord_\lambda f = ord_\gamma f$ for all arcs $\lambda \subset T(\gamma', \gamma'')$. Then γ is a generic arc of a μ'-Hölder triangle $T(\gamma', \gamma'')$, thus $\nu_T(\gamma, f) \leq \mu'$. Since $\nu_T(\gamma, f) \geq \max(\mu', \mu'')$, we have $\nu_T(\gamma, f) = \max(\mu', \mu'')$ in this case.

To show that $D_T(\gamma, f)$ is a closed perfect zone, note first that its order is $\nu = \nu_T(\gamma, f)$ and, unless $\nu = \infty$ and $D_T(\gamma, f) = \{\gamma\}$ is by definition closed perfect, γ is a generic arc of a ν-Hölder triangle $\tilde{T} = T(\gamma', \gamma'') \subset T$ such that $tord_\lambda f = tord_\gamma f$ for any arc $\lambda \subset \tilde{T}$. Then, since $tord(\gamma, \gamma') = tord(\gamma, \gamma'') = \nu < \infty$, there is a generic arc λ of $\tilde{T}$ such that $tord(\lambda, \gamma) = \nu$, thus $\bar{T} = T(\lambda, \gamma)$ is a ν-Hölder triangle and $V(\bar{T}) \subset D_T(\gamma, f)$. This implies that $D_T(\gamma, f)$ is a closed zone. If λ' and λ'' are any two arcs in $D_T(\gamma, f)$, then there are two Hölder triangles $T' \subset T$ and $T'' \subset T$ containing γ such that $\lambda' \in G(T')$ and $\lambda'' \in G(T'')$. Then both λ' and λ'' are generic arcs of $T' \cup T''$, thus $D_T(\gamma, f)$ is a perfect zone. $\square$

Remark 2.28. Let $h : T \to T'$ be an inner bi-Lipschitz homeomorphism, and let $f(x) = g(h(x))$ where g is a Lipschitz function defined on T'. Then $\mu_T(\gamma, f) = \mu_{T'}(h(\gamma), g)$, $\nu_T(\gamma, f) = \nu_{T'}(h(\gamma), g)$, $h(W_T(\gamma, f)) = W_{T'}(h(\gamma), g)$ and $h(D_T(\gamma, f)) = D_{T'}(h(\gamma), g)$, for any arc $\gamma \in V(T)$.

Lemma 2.29. *A zone $Z \subset V(X)$ is perfect if and only if, for any two arcs γ and γ' in Z, there exists a Hölder triangle $T \subset X$ such that $V(T) \subset Z$, and an inner bi-Lipschitz automorphism $h : X \to X$ such that $h(\gamma) = \gamma'$ and $h(x) = x$ for all $x \in X \setminus T$.*

Proof. Let $Z \subset V(X)$ be a perfect zone and γ, γ' two arcs in Z. Let $T = T(\gamma_1, \gamma_2)$ be a β-Hölder triangle such that $V(T) \subset Z$ and both γ and γ' are generic arcs in T. Then $T = T(\gamma_1, \gamma) \cup T(\gamma, \gamma_2)$ and $T = T(\gamma_1, \gamma') \cup T(\gamma', \gamma_2)$ are two decompositions of T into β-Hölder triangles. Let $h_1 : T(\gamma_1, \gamma) \to T(\gamma_1, \gamma')$ and $h_2 : T(\gamma, \gamma_2) \to T(\gamma', \gamma_2)$ be inner bi-Lipschitz homeomorphisms, such that $h_1|_{\gamma_1 = Id}$, $h_2|_{\gamma_2} = Id$ and $h_1|_\gamma = h_2|_\gamma$. Then the mapping $h : T \to T$ such that $h = h_1$ on $T(\gamma_1, \gamma)$ and $h = h_2$ on $T(\gamma, \gamma_2)$ is an inner bi-Lipschitz homeomorphism such that $h|_{\gamma_1} = Id$, $h|_{\gamma_2} = Id$ and $h(\gamma) = \gamma'$). Thus h can be extended by identity outside T to an inner bi-Lipschitz homeomorphism $X \to X$ preserving Z. $\square$

Proposition 2.30. *Let f be a non-negative Lipschitz function defined on a normally embedded Hölder triangle $T = T(\gamma_1, \gamma_2)$, oriented from γ_1 to γ_2. There exists a unique finite family $\{D_\ell\}_{\ell=0}^p$ of disjoint zones $D_\ell \subset V(T)$, the pizza zones associated with f, with the following properties:*

1. *The singular zones $D_0 = \{\gamma_1\}$ and $D_p = \{\gamma_2\}$ are the boundary arcs of T.*
2. *For any arc $\gamma \in D_\ell$, $D_\ell = D_T(\gamma, f)$ is a closed perfect ν_ℓ-zone, where $\nu_\ell = \nu_T(\gamma, f)$. In particular, D_ℓ is a q_ℓ-order zone for f, where $q_\ell = ord_\gamma f$ for $\gamma \in D_\ell$. Moreover, D_ℓ is a maximal q_ℓ-order zone for f of order ν_ℓ: if $Z \subset V(T)$ is a q_ℓ-order zone for f containing D_ℓ and $\lambda \in Z$ is an arc such that $tord(\lambda, D_\ell) \geq \nu_\ell$, then $\lambda \in D_\ell$.*
3. *Any choice of arcs $\lambda_\ell \in D_\ell$ defines a minimal pizza $\{T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)\}_{\ell=1}^p$ on T associated with f.*
4. *Any minimal pizza on T associated with f can be obtained as a decomposition $\{T_\ell\}$ of T defined by some choice of arcs $\lambda_\ell \in D_\ell$.*

Proof. Consider a decomposition $\{T_\ell\}_{\ell=1}^p$, of T into β_ℓ-Hölder triangles $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$ which is a minimal pizza for f. Let $Q_\ell \subset \mathbb{F} \cup \{\infty\}$ be the set (either a point or a closed interval) of values $tord_\gamma f$ for $\gamma \subset T_\ell$, and let $\mu_\ell : Q_\ell \to \mathbb{F} \cup \{\infty\}$ be the affine width function for f on T_ℓ (a constant if Q_ℓ is a point). We assume that λ_0 and λ_p are the boundary arcs of T, and that $T_\ell \cap T_{\ell+1} = \lambda_\ell$ for $1 \le \ell < p$.

Since each boundary arc of T is also a boundary arc of a pizza slice for any pizza decomposition of T, we can define singular zones $D_0 = \{\lambda_0\}$ and $D_p = \{\lambda_p\}$.

If the germ at zero of the set $S = \{x \in T,\ f(x) = 0\}$ is non-empty, it is a union of finitely many germs isolated arcs and germs of maximal in S Hölder triangles S_j. Each isolated arc of S, and each boundary arc of one of the triangles S_j, must be a boundary arc of a pizza slice for any minimal pizza on T associated with f. In particular, such an arc λ must be one of the arcs λ_ℓ, and the singular zone $\{\lambda\}$ must be one of the zones D_ℓ.

Assume now that $0 < \ell < p$ and $q_\ell = ord_{\lambda_\ell} f < \infty$. Consider the depth zone $D_\ell = D_T(\lambda_\ell, f)$ (see Definition 2.24). Then D_ℓ is a closed perfect zone of order $\nu_\ell = \nu_T(\lambda_\ell, f)$, which is also a q_ℓ-order zone for f. Moreover, if $\lambda \subset T_\ell$ is an arc such that $tord(\lambda, \lambda_\ell) \ge \nu_\ell$ and $ord_\gamma f = q_\ell$ for any arc $\gamma \subset T(\lambda_\ell, \lambda)$, then $\lambda \in D_\ell$ by Definition 2.24. The same argument works for $\lambda \subset T_{\ell-1}$. Thus D_ℓ is a maximal q_ℓ-order zone for f of order ν_ℓ.

We claim that, if the arc λ_ℓ is replaced by any other arc $\theta \in D_\ell$ and the Hölder triangles $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$ and $T_{\ell+1} = T(\lambda_\ell, \lambda_{\ell+1})$ with the common arc λ_ℓ are replaced by the Hölder triangles $T(\lambda_{\ell-1}, \theta)$ and $T(\theta, \lambda_{\ell+1})$ with the common arc θ, the resulting decomposition of T is again a minimal pizza on T associated with f. Indeed, since D_ℓ is a perfect zone, and also a q_ℓ-order zone for f, by Lemma 2.29 one can construct an inner bi-Lipschitz map $\phi : T \to T$, such that $\phi(\lambda_\ell) = \theta$ and $\phi(\gamma) = \gamma$ for any arc $\gamma \in V(T) \setminus D_\ell$. In particular, $ord_{\phi(\gamma)} f = ord_\gamma f$ for each arc $\gamma \subset T$, thus ϕ transforms the function f into a v-equivalent function. This implies that ϕ preserves all zones D_ℓ, and that decomposition $\{\phi(T_\ell)\}$ defines a minimal pizza on T associated with f. Replacing all arcs λ_ℓ with some other arcs $\theta_\ell \in D_\ell$, for $\ell = 0, \ldots, p$, we see that any choice of arcs $\lambda_\ell \in D_\ell$ results in a minimal pizza on T associated with f.

On the other hand, given a minimal pizza $\{T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)\}$ on T associated with f, consider any other minimal pizza $\{T'_\ell = T(\theta_{\ell-1}, \theta_\ell)\}$ on T associated with f. By the Lipschitz contact invariance of a minimal pizza (see Theorem 2.19) there exists an inner bi-Lipschitz homeomorphism $h : T \to T$ such that $h(\lambda_\ell) = \theta_\ell$ and $h(T_\ell) = T'_\ell$ for all ℓ, and also such that $ord_{h(\gamma)} f = ord_\gamma f$ for any arc $\gamma \subset T$. Thus h transforms f into a function of the same contact (see Definition 2.2 from [4]) . Since the zones D_ℓ are Lipschitz invariant, we have $h(D_\ell) = D_\ell$ for all ℓ, thus $\theta_\ell \in D_\ell$. This proves that any minimal pizza can be obtained by some choice of arcs $\lambda_\ell \in D_\ell$. $\square$

Corollary 2.31. *Let $\{D_\ell\}_{\ell=0}^p$ be the pizza zones of a minimal pizza $\{T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)\}_{\ell=1}^p$ on T associated with f, as in Proposition 2.30. For each $\ell = 1, \ldots, p$, the set $Y_\ell = D_{\ell-1} \cup D_\ell \cup V(T_\ell)$ is a pizza slice zone associated with f, independent of the choice of arcs $\lambda_\ell \in D_\ell$. Moreover, Y_ℓ is a maximal pizza slice zone: if a pizza slice zone $Y \subset V(T)$ associated with f contains Y_ℓ then $Y = Y_\ell$.*

3. Elementary pairs of normally embedded Hölder triangles

Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded β-Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively.

Definition 3.1. A pair (γ, γ') of arcs $\gamma \subset T$ and $\gamma' \subset T'$ is *regular* if

$$tord(\gamma, T') = tord(\gamma, \gamma') = tord(\gamma', T). \tag{4}$$

Proposition 3.2. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded β-Hölder triangles. Let $f(x) = dist(x, T')$ be the distance from $x \in T$ to T', and let $g(x') = dist(x', T)$ be the distance from $x' \in T'$ to T. Let $\Gamma \subset T \times \mathbb{R}$ and $\Gamma' \subset T' \times \mathbb{R}$ be the graphs of the functions $f(x)$ and $g(x')$. Then the following conditions are equivalent:*

1. There is a homeomorphism $H : T \cup T' \to T \cup \Gamma$, bi-Lipschitz with respect to the outer metric, such that $H(\gamma_1) = \gamma_1$ and $H(\gamma_2) = \gamma_2$.

2. There is a homeomorphism $H' : T \cup T' \to T' \cup \Gamma'$, bi-Lipschitz with respect to the outer metric, such that $H'(\gamma'_1) = \gamma'_1$ and $H'(\gamma'_2) = \gamma'_2$.

3. There exists a bi-Lipschitz homeomorphism $h : T \to T'$ such that $h(\gamma_1) = \gamma'_1$, $h(\gamma_2) = \gamma'_2$ and $tord(\gamma, h(\gamma)) = tord(\gamma, T')$ for any arc $\gamma \subset T$.

4. There exists a bi-Lipschitz homeomorphism $h' : T' \to T$ such that $h'(\gamma'_1) = \gamma_1$, $h'(\gamma'_2) = \gamma_2$ and $tord(\gamma', h'(\gamma')) = tord(\gamma', T)$ for any arc $\gamma' \subset T'$.

5. There exists a bi-Lipschitz homeomorphism $h : T \to T'$ such that $h(\gamma_1) = \gamma'_1$, $h(\gamma_2) = \gamma'_2$, and the pair of arcs $(\gamma, h(\gamma))$ is regular for any arc $\gamma \subset T$.

Proof. If condition 1 is satisfied, we may assume that $H(T) = T$ and $H(T') = \Gamma$. Since f is a Lipschitz function on T and H is an outer bi-Lipschitz homeomorphism, we have $tord(\gamma, T') = tord(H(\gamma), \Gamma) = ord_{H(\gamma)} f = tord(H(\gamma), f(H(\gamma))$ for any arc $\gamma \subset T$. Since H^{-1} is also an outer bi-Lipschitz homeomorphism, the mapping $h : T \to T'$ defined as $h(x) = H^{-1}((H(x), f(H(x)))$ is a bi-Lipschitz homeomorphism satisfying condition 5, which implies conditions 3 and 4. Conversely, given a homeomorphism $h : T \to T'$ satisfying condition 3, the mapping $H : T \cup T' \to T \cup \Gamma$ which is the identity on T and defined as $H(x') = (h^{-1}x', f(h^{-1}(x')))$for $x' \in T'$ satisfies condition 1. Thus conditions 1, 3 and 5 are equivalent.

Similarly, conditions 2, 4 and 5 are equivalent.

If conditions 1 and 3 are satisfied, we may assume that $T' = \Gamma$ and $h(x) = (x, f(x))$ for $x \in T$. Then $tord(\gamma, T') = tord(\gamma', T)$ for any arcs $\gamma \subset T$ and $\gamma' = \{(x, f(x)) : x \in \gamma\} \subset T'$. Thus $h' = h^{-1} : T' \to T$ satisfies condition 2. This implies that all five conditions are equivalent. □

If conditions of Proposition 3.2 are satisfied then the pairs of arcs (γ_1, γ'_1) and (γ_2, γ'_2) are regular:

$$(5)\quad tord(\gamma_1, T') = tord(\gamma_1, \gamma'_1) = tord(\gamma'_1, T), \quad tord(\gamma_2, T') = tord(\gamma_2, \gamma'_2) = tord(\gamma'_2, T).$$

In general, the opposite does not hold. However, Theorem 3.20 below states that conditions of Proposition 3.2 are satisfied if T is elementary with respect to f and (5) holds. The following Proposition from [7] is an important step in the proof of Theorem 3.20.

Proposition 3.3. (see [7, Proposition 2.20]) *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be normally embedded β-Hölder triangles such that $tord(\gamma_1, \gamma'_1) \geq \alpha$, $tord(\gamma_2, \gamma'_2) \geq \alpha$, and $tord(\gamma, T') \geq \alpha$ for all arcs $\gamma \subset T$, for some $\alpha > \beta$. Then there is a bi-Lipschitz homeomorphism $h : T \to T'$ such that $h(\gamma_1) = \gamma'_1$, $h(\gamma_2) = \gamma'_2$, and $tord(h(\gamma), \gamma) \geq \alpha$ for any arc $\gamma \subset T$.*

Remark 3.4. If a β-Hölder triangle $T = T(\gamma_1, \gamma_2)$ and a β'-Hölder triangle $T' = T(\gamma'_1, \gamma'_2)$ are normally embedded and satisfy (5) then $\beta' = \beta$, unless $tord(T, T') \leq \min(\beta, \beta')$.

Lemma 3.5. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be normally embedded Hölder triangles. Let $\lambda_1 \neq \lambda_2$ be two arcs in T, and let $\theta_1 \subset T'$, $\theta_2 \subset T'$ and $\theta \subset T(\theta_1, \theta_2) \subset T'$ be* *three arcs such that* $tord(\theta, T) = q < \min(tord(\lambda_1, \theta_1), tord(\lambda_2, \theta_2))$. *Then there is an arc* $\lambda \subset T(\lambda_1, \lambda_2) \subset T$ *such that* $tord(\lambda, T') \leq q$.

Proof. We may assume that $\theta_1 \subset T'_1 = T(\gamma'_1, \theta)$ and $\theta_2 \subset T'_2 = T(\theta, \gamma'_2)$. For $x \in T$, let $f_1(x) = dist(x, T'_1)$ and $f_2(x) = dist(x, T'_2)$. Then $f(x) = dist(x, T') = \min(f_1(x), f_2(x))$. Since $tord(\theta, \lambda_1) \leq tord(\theta, T) = q$ and $tord(\lambda_1, \theta_1) > q$, we have by the non-Archimedean property $tord(\theta_1, \theta) = \min(tord(\lambda_1, \theta_1), tord(\lambda_1, \theta)) \leq q$. Since T' is normally embedded, we have $tord(\theta_1, T'_2) = tord(\theta_1, \theta) \leq q$. Since $tord(\lambda_1, \theta_1) > q$, this implies $tord(\lambda_1, T'_2) = \min(tord(\lambda_1, \theta_1), tord(\theta_1, T'_2)) \leq q$ by the non-Archimedean property. Thus $f|_{\lambda_1} = f_1|_{\lambda_1}$. Similarly, $f|_{\lambda_2} = f_2|_{\lambda_2}$, thus there is an arc $\lambda \subset T(\lambda_1, \lambda_2)$ such that $f|_\lambda = f_1|_\lambda = f_2|_\lambda$ (see Fig. 1). Then $tord(\lambda, T') = ord_\lambda f \leq q$, otherwise we would have $tord(\lambda, T'_1) = tord(\lambda, T'_2) > q$. Since $tord(\lambda, \theta) \leq tord(\theta, T) = q$, this would contradict to $T' = T'_1 \cup T'_2$ being normally embedded. □

Corollary 3.6. *Let* T *and* T' *be normally embedded Hölder triangles. Let* $\tilde{T} = T(\lambda_1, \lambda_2) \subset T$ *be a* β*-Hölder triangle such that* $tord(\gamma, T') = q > \beta$ *for any arc* $\gamma \subset \tilde{T}$. *If* $\tilde{T}' = T(\theta_1, \theta_2) \subset T'$ *is a* β*-Hölder triangle such that* $tord(\theta_1, \lambda_1) = tord(\theta_2, \lambda_2) = q$ *then, for any arc* $\theta \subset T'$ *such that* $tord(\theta, \theta_1) < q$ *and* $tord(\theta, \theta_2) < q$, *we have* $tord(\theta, T) = q$.

Proof. Lemma 3.5 implies that $tord(\theta, T) \geq tord(\theta, \tilde{T}) \geq q$ for any arc $\theta \subset \tilde{T}'$. If $\theta \subset \tilde{T}'$ is an arc such that $tord(\theta, \theta_1) < q$ and $tord(\theta, \theta_2) < q$, Proposition 3.3 implies that $tord(\theta, T \setminus \tilde{T}) < q$. If $tord(\theta, T) > q$ and $\gamma \subset T$ is an arc such that $tord(\gamma, \theta) > q$, then $\gamma \subset \tilde{T}$ and $tord(\gamma, T') > q$, a contradiction. Thus $tord(\theta, T) = q$. □

Definition 3.7. Let T and T' be normally embedded oriented Hölder triangles. A pair of β-Hölder triangles $\tilde{T} = T(\lambda_1, \lambda_2) \subset T$ and $\tilde{T}' = T(\theta_1, \theta_2) \subset T'$ in Corollary 3.6 is called *positively oriented* if their orientations induced from T and T' are either both the same as their orientations from λ_1 to λ_2 and from θ_1 to θ_2 or both opposite. Otherwise, $\tilde{T}$ and $\tilde{T}'$ is called a *negatively oriented* pair of Hölder triangles.

Remark 3.8. Any pair of α-Hölder triangles $T(\lambda'_1, \lambda'_2) \subset \tilde{T}$ and $T(\theta'_1, \theta'_2) \subset \tilde{T}'$, where $\alpha < q$, satisfying conditions $tord(\theta'_1, \lambda'_1) = tord(\theta'_2, \lambda'_2) = q$ is positively (resp., negatively) oriented if, and only if, the pair $(\tilde{T}, \tilde{T}')$ is positively (resp., negatively) oriented.

Proposition 3.9. *Let* T *and* T' *be normally embedded Hölder triangles with the distance functions* $f(x) = dist(x, T')$ *and* $g(x') = dist(x', T)$. *Let* $Z \subset V(T)$ *be a maximal* q*-order zone for* f *such that* $\mu(Z) < q$. *Then there exists a unique maximal* q*-order zone* $Z' \subset V(T')$ *for* g *such that* $\mu(Z') = \mu(Z)$ *and, for any arc* $\gamma \in Z$ *such that* $\nu_Z(\gamma, f) < q$ *and any arc* $\gamma' \subset T'$ *such that* $tord(\gamma, \gamma') = q$, *we have* $\gamma' \subset Z'$ *and* $\nu_{Z'}(\gamma', g) = \nu_Z(\gamma, f)$. *Conversely, if* $\gamma' \in Z'$ *is any arc such that* $\nu_{Z'}(\gamma', g) < q$ *then, for any arc* $\gamma \subset T$ *such that* $tord(\gamma, \gamma') = q$, *we have* $\gamma \subset Z$ *and* $\nu_Z(\gamma, f) = \nu_{Z'}(\gamma', g)$.

Proof. Let $\tilde{Z} \subset Z$ be the set of all arcs $\gamma \subset Z$ such that $\nu_Z(\gamma, f) < q$. Let λ_1 and λ_2 be any two arcs in $\tilde{Z}$ such that $\beta = tord(\lambda_1, \lambda_2) < q$. Consider the β-Hölder triangle $\tilde{T} = T(\lambda_1, \lambda_2) \subset T$. Since Z is a zone, we have $V(\tilde{T}) \subset Z$, thus $tord(\gamma, T') = q$ for any arc $\gamma \subset \tilde{T}$. Also, $\nu_Z(\gamma, f) \leq \max(\nu_Z(\lambda_1, f), \nu_Z(\lambda_2, f)) < q$ for any arc $\gamma \subset \tilde{T}$, thus $V(\tilde{T}) \subset \tilde{Z}$. This implies that $\tilde{Z}$ is a q-order zone for f. Let $\tilde{Z}'$ be the set of all arcs $\gamma' \subset T'$ such that $tord(\gamma, \gamma') = q$ for some arc $\gamma \subset \tilde{Z}$.

Let λ'_1 and λ'_2 be any two arcs in T' such that $tord(\lambda_1, \lambda'_1) = tord(\lambda_2, \lambda'_2) = q$. Since $\beta < q$, $\tilde{T}' = T(\lambda'_1, \lambda'_2) \subset T'$ is a β-Hölder triangle. Since $\lambda_1 \in \tilde{Z}$ and $\lambda_2 \in \tilde{Z}$, we have

$V(\tilde{T}') \subset \tilde{Z}'$. It follows from Proposition 3.3 that $tord(\gamma, T'\setminus\tilde{T}') < q$ for any arc $\gamma \subset \tilde{T}$, thus $tord(\gamma, \tilde{T}') = q$ for any arc $\gamma \subset \tilde{T}$. Corollary 3.6 implies that $tord(\theta, \tilde{T}) = q$ for any arc $\theta \subset \tilde{T}'$. This implies that $\tilde{Z}'$ is a q-order zone for g. It follows from the non-Archimedean property that $\nu_{\tilde{Z}'}(\gamma', g) < q$ for any arc $\gamma' \in \tilde{Z}'$.

Let Z' be the maximal q-zone for g containing $\tilde{Z}'$. By the construction this zone is unique. Let us show that $\nu_{Z'}(\gamma', g) \geq q$ for any arc $\gamma' \in Z' \setminus \tilde{Z}'$. If $\gamma' \in Z' \setminus \tilde{Z}'$ and $\nu_{Z'}(\gamma', g) < q$, applying the same arguments as above to Z' and g instead of Z and f, we can show that any arc $\gamma \subset T$ such that $tord(\gamma, \gamma') = q$ belongs to Z and $\nu_Z(\gamma, f) < q$. Thus $\gamma \in \tilde{Z}$, which implies $\gamma' \in \tilde{Z}'$, a contradiction. The equality $\nu_Z(\gamma, f) = \nu_{Z'}(\gamma', g)$ follows from the non-Archimedean property. □

Corollary 3.10. *Let T and T' be normally embedded Hölder triangles with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$. For any $q \in \mathbb{F}$, the finite set L_q of maximal q-order zones $Z \subset V(T)$ for f such that $\mu(Z) < q$ is nonempty if, and only if, the set L'_q of maximal q-order zones $Z' \subset V(T')$ for g such that $\mu(Z') < q$ is nonempty, and there is a canonical one-to-one correspondence $Z' = \tau_q(Z)$ between the sets L_q and L'_q such that $tord(Z, \tau_q(Z)) = q$.*

Proof. The finiteness of the set L_q follows from the fact that, for a given $q \in \mathbb{F}$, each pizza slice of a pizza on T associated with f contains at most one zone from L_q. □

Lemma 3.11. *Let T and T' be normally embedded oriented Hölder triangles. Let $Z \subset V(T)$ and $Z' \subset V(T')$ be maximal q-order zones for f and g respectively, of orders $\mu(Z) = \mu(Z') < q$, related as in Proposition 3.9 and Corollary 3.10. Then the pairs $(\tilde{T}, \tilde{T}')$ of Hölder triangles $\tilde{T} \subset T$ and $\tilde{T}' \subset T'$ related as in Corollary 3.6, such that $V(\tilde{T}) \subset Z$ and $V(\tilde{T}') \subset Z'$, are either all positively oriented or all negatively oriented.*

Proof. This follows from Remark 3.8, since for any two pairs of Hölder triangles in Lemma 3.11 there is a larger pair of Hölder triangles containing both of them and satisfying conditions of Corollary 3.6. □

Definition 3.12. The pair of zones $Z \subset V(T)$ and $Z' \subset V(T')$ in Lemma 3.11 is called *positively oriented* (resp., *negatively oriented*) if the pairs $(\tilde{T}, \tilde{T}')$ of Hölder triangles $\tilde{T} \subset T$ and $\tilde{T}' \subset T'$ in Lemma 3.11 are positively oriented (resp., negatively oriented).

Lemma 3.13. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded Hölder triangles, such that T is elementary with respect to $f(x) = dist(x, T')$ and $tord(\gamma_1, \gamma'_1) = tord(T, T')$. Then T' is elementary with respect to $g(x') = dist(x', T)$.*

Proof. We have to show that, for any Hölder triangle $T'' = T(\theta_1, \theta_2) \subset T'$ such that $tord(\theta_1, T) = tord(\theta_2, T) = q$, we have $tord(\gamma', T) = q$ for each arc $\gamma' \subset T''$. Let us show first that $q' = tord(\gamma', T) \geq q$ for each arc $\gamma' \subset T''$. If $q' < q$, let λ_1 and λ_2 be arcs in T such that $tord(\lambda_1, \theta_1) = tord(\lambda_2, \theta_2) = q$. Lemma 3.5 implies that there is an arc $\lambda \subset T(\lambda_1, \lambda_2)$ such that $tord(\lambda, T') \leq q' < q$, a contradiction with T being elementary with respect to f.

Suppose now that $q' > q$. We may assume that $\theta_1 \subset T(\gamma'_1, \gamma') \subset T'$. Since $tord(\gamma_1, \gamma'_1) = tord(T, T')$, we have $tord(\gamma_1, \gamma'_1) \geq q'$. Let $\gamma \subset T$ be an arc such that $tord(\gamma, \gamma') = q'$ (see Fig. 2). Then Lemma 3.5 applied to $T(\gamma_1, \gamma) \subset T$ and $T(\gamma'_1, \gamma') \subset T'$ implies that there is an arc $\lambda \subset T(\gamma_1, \gamma)$ such that $tord(\lambda, T') \leq q$, a contradiction with T being elementary with respect to f. □

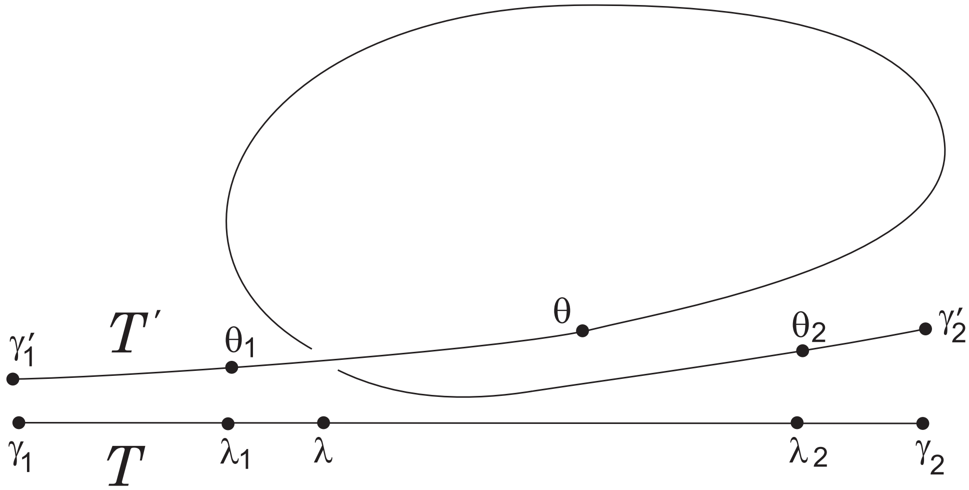

FIGURE 1. Illustration to the proof of Lemma 3.5.

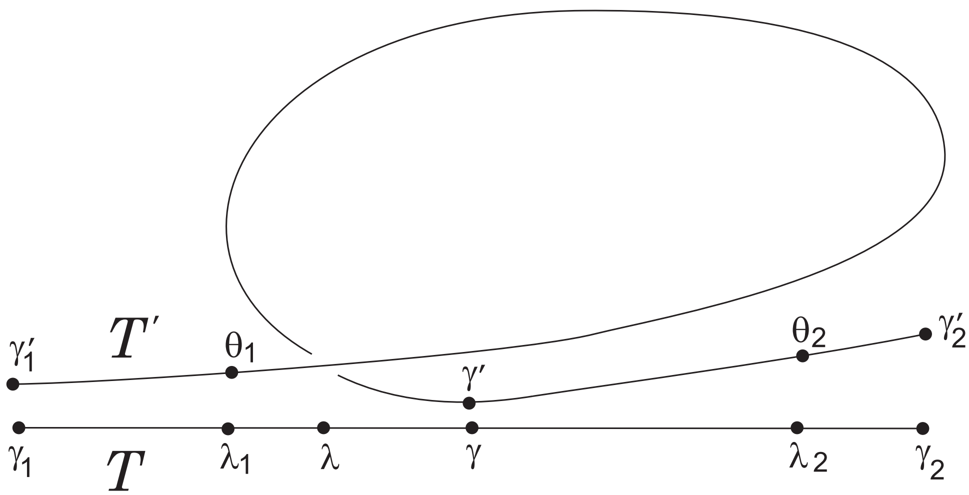

FIGURE 2. Illustration to the proof of Lemma 3.13.

Corollary 3.14. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$ be normally embedded Hölder triangles, such that T is elementary with respect to $f(x) = dist(x, T')$ and $tord(\gamma_1, \gamma_1') = tord(T, T')$. Then, for any two Hölder triangles $\tilde{T} = T(\gamma_1, \lambda) \subset T$ and $\tilde{T}' = T(\gamma_1', \lambda') \subset T'$, $\tilde{T}$ is elementary with respect to $\tilde{f}(x) = dist(x, \tilde{T}')$ and $\tilde{T}'$ is elementary with respect to $\tilde{g}(x') = dist(x', \tilde{T})$.*

Proof. Since $\tilde{T}$ is elementary with respect to $f|_{\tilde{T}}$, Lemma 3.13 applied to $\tilde{T}$ instead of T implies that T' is elementary with respect to $h(x') = dist(x', \tilde{T})$. Thus $\tilde{T}'$ is elementary with respect to $\tilde{g} = h|_{\tilde{T}'}$. Lemma 3.13 applied to $\tilde{T}'$ instead of T and $\tilde{T}$ instead of T' implies that $\tilde{T}$ is elementary with respect to $\tilde{f}$. □

Lemma 3.15. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$ be normally embedded β-Hölder triangles satisfying (5). Suppose that T is a pizza slice associated with $f(x) = dist(x, T')$. Then conditions of Proposition 3.2 are satisfied for T and T'. Moreover, $\mu_{T,f} \equiv \mu_{T',g}$, where $\mu_{T,f}(q)$ and $\mu_{T',g}(q)$ are the width functions defined on $Q_f(T) = Q_g(T')$.*

Proof. Since the five conditions of Proposition 3.2 are equivalent, it is enough to prove condition 3: there is a bi-Lipschitz homeomorphism $h : T \to T'$ such that $h(\gamma_1) = \gamma_1'$, $h(\gamma_2) = \gamma_2'$ and $tord(\gamma, h(\gamma)) = tord(\gamma, T')$ for each arc $\gamma \subset T$.

Let $Q = Q_f(T)$, and let the width function $\mu(q) = \mu_{T,f}(q) : Q \to \mathbb{F} \cup \{\infty\}$ be affine, $\mu(q) = aq + b$. We consider the following cases: (1) $Q = \{\alpha\}$ where $\alpha \le \beta$, (2) $Q = \{\alpha\}$ where $\alpha > \beta$, (3) $\mu(q) \equiv q$, (4) $\mu(q) < q$ for all $q \in Q$, (5) $\mu(q) = q$ only for the maximal value of $\mu(q)$, (6) $\mu(q) = q$ only for the minimal value $\mu(q) = \beta$.

Case 1. Any bi-Lipschitz homeomorphism $h : T \to T'$ such that $h(\gamma_1) = \gamma'_1$ and $h(\gamma_2) = \gamma'_2$ satisfies $tord(\gamma, h(\gamma)) = tord(\gamma, T') = \alpha$ for all arcs $\gamma \in V(T)$.

Case 2. It follows from [7, Proposition 2.20] (see Proposition 3.3) that there is a bi-Lipschitz homeomorphism $h : T \to T'$ such that $tord(\gamma, h(\gamma)) \ge \alpha$ for any arc $\gamma \subset T$. Since $tord(\gamma, h(\gamma)) \le tord(\gamma, T') = \alpha$ for any arc $\gamma \subset T$, we have $tord(\gamma, h(\gamma)) = \alpha$.

Case 3. We may assume that Q is not a point and $q_1 = tord(\gamma_1, T')$ is the maximal value of $q \in Q$. Then $q = ord_\gamma f = \mu_T(\gamma, f) = tord(\gamma, \gamma_1)$ for all arcs $\gamma \subset T$ such that $tord(\gamma, \gamma_1) \le q_1$, otherwise $ord_\gamma f = q_1$. Any bi-Lipschitz homeomorphism $h : T \to T'$ such that $h(\gamma_1) = \gamma'_1$ satisfies $tord(h(\gamma), \gamma'_1) = tord(\gamma, \gamma_1)$ for all arcs $\gamma \subset T$. Thus $q = ord_\gamma f = \mu_T(\gamma, f) = tord(\gamma, \gamma_1) = tord(h(\gamma), \gamma'_1)$ for all $\gamma \subset T$ such that $tord(\gamma, \gamma_1) \le q_1$. Since $tord(\gamma_1, \gamma'_1) = q_1 \ge q$, this implies that $tord(\gamma, h(\gamma)) \ge q$. If $tord(\gamma, h(\gamma)) > q$ then $tord(\gamma, T') > q$, a contradiction. Thus $tord(\gamma, h(\gamma)) = q$ for all $\gamma \subset T$ such that $tord(\gamma, \gamma_1) \le q_1$. Otherwise, if $tord(\gamma, \gamma_1) > q_1$, then $tord(h(\gamma), \gamma'_1) > q_1$, thus $tord(\gamma, h(\gamma)) = tord(\gamma_1, \gamma'_1) = q_1 = tord(\gamma, T')$.

Case 4. Using the same arguments as in the proof of [7, Proposition 2.20], we assume that $T' = T_\beta \subset \mathbb{R}^2$ is a standard β-Hölder triangle (1), $T \cup T' \subset \mathbb{R}^n$, and $\pi : T \to \mathbb{R}^2$ is an orthogonal projection. We may also assume that Q is not a point, and that $\mu(q_1)$, where $q_1 = tord(\gamma_1, T')$, is the maximal value of $\mu(q)$ for $q \in Q$. Then $\mu_T(\gamma, f) = tord(\gamma, \gamma_1)$ for all arcs $\gamma \subset T$ such that $tord(\gamma, \gamma_1) \le \mu(q_1)$, otherwise $ord_\gamma f = q_1$.

The set $S \subset T$ where π is not smooth and orientation-preserving is a finite union of isolated arcs and β_j-Hölder triangles $T_j = T(\lambda_j, \lambda'_j) \subset T$. We want to show that f has the same order q_j on each arc $\gamma \subset T_j$. It is enough to show that $ord_{\lambda_j} f = ord_{\lambda'_j} f$. We may assume that $\lambda'_j \subset T(\gamma_1, \lambda_j) \subset T$, thus $\mu_j = \mu_T(\lambda_j, f) \le \mu(\lambda'_j, f)$. If $ord_{\lambda_j} f = q_j \ne ord_{\lambda'_j} f$ then $\beta_j \le \mu_j = tord(\lambda_j, \gamma_1)$. Since T_j is orientation-reversing and T is normally embedded, we have $\beta_j \ge q_j$, a contradiction with the condition $\mu_j < q_j$. Thus $ord_{\lambda_j} f = ord_{\lambda'_j} f = q_j \le \beta_j < \mu_j$, and there is a μ_j-Hölder triangle $\tilde{T}_j \subset T$ containing T_j such that $ord_\gamma f = q_j$ for each arc $\gamma \subset \tilde{T}_j$.

It follows from [7, Proposition 2.20] that there is a bi-Lipschitz orientation-preserving homeomorphism $h_j : \tilde{T}_j \to \pi(\tilde{T}_j) \cap T'$ such that $tord(\gamma, h_j(\gamma)) = q_j$ for each arc $\gamma \subset \tilde{T}_j$. One can choose triangles $\tilde{T}_j$ so that they are all disjoint. Replacing projection π with the homeomorphisms h_j on each triangle $\tilde{T}_j$, a bi-Lipschitz homeomorphism $h : T \to T'$ can be obtained, such that $tord(\gamma, h(\gamma)) = tord(\gamma, T')$ for each arc $\gamma \subset T$.

Case 5. Assuming that $\mu(q_1) = q_1 = tord(\gamma_1, \gamma'_1)$ is the maximal value of $\mu(q)$, for any arc $\gamma \subset T$ such that $tord(\gamma, T') = q_1$ we have $tord(\gamma, \gamma_1) \ge \mu(\gamma) = q_1$, thus $tord(\gamma, h(\gamma)) = tord(\gamma_1, \gamma'_1) = q_1$. For any arc $\gamma \subset T$ such that $q = tord(\gamma, T') > \mu(q) = tord(\gamma, \gamma_1)$, the same arguments as in Case 4 apply.

Case 6. The same arguments as in Case 4 imply that, for any triangle $T_j \subset T$ containing an arc γ such that $ord_\gamma f > \beta$ and $\pi|_T$ is orientation-reversing, the order q_j of f is the same on all arcs of T_j, and there is a μ_j-triangle $\tilde{T}_j$ containing T_j, where $\mu_j > \beta$, such that $\pi|_{\tilde{T}_j}$ can be replaced with a bi-Lipschitz orientation-preserving homeomorphism $h_j : \tilde{T}_j \to \pi(\tilde{T}_j) \cap T'$, such that $tord(\gamma, h_j(\gamma)) = q_j$ for each arc $\gamma \subset \tilde{T}_j$. This allows one to find β-Hölder triangles $\tilde{T} = T(\gamma_1, \tilde{\gamma}) \subset T$ and $\tilde{T}' = T(\gamma'_1, \tilde{\gamma}') \subset T'$ such that $\bar{T} = T(\tilde{\gamma}, \gamma_2) \subset T$ and $\bar{T}' = T(\tilde{\gamma}', \gamma'_2) \subset T'$ are also β-Hölder triangles, $ord_\gamma f = \beta$ for each arc $\gamma \subset \bar{T}$, and to obtain a bi-Lipschitz homeomorphism $\tilde{h} : \tilde{T} \to \tilde{T}'$ such that $tord(\gamma, h(\gamma)) = tord(\gamma, T')$ for each arc $\gamma \subset \tilde{T}$. After that, $\tilde{h}$ combined with any bi-Lipschitz homeomorphism $\bar{h} : \bar{T} \to \bar{T}'$, such that $\bar{h}(\tilde{\gamma}) = \tilde{\gamma}'$ and $\bar{h}(\gamma_2) = \gamma'_2$, defines a bi-Lipschitz homeomorphism $h : T \to T'$ such that $tord(\gamma, h(\gamma)) = tord(\gamma, T')$ for each arc $\gamma \subset T$.

The existence of a mapping $h : T \to T'$ satisfying condition 5 of Proposition 3.2 implies that $Q_f(T) = Q_g(T')$ and $\mu_{T,f} \equiv \mu_{T',g}$. □

Definition 3.16. Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded oriented β-Hölder triangles satisfying (5), such that T is a pizza slice associated with $f(x) = dist(x, T')$ and $tord(T, T') = tord(\gamma_1, \gamma'_1) > \beta$. The pair (T, T') is called *positively oriented* if either T is oriented from γ_1 to γ_2 and T' from γ'_1 to γ'_2, or T is oriented from γ_2 to γ_1 and T' from γ'_2 to γ'_1. Otherwise, the pair (T, T') is called *negatively oriented*.

Lemma 3.17. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded β-Hölder triangles in Definition 3.16 such that $\mu_{T,f}(q) \not\equiv q$. For $q \in Q_T(f)$ such that $\mu(q) < q$, let $Z_q \subset V(T)$ and $Z'_q \subset V(T')$ be the maximal q-order zones for $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ respectively. Then the pair of zones (Z_q, Z'_q) is positively oriented if the pair of Hölder triangles (T, T') is positively oriented, and negatively oriented otherwise.*

Lemma 3.18. *Let a β-Hölder triangle $T = T(\gamma_1, \gamma_2)$ and a β'-Hölder triangle $T' = T(\gamma'_1, \gamma'_2)$ be normally embedded, where $\beta \geq \beta'$, $q_1 = tord(\gamma_1, \gamma'_1) = tord(T, T') > \beta$ and $q_2 = tord(\gamma_2, T') \geq \beta$. If T is a pizza slice associated with $f(x) = dist(x, T')$ then there is an arc $\theta \subset T'$ such that $tord(\gamma'_1, \theta) = \beta$, and*

$$tord(\gamma_2, \theta) = tord(\theta, T) = q_2, \tag{6}$$

thus conditions (5) are satisfied for triangles T and $T(\gamma'_1, \theta) \subset T'$. Moreover, if $q_2 > \beta$ then $tord(\gamma'_1, \theta) = \beta$ for any arc $\theta \subset T'$ satisfying condition (6), and if $q_2 = \beta$ then any arc $\theta \subset T'$ such that $tord(\gamma'_1, \theta) = \beta$ satisfies condition (6).

Proof. Let $\theta \subset T'$ be an arc such that $tord(\gamma_2, \theta) = q_2$. Note first that $\alpha = tord(\theta, T) \geq q_2$. Suppose that $\alpha > q_2$, and let $\lambda \subset T$ be an arc such that $tord(\lambda, \theta) = \alpha$,thus $q' = tord(\lambda, T') \geq \alpha > q_2$. Let $\mu_T(q)$ be the affine width function of T. Note that μ cannot be constant, since $q_1 \geq \alpha > q_2$, If the maximum of $\mu_T(q)$ is at $q = q_2$ then $tord(\lambda, \gamma_2) = \mu(q') \leq \mu(\alpha) < \mu(q_2) \leq q_2$. Since $q_2 = tord(\gamma_2, \theta)$ and $\alpha = tord(\lambda, \theta) > q_2$, this contradicts the non-Archimedean property. Thus the maximum of $\mu_T(q)$ is at $q = q_1$ and its minimum is $\mu(q_2) = \beta$.

If $q_2 > \beta$ then $tord(\theta, \gamma'_1) = tord(\gamma_1, \gamma_2) = \beta$ for any arc $\theta \subset T'$ satisfying condition (6). However, since $tord(\gamma_1, \lambda) > \beta$, $tord(\gamma_1, \gamma'_1) = \mu(q_1) > \beta$ and $\alpha = tord(\lambda, \theta) > q_2 \geq \beta$, condition $tord(\gamma'_1, \theta) = \beta$ cannot be satisfied, a contradiction. Thus $\alpha = q_2$ in this case.

Otherwise, if $q_2 = \beta$, then any arc $\theta \subset T'$ such that $\alpha = tord(\theta, T) > \beta$ satisfies $tord(\theta, \gamma'_1) \geq \min(\alpha, q_1) > \beta$, thus any arc $\theta \subset T'$ such that $tord(\gamma'_1, \theta) = \beta$ satisfies condition (6). □

Proposition 3.19. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be normally embedded β-Hölder triangles with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$, such that T is elementary with respect to f and $tord(\gamma_1, \gamma'_1) = tord(T, T')$. Let $\lambda \subset T$ and $\lambda' \subset T'$*

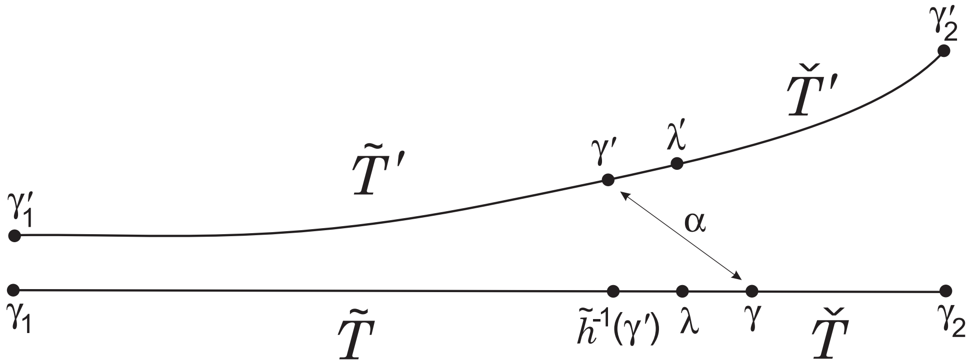

FIGURE 3. Illustration to the proof of Proposition 3.19.

be a regular pair of arcs such that $\tilde{T} = T(\gamma_1, \lambda)$ *and* $\tilde{T}' = T(\gamma_1', \lambda')$ *are* $\tilde{\beta}$*-Hölder triangles and* $\check{T} = T(\lambda, \gamma_2)$ *and* $\check{T}' = T(\lambda', \gamma_2')$ *are* $\check{\beta}$*-Hölder triangles. If both pairs* $(\tilde{T}, \tilde{T}')$ *and* $(\check{T}, \check{T}')$ *satisfy conditions of Proposition 3.2, then the pair* (T, T') *satisfies conditions of Proposition 3.2.*

Proof. Since conditions 1 - 5 of Proposition 3.2 are equivalent, it is enough to prove condition 3, i.e., to find a bi-Lipschitz homeomorphism $h : T \to T'$ such that $h(\gamma_1) = \gamma_1'$, $h(\gamma_2) = \gamma_2'$ and $tord(\gamma, h(\gamma)) = tord(\gamma, T')$ for each arc $\gamma \subset T$. Conditions of Proposition 3.19 imply that there is a bi-Lipschitz homeomorphism $\tilde{h} : \tilde{T} \to \tilde{T}'$ such that $\tilde{h}(\gamma_1) = \gamma_1'$, $\tilde{h}(\lambda) = \lambda'$ and $tord(\gamma, \tilde{h}(\gamma)) = tord(\gamma, \tilde{T}') = tord(\tilde{h}(\gamma, \tilde{T})$ for each arc $\gamma \subset \tilde{T}$, and also a bi-Lipschitz homeomorphism $\check{h} : \check{T} \to \check{T}'$ such that $\check{h}(\lambda) = \lambda'$, $\check{h}(\gamma_2) = \gamma_2'$ and $tord(\gamma, \check{h}(\gamma)) = tord(\gamma, \check{T}') = tord(\check{h}(\gamma), \check{T})$ for each arc $\gamma \subset \check{T}$. We may assume that $\tilde{h}(x) = \check{h}(x)$ for $x \in \lambda$.

We claim that a bi-Lipschitz homeomorphism $h : T \to T'$ with the necessary properties can be defined as $h(x) = \tilde{h}(x)$ for $x \in \tilde{T}$ and $h(x) = \check{h}(x)$ for $x \in \check{T}$. It is enough to show that $tord(\gamma, T') = tord(\gamma, \tilde{T}')$ for any arc $\gamma \subset \tilde{T}$ and $tord(\gamma, T') = tord(\gamma, \check{T}')$ for any arc $\gamma \subset \check{T}$.

Since T is elementary with respect to f, Lemma 3.13 implies that T' is elementary with respect to g. Corollary 3.14 implies that $\tilde{T}$ is elementary with respect to $\tilde{f}$, $\tilde{T}'$ is elementary with respect to $\tilde{g}$, $\check{T}$ is elementary with respect to $\check{f}$ and $\check{T}'$ is elementary with respect to $\check{g}$.

Let $\gamma \subset \tilde{T}$. Then $\alpha = tord(\gamma, T') = max(tord(\gamma, \tilde{T}'), tord(\gamma, \check{T}') \geq tord(\gamma, \tilde{T}')$. If $\alpha > tord(\gamma, \tilde{T}')$ then there is an arc $\gamma' \subset \check{T}'$ such that $tord(\gamma, \gamma') = tord(\gamma, \check{T}') = \alpha$, thus $tord(\gamma', T) \geq \alpha$. Then

$$\alpha > tord(\gamma, \tilde{h}(\gamma)) = tord(\tilde{h}(\gamma), \tilde{T}) \geq tord(\lambda', \tilde{T}) = tord(\lambda, \lambda') = tord(\lambda', \check{T})$$

Thus $tord(\gamma', T) > tord(\lambda', T)$, a contradiction with T' being elementary with respect to g.

Let $\gamma \subset \check{T}$. Then $\alpha = tord(\gamma, T') \leq tord(\lambda, T') = tord(\lambda, \lambda')$. If $\alpha > tord(\gamma, \check{T}')$ then there is an arc $\gamma' \subset \tilde{T}'$ such that $tord(\gamma, \gamma') = \alpha > tord(\gamma, \check{T}' \geq tord(\gamma, \lambda')$ (see Fig. 3). This implies that $tord(\gamma', \lambda') = tord(\gamma, \lambda')$. Since $tord(\tilde{h}^{-1}(\gamma'), \gamma) \leq tord(\tilde{h}^{-1}(\gamma'), \lambda) = tord(\gamma', \lambda')$, we have $tord(\tilde{h}^{-1}(\gamma'), \gamma') = tord(\tilde{h}^{-1}(\gamma'), \gamma) < \alpha$. Since T is elementary with respect to f, we have $tord(\tilde{h}^{-1}(\gamma'), \gamma') \geq tord(\lambda, \lambda') \geq \alpha$, a contradiction with condition $tord(\tilde{h}^{-1}(\gamma'), \gamma) < \alpha$. Thus $tord(\gamma, T') = tord(\gamma, \check{T})$. $\square$

Theorem 3.20. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be normally embedded β-Hölder triangles satisfying (5). Let $f(x) = dist(x, T')$ for $x \in T$ and $g(x') = dist(x', T)$ for $x' \in T'$. If T is elementary with respect to f then T and T' satisfy conditions of Proposition 3.2. Moreover, $\mu_{T,f} \equiv \mu_{T',g}$, where $\mu_{T,f}(q)$ and $\mu_{T',g}(q)$ are the width functions defined on $Q_f(T) = Q_g(T')$.*

Proof. Since the triangles are elementary, we may assume that $tord(\gamma_1, \gamma'_1) = tord(T, T')$. Let $\{T_i\}_{i=1}^p$ be a minimal pizza decomposition of T associated with f, where each pizza slice $T_i = T(\lambda_{i-1}, \lambda_i)$ is a β_i-Hölder triangle, $\lambda_0 = \gamma_1$ and $\lambda_p = \gamma_2$. We proceed by induction on the number p of pizza slices. The case $p = 1$ follows from Lemma 3.15. If $p > 1$ then $tord(\lambda_1, T') > \beta$, otherwise $\{T_i\}$ would not be a minimal pizza decomposition. It follows from Lemma 3.18 applied to T_1 that there is an arc $\theta_1 \subset T'$ such that $tord(\gamma'_1, \theta) = \beta_1$ and conditions of Proposition 3.2 are satisfied for T_1 and $T'_1 = T(\gamma'_1, \theta_1)$.

Let $\check{T} = T(\lambda_1, \gamma_2)$ and $\check{T}' = T(\theta_1, \gamma'_2)$. Since $tord(\lambda_1, T') > \beta$, $\check{T}$ and $\check{T}'$ have the same exponents (see Remark 3.4). The same arguments as in the proof of Proposition 3.19 show that $tord(\gamma, \check{T}') = tord(\gamma, T')$ for any arc $\gamma \subset \check{T}$, thus $\{T_i\}_{i=2}^p$ is a minimal pizza decomposition of $\check{T}$ associated with the function $\check{f}(x) = dist(x, \check{T}')$. By the inductional hypothesis, Hölder triangles $\check{T}$ and $\check{T}'$ satisfy conditions of Proposition 3.2. Proposition 3.19 implies that $T = T_1 \cup \check{T}$ and $T' = T'_1 \cup \check{T}'$ satisfy conditions of Proposition 3.2. The existence of a mapping $h : T \to T'$ satisfying condition 5 of Proposition 3.2 implies that $Q_f(T) = Q_g(T')$ and $\mu_{T,f} \equiv \mu_{T',g}$. □

4. The $\sigma\tau$-pizza invariant.

If two normally embedded Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ satisfying condition (5) are not elementary with respect to the distance functions, then $T \cup T'$ may be not outer bi-Lipschitz equivalent to the union of T and a graph of a function defined on T (see Fig. 4). In any case, a minimal pizza on T associated with the function $f(x) = dist(x, T')$, and a minimal pizza on T' associated with the function $g(x') = dist(x', T)$, are outer Lipschitz invariants of the pair (T, T'). The following example shows that two pairs (T, T') and $(\tilde{T}, \tilde{T}')$ of normally embedded triangles satisfying condition (5) may be not outer bi-Lipschitz equivalent even when the minimal pizzas on T and T' are equivalent to the minimal pizzas on $\tilde{T}$ and $\tilde{T}'$ respectively.

Example 4.1. The links of two normally embedded Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ are shown in Fig. 5. Triangle T is partitioned by the arcs λ_1, λ_2, λ_3, λ_4 into Hölder triangles $T_1 = T(\gamma_1, \lambda_1)$, $T_2 = T(\lambda_1, \lambda_2)$, $T_3 = T(\lambda_2, \lambda_3)$, $T_4 = T(\lambda_3, \lambda_4)$, $T_5 = T(\lambda_4, \gamma_2)$ with exponents μ_2, q_2, μ_1, q_2, μ_2, respectively, and triangle T' is partitioned by the arcs λ'_1, λ'_2, λ'_3, λ'_4 into Hölder triangles $T'_1 = T(\gamma'_1, \lambda'_1)$, $T'_2 = T(\lambda'_1, \lambda'_2)$, $T'_3 = T(\lambda'_2, \lambda'_3)$, $T'_4 = T(\lambda'_3, \lambda'_4)$, $T'_5 = T(\lambda'_4, \gamma'_2)$ with exponents μ_2, q_2, μ_1, q_2, μ_2, respectively, so that the following holds:

$tord(\gamma, T') = q_2$ for any arc $\gamma \subset T_1$, $tord(\gamma, T') = q_1$ for any arc $\gamma \subset T_3$, $tord(\gamma, T') = q_2$ for any arc $\gamma \subset T_5$; $tord(\gamma, T') = tord(\gamma, \lambda'_2)$ for any arc $\gamma \subset T_2$, $tord(\gamma, T') = tord(\gamma, \lambda'_3)$ for any arc $\gamma \subset T_4$; $tord(\gamma', T) = q_2$ for any arc $\gamma' \subset T'_1$, $tord(\gamma', T) = q_1$ for any arc $\gamma' \subset T'_3$, $tord(\gamma', T) = q_2$ for any arc $\gamma' \subset T'_5$; $tord(\gamma', T) = tord(\gamma', \lambda_2)$ for any arc $\gamma' \subset T'_2$, $tord(\gamma', T) = tord(\gamma', \lambda_3)$ for any arc $\gamma' \subset T'_4$.

In particular, T and T' satisfy condition (5):

$$tord(\gamma_1, T') = tord(\gamma_1, \gamma'_1) = tord(\gamma'_1, T) = tord(\gamma_2, T') = tord(\gamma_2, \gamma'_2) = tord(\gamma'_2, T) = q_2.$$

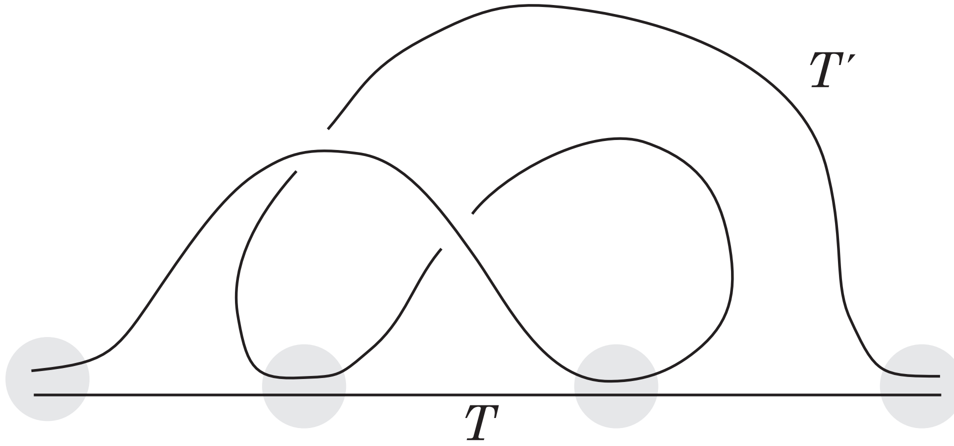

FIGURE 4. Two normally embedded β-Hölder triangles, not elementary with respect to the distance functions. Shaded disks indicate zones with the tangency order higher than β.

Assuming $q_1 > \mu_1 \geq q_2 > \mu_2$, the arcs $\lambda_1, \ldots, \lambda_4$ define a minimal pizza decomposition of T associated with the function $f(x) = dist(x, T')$.

Although T is not elementary with respect to $f(x)$, the union $T \cup T'$ is outer bi-Lipschitz equivalent to the union of T and the graph of $f(x)$. In particular, a minimal pizza on T' associated with the function $g(x') = dist(x', T)$ is equivalent to a minimal pizza on T associated with the function $f(x)$.

Hölder triangles $\tilde{T}$ and $\tilde{T}'$ in Fig. 6 are also normally embedded and satisfy condition (5):

$$tord(\gamma_1, \tilde{T}') = tord(\gamma_1, \gamma_1') = tord(\gamma_1', \tilde{T}) = tord(\gamma_2, \tilde{T}') = tord(\gamma_2, \gamma_2') = tord(\gamma_2', \tilde{T}) = q_2.$$

Triangle $\tilde{T}$ is partitioned by the arcs λ_1, λ_2, λ_3, λ_4 into Hölder triangles $\tilde{T}_1 = T(\gamma_1, \lambda_1)$, $\tilde{T}_2 = T(\lambda_1, \lambda_2)$, $\tilde{T}_3 = T(\lambda_2, \lambda_3)$, $\tilde{T}_4 = T(\lambda_3, \lambda_4)$, $\tilde{T}_5 = T(\lambda_4, \gamma_2)$, and triangle $\tilde{T}'$ is partitioned by the arcs λ_1', λ_2', λ_3', λ_4' into Hölder triangles $\tilde{T}_1' = T(\gamma_1', \lambda_1')$, $\tilde{T}_2' = T(\lambda_1', \lambda_2')$, $\tilde{T}_3' = T(\lambda_2', \lambda_3')$, $\tilde{T}_4' = T(\lambda_3', \lambda_4')$, $\tilde{T}_5' = T(\lambda_4', \gamma_2')$. Conditions satisfied by these triangles are the same as for those in Fig. 5, except $tord(\gamma, \tilde{T}') = tord(\gamma, \lambda_3')$ for any arc $\gamma \subset \tilde{T}_2$, $tord(\gamma, \tilde{T}') = tord(\gamma, \lambda_2')$ for any arc $\gamma \subset \tilde{T}_4$; $tord(\gamma', \tilde{T}) = tord(\gamma', \lambda_3)$ for any arc $\gamma' \subset \tilde{T}_2'$, $tord(\gamma', \tilde{T}) = tord(\gamma', \lambda_2)$ for any arc $\gamma' \subset \tilde{T}_4'$. Assuming $q_1 > \mu_1 \geq q_2 > \mu_2$, the arcs $\lambda_1, \ldots, \lambda_4$ define a minimal pizza decomposition of $\tilde{T}$ associated with the function $\tilde{f}(x) = dist(x, \tilde{T}')$.

One can show that minimal pizzas on T and $\tilde{T}$ are equivalent, and a minimal pizza on $\tilde{T}'$ associated with the function $\tilde{g}(x') = dist(x', \tilde{T})$ is equivalent to a minimal pizza on $\tilde{T}$ associated with the function $\tilde{f}(x)$. Thus minimal pizzas on T' and $\tilde{T}'$ are also equivalent.

However, the union $\tilde{T} \cup \tilde{T}'$ is not outer bi-Lipschitz equivalent to the union of $\tilde{T}$ and the graph of $\tilde{f}(x)$. In particular, $\tilde{T} \cup \tilde{T}'$ is not outer bi-Lipschitz equivalent to $T \cup T'$.

Definition 4.2. Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$ be normally embedded β-Hölder triangles, oriented from γ_1 to γ_2 and from γ_1' to γ_2' respectively, satisfying condition (5). Let $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ be the distance functions defined on T and T' respectively. Let $D_\ell \subset V(T)$, for $\ell = 0, \ldots, p$, be the pizza zones of a minimal pizza on T associated with $f(x)$, ordered according to the orientation of T, and let $q_\ell =$

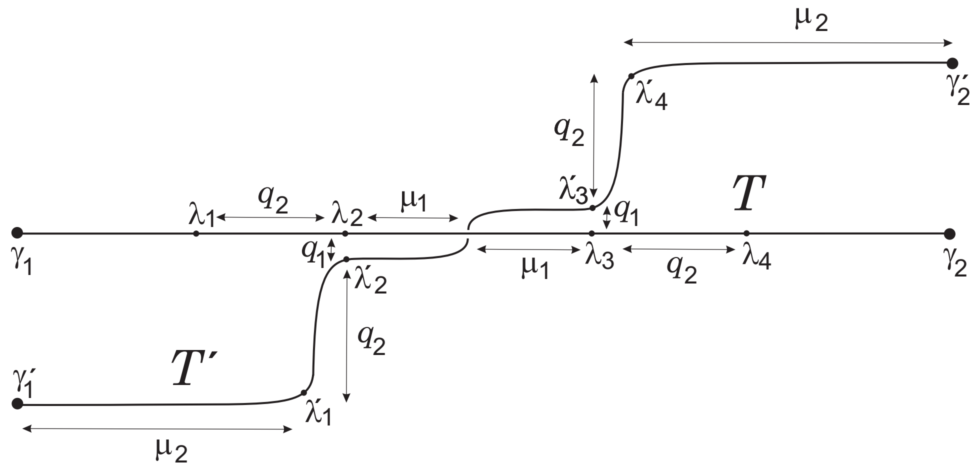

FIGURE 5. Two normally embedded Hölder triangles T and T' in Example 4.1.

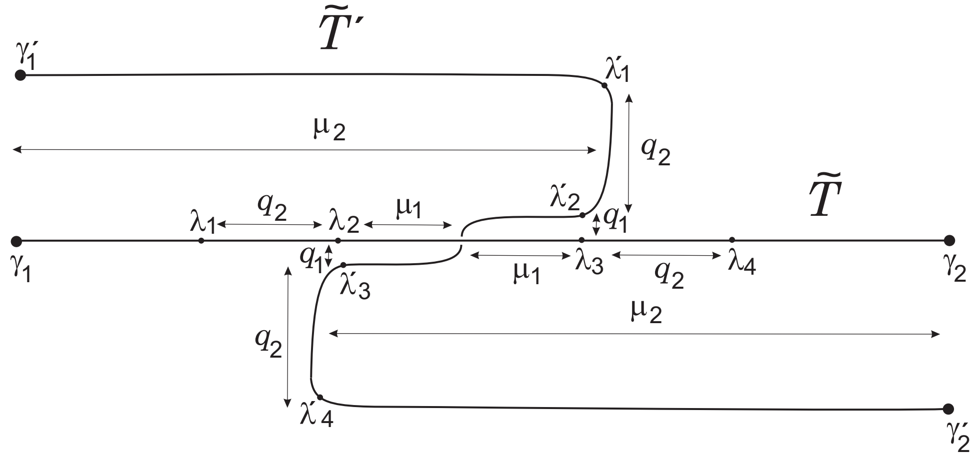

FIGURE 6. Two normally embedded Hölder triangles $\tilde{T}$ and $\tilde{T}'$ in Example 4.1.

$tord(D_\ell, T') = ord_\gamma f$ for any arc $\gamma \in D_\ell$. A zone D_ℓ is called a *maximal exponent zone* for $f(x)$ (or simply a *maximum zone*) if either $0 < \ell < p$ and $q_\ell \geq \max(q_{\ell-1}, q_{\ell+1})$, or $\ell = 0$ and $\beta < q_0 \geq q_1$, or $\ell = p$ and $\beta < q_p \geq q_{p-1}$. If a zone D_ℓ is not a maximum zone, it is called a *minimal exponent zone* for $f(x)$ (or simply a *minimum zone*) if either $0 < \ell < p$ and $q_\ell \leq \min(q_{\ell-1}, q_{\ell+1})$, or $\ell = 0$ and $q_0 \leq q_1$, or $\ell = p$ and $q_p \leq q_{p-1}$. Maximum and minimum pizza zones $D'_{\ell'} \subset V(T')$ for a minimal pizza on T' associated with $g(x')$ are defined similarly, exchanging T and T'.

Remark 4.3. Each of the singular pizza zones $D_0 = \{\gamma_1\}$ and $D_p = \{\gamma_2\}$ is either a maximum or a minimum zone. When $p = 1$ and $q_0 = q_1 > \beta$, both D_0 and D_1 are maximum zones. When $p = 1$ and $q_0 = q_1 \leq \beta$, both D_0 and D_1 are minimum zones. If $p > 1$ and $0 < \ell < p$, then $q_\ell > \min(q_{\ell-1}, q_{\ell+1})$ if D_ℓ is a maximum zone, $q_\ell < \max(q_{\ell-1}, q_{\ell+1})$ if D_ℓ is a minimum zone.

Proposition 4.4. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, satisfying condition (5). Let $\{M_i\}_{i=1}^m$ and $\{M'_{i'}\}_{i'=1}^{m'}$ be the maximum zones in $V(T)$ and $V(T')$ for the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ respectively, ordered according to the orientations of T and T'. Let $\bar{q}_i = tord(M_i, T')$ and $\bar{q}'_{i'} = tord(M'_{i'}, T)$. Then $m' = m$, and* *there is a canonical one-to-one correspondence* $i' = \sigma(i)$ *between the zones* M_i *and* $M'_{i'}$, *such that* $\mu(M'_{i'}) = \mu(M_i)$ *and* $tord(M_i, M'_{i'}) = \bar{q}_i = \bar{q}'_{i'}$. *If* $\{\gamma_1\} = M_1$ *is a maximum zone then* $M'_1 = \{\gamma'_1\}$ *and* $\sigma(1) = 1$. *If* $\{\gamma_2\} = M_m$ *is a maximum zone then* $M'_m = \{\gamma'_2\}$ *and* $\sigma(m) = m$.

Proof. The case $p = 1$ follows from Lemma 3.15, thus we may assume $p > 1$.

Let us choose any arcs $\lambda_i \in D_i$, so that $\{T_i = T(\lambda_{i-1}, \lambda_i)\}$ is a minimal pizza on T associated with the function $f(x)$. Let $q_i = ord_{\lambda_i} f$ for $i = 0, \ldots, p$, and let $\beta_i = tord(\lambda_{i-1}, \lambda_i)$ be the exponent of a pizza slice T_i, for $i = 1, \ldots, p$.

Consider first the case when $\{\gamma_1\} = \{\lambda_0\}$ is a maximum zone for $f(x)$. Then $q_0 = tord(\gamma_1, \gamma'_1)$ and $q_1 = ord_{\lambda_1} f \le q_0$. If $q_1 > \beta_1$, it follows from Lemma 3.15 that, for any arc $\lambda' \subset T'$ such that $tord(\lambda_1, \lambda') = q_1$, Hölder triangles T_1 and $T'_1 = T(\gamma'_1, \lambda')$ satisfy (5) and conditions of Proposition 3.2. If γ'_1 is not a maximum zone for $g(x')$ then, for any arc λ'_1 such that $T'_1 = T(\gamma'_1, \lambda'_1)$ is a pizza slice for a minimal pizza associated with $g(x')$, we have $q'_1 = tord(\lambda'_1, T) > q_0$. Let $\lambda \subset T$ be any arc such that $tord(\lambda, \lambda'_1) = q'_1$. Then Lemma 3.15 applied to T'_1 and $\bar{T} = T(\gamma_1, \lambda)$ implies that $\bar{T}$ is a pizza slice for $f(x)$. Since $tord(\gamma, T') \le q_0$ for any arc $\gamma \subset T_1$, we have $T_1 \subset \bar{T}$, a contradiction with T_1 being a pizza slice for a minimal pizza associated with $f(x)$.

Similarly, if $\{\gamma_2\}$ is a maximum zone for $f(x)$ then $\{\gamma'_2\}$ is a maximum zone for $g(x')$.

Suppose next that $p > 1$ and M_i is a maximum zone for $f(x)$, where $0 < i < p$. Let $\lambda' \subset T'$ be any arc such that $tord(\lambda_i, \lambda') = q_i$. We are going to show that λ' belongs to a maximum zone D' for a minimal pizza on T' associated with $g(x')$.

Note first that, if λ' belongs to a pizza zone D' for a minimal pizza on T' associated with $g(x')$, the same arguments as those for γ_1 and γ'_1 show that D' is a maximum zone for $g(x')$.

Suppose that λ' does not belong to a pizza zone. Let D'_{j-1} and D'_j be two adjacent pizza zones for a minimal pizza on T' associated with $g(x')$, $\lambda'_{j-1} \in D'_{j-1}$ and $\lambda'_j \in D'_j$, $\lambda' \subset T'_j = T(\lambda'_{j-1}, \lambda'_j)$ and $\lambda' \notin D'_{j-1} \cup D'_j$. The same arguments as those for γ_1 and γ'_1 show that $tord(\lambda'_{j-1}, T) \le q_i$ and $tord(\lambda'_j, T) \le q_i$. Since T'_j is a pizza slice, at least one of these inequalities is an equality, say $tord(\lambda'_j, T) = q_i$. Let $\mu'_j = \nu(\lambda'_j) \le q_i$ be the order of the pizza zone D'_j. Since any arc $\gamma' \subset T'_j$ such that $tord(\gamma', \lambda'_j) \ge \nu(\lambda'_j)$ belongs to D'_j and $\lambda' \notin D'_j$, we have $tord(\lambda', \lambda'_j) < \mu'_j \le q_i$. Thus $\bar{T}' = T(\lambda', \lambda'_j)$ is a $\bar{\beta}$-Hölder triangle, where $\bar{\beta} < q_i$, such that $tord(\gamma', T) = q_i$ for any arc $\gamma' \subset \bar{T}'$. □

Definition 4.5. Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, satisfying condition (5). Let $\{M_i\}_{i=1}^m$ and $\{M'_{i'}\}_{i'=1}^{m'}$ be the maximum zones in $V(T)$ and $V(T')$ for the functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ respectively, ordered according to the orientations of T and T'. According to Proposition 4.4, we have $m' = m$, and there is a canonical permutation σ of the set $\{1, \ldots, m\}$, the *characteristic permutation* of the pair T and T', such that $tord(M_i, M'_{\sigma(i)}) = tord(M_i, T') = tord(M'_{\sigma(i)}, T)$.

Definition 4.6. Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, satisfying condition (5). Let $D_\ell \subset V(T)$, for $\ell = 0, \ldots, p$, be the pizza zones of a minimal pizza $\{T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)\}$ on T associated with the distance function $f(x) = dist(x, T')$, ordered according to the orientation of T. For $\ell = 1, \ldots, p$, let $Y_\ell = D_{\ell-1} \cup D_\ell \cup V(T_\ell)$ be the maximal pizza slice zones in $V(T)$ associated with f (see Corollary 2.31). Let $Q_\ell = Q_f(Y_\ell) = [q_{\ell-1}, q_\ell]$, where $q_\ell = ord_{\lambda_\ell} f$, and $\mu_\ell = \mu_{Y_\ell,f} : Q_\ell \to \mathbb{F}\cup\{\infty\}$ be the corresponding exponent intervals and affine width functions (see Definition 2.12). We say that a zone Y_ℓ, and a pizza slice $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$ where $\lambda_{\ell-1} \in D_{\ell-1}$ and $\lambda_\ell \in D_\ell$, is *transversal* if $\mu_\ell(q) \equiv q$, and *non-transversal* otherwise.

Proposition 4.7. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, satisfying condition (5). Let D_ℓ, Y_ℓ, $Q_\ell = [q_{\ell-1}, q_\ell]$ and μ_ℓ be as in Definition 4.6. Let $D'_{\ell'}$, for $\ell' = 0,\ldots,p'$, be the pizza zones of a minimal pizza on T' associated with $g(x') = dist(x', T)$, ordered according to the orientation of T'. Let $Y'_{\ell'} \subset V(T')$, $Q'_{\ell'} = Q_g(Y'_{\ell'}) = [q_{\ell'-1}, q_{\ell'}] \subset \mathbb{F}\cup\{\infty\}$ and $\mu'_{\ell'} : Q'_{\ell'} \to \mathbb{F}\cup\{\infty\}$ be the corresponding maximal pizza slice zones, exponent intervals and affine width functions. Then, for each index ℓ such that the pizza slice zone Y_ℓ is non-transversal, there is a unique index $\ell' = \tau(\ell)$ such that $Q'_{\ell'} = Q_\ell$, $\mu'_{\ell'} \equiv \mu_\ell$ and one of the following two conditions holds:*

$$(7)\quad tord(D_\ell, T') = tord(D_\ell, D'_{\ell'}) = tord(D'_{\ell'}, T),\qquad tord(D_{\ell-1}, T') = tord(D_{\ell-1}, D'_{\ell'-1}) = tord(D'_{\ell'-1}, T);$$

$$(8)\quad tord(D_\ell, T') = tord(D_\ell, D'_{\ell'-1}) = tord(D'_{\ell'-1}, T),\qquad tord(D_{\ell-1}, T') = tord(D_{\ell-1}, D'_{\ell'}) = tord(D'_{\ell'}, T).$$

Proof. Let $Y_\ell \subset V(T)$ be a non-transversal maximal pizza slice zone for a minimal pizza associated with f. For each $q \in Q_\ell$ let $Z_q \subset Y_\ell$ be the maximal q-order zone for f. If $Q_\ell = \{q_\ell\}$ is a point, then $q_\ell > \mu_\ell$ since Y_ℓ is non-transversal. It follows from Proposition 3.9 and Corollary 3.10 that there is a unique maximal q_ℓ-order zone $Z' \subset V(T')$ for g, of order μ_ℓ, containing all arcs $\gamma' \subset V(T')$ such that $tord(\gamma', Y_\ell) = ord_{\gamma'} g = q_\ell$. Let us show that Z' is a maximal pizza slice zone for g. Let $Z' \subset Y'_{\ell'}$ where $Y'_{\ell'}$ is a maximal pizza slice zone for a minimal pizza associated with g, of order $\mu'_{\ell'}$. If $Z' \neq Y'_{\ell'}$ then either $Q'_{\ell'} = \{q_\ell\}$ is a point but $\mu_\ell > \mu'_{\ell'}$ or $Q'_{\ell'}$ is not a point.

In the first case, there is a β'-Hölder triangle $\tilde T' = T(\tilde\gamma'_1, \tilde\gamma'_2)$ such that $\beta' < \mu_\ell$, $V(\tilde T') \subset Y'_{\ell'}$ and $Z' \cap V(T')$ is a μ_ℓ-zone. Let $\tilde T = T(\tilde\gamma_1, \tilde\gamma_2) \subset T$ be a β'-Hölder triangle, where $\tilde\gamma_1$ and $\tilde\gamma_2$ are two arcs in T such that $tord(\tilde\gamma_1, \tilde\gamma'_1) = tord(\tilde\gamma_2, \tilde\gamma'_2) = q_\ell$. Since $Y'_{\ell'}$ is a pizza slice zone, $\tilde T'$ is elementary with respect to g, and the pair $(\tilde T, \tilde T')$ satisfies (5). It follows from Theorem 3.20 applied to $\tilde T'$ that $\tilde T$ is elementary with respect to f, $Q_f(\tilde T) = \{q_\ell\}$ and $Z \cap V(\tilde T)$ is a μ_ℓ-zone. Since $\beta' < \mu_\ell$, this contradicts the assumption that Y_ℓ is a minimal pizza slice zone.

The arguments for the second case, when Q_ℓ is a point but $Q'_{\ell'}$ is not a point, are similar: one can find a Hölder triangle $\tilde T' \subset T'$ such that $Q_g(\tilde T') = Q'_{\ell'}$ is not a point, $\tilde T'$ is elementary with respect to g, and $V(\tilde T') \cap Z'$ is a μ_ℓ-zone. Then there is a Hölder triangle $\tilde T \subset T$ such that the pair $(\tilde T, \tilde T')$ satisfies (5). Theorem 3.20 applied to $\tilde T'$ implies that $Q_f(\tilde T)$ is not a point, while the width function of $\tilde T$ is affine, a contradiction with the assumption that Y_ℓ is a minimal pizza slice zone.

Suppose now that $Q_\ell = [q_{\ell-1}, q_\ell]$ is not a point. Then Proposition 3.9 and Corollary 3.10 applies to each q-order zone $Z_q \subset Y_\ell$ for f when $q \in \dot Q_\ell = (q_{\ell-1}, q_\ell)$, but may be not applicable when $q = q_{\ell-1}$ or $q = q_\ell$ if $\mu_\ell(q) = q$. For $q \in \dot Q_\ell$, let $Z'_q \subset V(T')$ be the q-order zone for g, of order $\mu_\ell(q)$, corresponding to Z_q. Then $Z = \bigcup_{q\in\dot Q_\ell} Z_q$ is a pizza slice zone for f, and $Z' = \bigcup_{q\in\dot Q_\ell} Z'_q$ is a pizza slice zone for g. Let $\subset Y'_{\ell'} \supset Z'$ be the maximal pizza slice zone for a minimal pizza associated with g, of order $\mu'_{\ell'}$. The same arguments as above show that $Q'_{\ell'} = Q_\ell$ and $\mu'_{\ell'} \equiv \mu_\ell$.

Note that the pairs of zones (Z_q, Z'_q) are either all positively oriented or all negatively oriented (see Definition 3.12). Accordingly, either (7) or (8) holds for the pairs of maximal pizza slice zones $(Y_\ell, Y'_{\ell'})$. □

Definition 4.8. Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, satisfying condition (5). Let $\{T_\ell\}$ and $\{T'_{\ell'}\}$ be minimal pizzas on T and T' for the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ respectively, ordered according to the orientations of T and T'. Then, according to Proposition 4.7, there is a canonical one-to-one correspondence $\ell' = \tau(\ell)$ between the sets of non-transversal pizza slices T_ℓ for a minimal pizza on T associated with $f(x) = dist(x, T')$, ordered according to the orientation of T, and the set of non-transversal pizza slices $T'_{\ell'}$ for a minimal pizza on T' associated with $g(x') = dist(x', T)$, ordered according to the orientation of T'. This defines a *characteristic correspondence* τ between the sets of non-transversal pizza slices of T and T'. In particular, these two sets have the same number of elements. We say that a pair of non-transversal pizza slice zones Y_ℓ and $Y'_{\ell'}$ where $\ell' = \tau(\ell)$, and a pair of non-transversal pizza slices T_ℓ and $T'_{\ell'}$, is *positively oriented* if (7) holds and *negatively oriented* otherwise (see Definition 3.16). Thus τ is a *signed* correspondence, with the signs $+$ and $-$ assigned to the positively and negatively oriented pairs of non-transversal pizza slice zones.

Remark 4.9. For each pair $(T_\ell, T'_{\ell'})$ of non-transversal pizza slices, where $\ell' = \tau(\ell)$, the signed correspondence τ defines a correspondence between the two pizza zones $D_{\ell-1}$ and D_ℓ and the two pizza zones $D'_{\ell'-1}$ and $D'_{\ell'}$, in the same (resp., opposite) order if the pair is positively (resp., negatively) oriented. This correspondence between a subset of pizza zones of T and a subset of pizza zones of T' may be not one-to-one: a pizza zone of T common to two non-transversal pizza slices may correspond to two different pizza zones of T', and two different pizza zones of T may correspond to the same pizza zone of T' (see Fig. 7). However, it is one-to-one on the set of those pizza zones which are also maximum zones (see Proposition 4.10).

Proposition 4.10. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be normally embedded Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, satisfying condition (5). Let $\{T_\ell\}$ and $\{T'_{\ell'}\}$ be minimal pizzas on T and T' for the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ respectively, ordered according to the orientations of T and T'. Let $(T_\ell, T'_{\ell'})$, where $\ell' = \tau(\ell)$, be a pair of non-transversal pizza slices such that one of the pizza zones of T_ℓ, say $D = D_\ell$, is a maximum zone $M_i \subset V(T)$. Then the corresponding pizza zone D' of $T'_{\ell'}$ (either $D' = D'_{\ell'}$ for a positively oriented pair $(T_\ell, T'_{\ell'})$ or $D' = D'_{\ell'-1}$ for a negatively oriented pair) is a maximum zone $M'_{i'} \subset V(T')$, where $i' = \sigma(i)$.*

Proof. If D is a boundary arc of T then the statement follows from Proposition 4.4, since D' is also a boundary arc of T' and a (singular) maximum zone.

If $D = D_\ell$ is not a boundary arc, and both maximal pizza slice zones Y_ℓ and $Y_{\ell+1}$ containing D_ℓ are non-transversal, then Proposition 4.7 implies that the corresponding zones in $V(T')$ are either $Y'_{\ell'}$ and $Y'_{\ell'+1}$ (if $(Y_\ell, Y'_{\ell'})$ is a positively oriented pair) or $Y'_{\ell'}$ and $Y'_{\ell'-1}$ (if $(Y_\ell, Y'_{\ell'})$ is a negatively oriented pair). In both cases, Proposition 4.7 implies

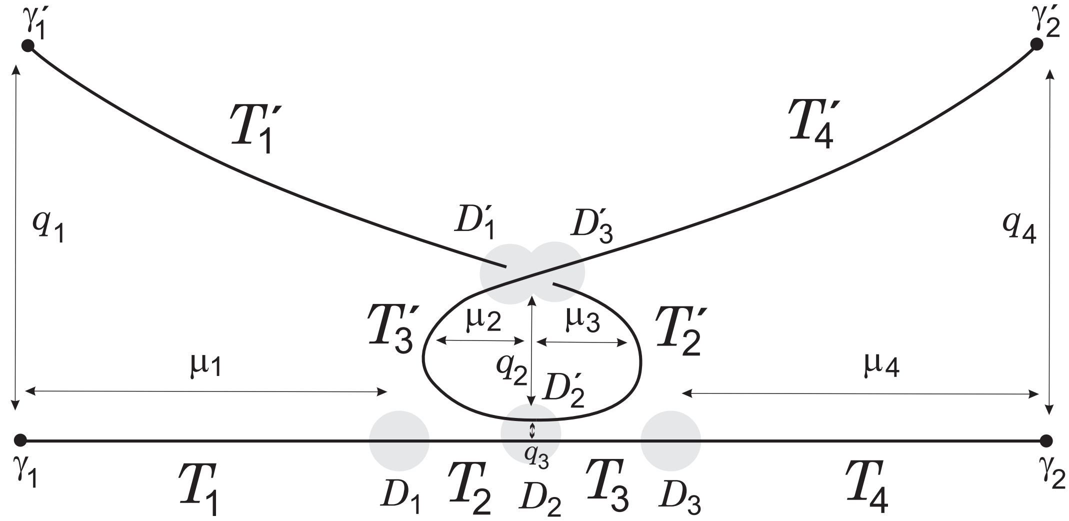

FIGURE 7. Two normally embedded Hölder triangles in Remark 4.9. Shaded disks indicate pizza zones of minimal pizzas on T and T'. Assuming $q_1 = q_4 > \mu_1 = \mu_4$ and $q_2 = \mu_2 = \mu_3 < q_3$, there are four non-transversal pairs of pizza slices: (T_1, T'_1), (T_2, T'_3), (T_3, T'_2), (T_4, T'_4). The correspondence $\tau(1)$ maps D_1 to D'_1, while $\tau(2)$ maps D_1 to D'_3.

that D' is a maximum zone such that $tord(D, D') = tord(D, T') = tord(D', T) = q_\ell$, thus $i' = \sigma(i)$.

Otherwise, if Y_ℓ is a non-transversal zone but $Y_{\ell+1}$ is transversal, Lemma 3.18 implies that there are two $\tilde\beta$-Hölder triangles $\tilde T = T(\tilde\gamma_1, \tilde\gamma_2) \subset T_{\ell+1}$ and $\tilde T' = T(\tilde\gamma'_1, \tilde\gamma'_2) \subset T'$ satisfying (5), where $\tilde\beta < q_\ell$, such that $\tilde T' \cap T'_\ell = \{\tilde\gamma'_1\}$, $\tilde\gamma_1 \in D$, $\tilde\gamma'_1 \in D'$ and $tord(\tilde\gamma_1, \tilde\gamma'_1) = q_\ell$. Theorem 3.20 applied to $\tilde T$ and $\tilde T'$ implies that D' is a maximum zone such that $tord(D, D') = tord(D, T') = tord(D', T) = q_\ell$, thus $i' = \sigma(i)$. □

Proposition 4.11. *Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, satisfying condition (5). Then the sign assigned to each pair $(T_\ell, T'_{\ell'})$ of non-transversal pizza slices such that $\ell' = \tau(\ell)$ is completely determined by the minimal pizzas $\{T_\ell\}$ and $\{T'_{\ell'}\}$, the characteristic permutation σ and the characteristic correspondence τ.*

Proof. Let $Y_\ell \subset V(T)$ and $Y'_{\ell'} \subset V(T')$ be two non-transversal pizza slice zones such that $\ell' = \tau(\ell)$. According to Definition 4.8, the pair $(Y_\ell, Y'_{\ell'})$ is positively oriented if (7) holds and negatively oriented if (8) holds. If $Q_\ell = [q_\ell, q_{\ell+1}]$ is not a point then $q_\ell \neq q_{\ell+1}$, thus $Q'_{\ell'} = [q'_{\ell'}, q'_{\ell'+1}]$ is also not a point, and the pair is positive when $q_\ell = q'_{\ell'}$ and negative otherwise. If $Q_\ell = \{q_\ell\}$ is a point then $\mu_\ell < q_\ell$, and each of the pizza zones D_ℓ and $D_{\ell+1}$ is either a maximum or a minimum zone. If, say, $D_\ell = M_i$ is a maximum zone, then the pair is positive when $D'_{\ell'} = M_{\sigma(i)}$ and negative otherwise.

The case when each of them contains a boundary arc is trivial, so we may assume that they are interior zones. If $Q_\ell = Q_{\ell'}$ is not a point then the sign is uniquely determined by the maxima of non-constant affine functions $\mu_\ell(q) \equiv \mu'_{\ell'}(q)$. If $Q_\ell = Q'_{\ell'} = \{q_\ell\} > \mu_\ell$ is a point, then the pizza zones $D_{\ell-1}$ and D_ℓ correspond to the pizza zones $D'_{\ell'-1}$ and $D'_{\ell'}$ in the same order if the sign is positive, and in the opposite order if the sign is negative. Note that each of these zones is either a maximum or a minimum zone, since on one side of each of them q is constant. The correspondence between the pizza slice zones sends a maximum pizza zone in $V(T)$ to a maximum pizza zone in $V(T')$, and a minimum pizza zone in $V(T)$ to a minimum pizza zone in $V(T')$. If one of the pizza zones in $V(T)$ is a maximum zone and another is a minimum zone, then the same is true for the pizza zones in $V(T')$, and the correspondence is uniquely defined. If both pizza zones are maximum zones then the correspondence is defined by σ. If both pizza zones in $V(T)$ are minimum zones, since $q_\ell > \mu_\ell$, the two maximum pizza zones in $V(T)$ closest to Y_ℓ are mapped by σ to the two maximum pizza zones in $V(T')$ closest to $Y'_{\ell'}$ in the same order if the sign is positive and in the opposite order if the sign is negative: one can only get from one side of Y_ℓ to another side through a part of T' where $q \le \mu_\ell$. □

Definition 4.12. Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ be two normally embedded Hölder triangles, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, satisfying condition (5). Let $\{T_\ell\}$ and $\{T'_{\ell'}\}$ be minimal pizzas on T and T' for the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ respectively, ordered according to the orientations of T and T'. A *$\sigma\tau$-pizza* on $T \cup T'$ is a triplet consisting of the pair of minimal pizzas $\{T_\ell\}$ and $\{T'_{\ell'}\}$, the characteristic permutation σ of the maximum pizza zones in $V(T)$ and $V(T')$, and the characteristic correspondence τ of the non-transversal pizza slices of T and T'. Two $\sigma\tau$-pizzas $(\{T_\ell\}, \{T'_{\ell'}\}, \sigma_T, \tau_T)$ on $T \cup T'$ and $(\{S_\ell\}, \{S'_{\ell'}\}, \sigma_S, \tau_S)$ on $S \cup S'$ are *combinatorially equivalent* if the pairs $(\{T_\ell\}, \{T'_{\ell'}\})$ and $(\{S_\ell\}, \{S'_{\ell'}\})$ are combinatorially equivalent, $\sigma_T = \sigma_S$ and $\tau_T = \tau_S$.

Theorem 4.13. *Let (T, T') and (S, S') be two oriented pairs of normally embedded Hölder triangles satisfying condition (5). If there is an orientation-preserving outer bi-Lipschitz homeomorphism $H : T \cup T' \to S \cup S'$ such that $H(T) = S$ and $H(T') = S'$, then the $\sigma\tau$-pizzas of the pairs (T, T') and (S, S') are combinatorially equivalent.*

Proof. Let $f_T(x) = dist(x, T')$, $g_T(x') = dist(x', T)$, $f_S(y) = dist(y, S')$ and $g_S(y') = dist(y', S)$ be the distance functions defined on T, T', S and S' respectively. Let $M_i \subset V(T)$ and $M'_{i'} \subset V(T')$ be the maximum zones for the pair (T, T'), and let $N_i \subset V(S)$ and $N'_{i'} \subset V(S')$ be the maximum zones for the pair (S, S').

Since H is an outer bi-Lipschitz homeomorphism, f_T is Lipschitz contact equivalent to f_S, and g_T is Lipschitz contact equivalent to g_S. Theorem 2.19 implies that the corresponding pairs of minimal pizzas $(\{T_\ell\}, \{T'_{\ell'}\})$ and $(\{S_\ell\}, \{S'_{\ell'}\})$ are combinatorially equivalent. Accordingly, H maps each maximum zone M_i to the maximum zone N_i, and each maximum zone $M'_{i'}$ to the maximum zone $N'_{i'}$. A pair of maximum zones $(M_i, M'_{i'})$, where $i' = \sigma_T(i)$, is mapped to the pair of maximum zones $(N_i, N'_{i'})$ preserving the order of contact between these zones. This implies that $i' = \sigma_S(i)$, thus the permutations σ_T and σ_S are equal.

Moreover, since H preserves the tangency orders between arcs, it maps each maximal pizza slice zone Y_ℓ of a minimal pizza on T associated with f_T to the maximal pizza slice zone Z_ℓ of a minimal pizza on S associated with f_S, and each maximal pizza slice zone $Y'_{\ell'}$ of a minimal pizza on T' associated with g_T to the maximal pizza slice zone $Z'_{\ell'}$ of a minimal pizza on S' associated with g_S, with the corresponding width functions preserved. Accordingly, if $(T_\ell, T'_{\ell'})$ is a non-transversal pair of pizza slices of minimal pizzas associated with f_T and g_T, where $\ell' = \tau_T(\ell)$, then $(H(T_\ell), H(T'_{\ell'}))$ is a non-transversal pair of pizza slices of minimal pizzas associated with f_S and g_S, such that $V(H(T_\ell)) \subset Z_\ell$ and $V(H(T'_{\ell'})) \subset Z'_{\ell'}$. This implies that $\ell' = \tau_S(\ell)$, thus the correspondences τ_T and τ_S are equal. Proposition 4.11 implies that τ_T and τ_S are equal also as signed correspondences. ☐

The following conjecture states that, conversely, two pairs of normally embedded Hölder triangles satisfying condition (5) with the same $\sigma\tau$-pizza invariant are outer bi-Lipschitz equivalent, thus the $\sigma\tau$-pizza is a complete combinatorial invariant of an outer bi-Lipschitz equivalence class of pairs of normally embedded Hölder triangles.

Conjecture 4.14. *Let (T, T') and (S, S') be two ordered oriented pairs of normally embedded Hölder triangles satisfying condition (5). If the $\sigma\tau$-pizza of the pair (T, T') is combinatorially equivalent to the $\sigma\tau$-pizza of the pair (S, S'), then there is an orientation-preserving outer bi-Lipschitz homeomorphism $H : T \cup T' \to S \cup S'$ such that $H(T) = S$ and $H(T') = S'$.*

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