The Wronski map and Grassmannians of real codimension 2 subspaces

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Abstract

We study the map which sends a pair of real polynomials (f_0, f_1) into their Wronski determinant $W(f_0, f_1)$. This map is closely related to a linear projection from a Grassmannian $G_{\mathbf{R}}(m, m+2)$ to the real projective space $\mathbf{RP}^{2m}$. We show that the degree of this projection is $\pm u((m+1)/2)$ where u is the m-th Catalan number. One application of this result is to the problem of describing all real rational functions of given degree $m+1$ with prescribed $2m$ critical points. A related question of control theory is also discussed.

1 Introduction

A recent result [3] says that if all critical points of a rational function are real then the function itself can be made real by postcomposition with a fractional-linear transformation. In the present paper we study the structure of the set of real rational functions whose critical points coincide with a given set symmetric with respect to the real axis.

We first recall what is known in general about rational functions with prescribed critical points. Two rational functions f and g will be called *equivalent*,

$$f \sim g \quad \text{if} \quad f = \ell \circ g, \quad \text{where } \ell \text{ is a fractional-linear transformation.} \tag{1}$$

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Equivalent rational functions have the same critical points. We will see that for a given set of $2m$ points in the complex plane there are finitely many classes of rational functions of degree $m+1$ having these critical points.

If $f = f_2/f_1$ is a non-constant rational function, then $f' = W(f_1, f_2)/f_1^2$, where

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

is the Wronski determinant. The equivalence relation (1) on rational functions corresponds[1] to the following equivalence relation on pairs of polynomials:

$$(f_1, f_2) \sim (g_1, g_2) \quad \text{if} \quad (f_1, f_2) = (g_1, g_2)A, \quad \text{where} \quad A \in GL(2, \mathbf{C}).$$

Notice that equivalent pairs of polynomials have proportional Wronski determinants. This suggests that the map $(f_1, f_2) \mapsto W(f_1, f_2)$ should be considered as a map from a Grassmann variety to a projective space.

We recall the relevant definitions, referring for the general background to [7, 8]. Let $\mathbf{F}$ be one of the fields $\mathbf{R}$ (real numbers) or $\mathbf{C}$ (complex numbers). We denote by $G_{\mathbf{F}} = G_{\mathbf{F}}(m, m+2)$, $m \geq 2$, the Grassmannian, that is the set of all linear subspaces of dimension m in $\mathbf{F}^{m+2}$. Such subspaces can be described as row spaces of $m \times (m+2)$ matrices K of maximal rank. Two such matrices K_1 and K_2 define the same element of $G_{\mathbf{F}}$ if $K_1 = UK_2$, where $U \in GL(m, \mathbf{F})$. So $G_{\mathbf{F}}$ is an algebraic manifold over $\mathbf{F}$ of dimension $2m$. The Plücker coordinates of a point in $G_{\mathbf{F}}$ represented by a matrix K are the full size minors of K. This defines an embedding of $G_{\mathbf{F}}$ to a projective space $\mathbf{FP}^N$, where

$$N = \binom{m+2}{m} - 1 = m(m+3)/2.$$

We usually identify $G_{\mathbf{F}}$ with its image under this embedding, which is called a *Grassmann variety*. It is a smooth algebraic variety in $\mathbf{FP}^N$.

Let $S \subset \mathbf{FP}^N$ be a projective subspace disjoint from $G_{\mathbf{F}}$, and $\dim_{\mathbf{F}} S = N - \dim G_{\mathbf{F}} - 1 = (m+1)(m-2)/2$. We consider the *central projection* $\pi_S : \mathbf{FP}^N \backslash S \to \mathbf{FP}^{2m}$, and its restriction to $G_{\mathbf{F}}$,

$$\phi_S = \pi_S|_{G_{\mathbf{F}}} : G_{\mathbf{F}} \to \mathbf{FP}^{2m}. \tag{2}$$

[1]The correspondence between non-constant rational functions and pairs of non-proportional polynomials is not bijective because a polynomial pair may have a common factor of positive degree.

Then ϕ_S is a finite regular map of projective varieties.

The following result goes back to Schubert (1886). For a modern proof, see [6] or [8, XIV, 7].

Theorem A *When* $\mathbf{F} = \mathbf{C}$, *the degree of* ϕ_S *is* $u(m+1)$, *where*

$$u(d) = \frac{1}{d}\begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix} \quad \textit{is the } d\textit{-th Catalan number.} \tag{3}$$

When $\mathbf{F} = \mathbf{C}$, the degree of ϕ_S is independent of S, and equals the number of intersections of $G_{\mathbf{C}}$ with a generic projective subspace $L \subset \mathbf{CP}^N$ of codimension $2m$. It is called the degree of $G_{\mathbf{C}}$.

Now we show that the map $(f_1, f_2) \mapsto W(f_1, f_2)$ defines a projection of the form (2) with a special choice of center S.

To explain our choice of the center of projection S_0, we consider the $2 \times (m+2)$ matrix of polynomials

$$E(z) = \begin{pmatrix} F(z) \\ F'(z) \end{pmatrix} = \begin{pmatrix} z^{m+1} & z^m & \dots & 1 \\ (m+1)z^m & mz^{m-1} & \dots & 0 \end{pmatrix}. \tag{4}$$

For a fixed z, the row space of this matrix represents the tangent line to the rational normal curve $F : \mathbf{FP}^1 \to \mathbf{FP}^{m+1}$ at the point $F(z)$. The space $\mathrm{Poly}_{\mathbf{F}}^{2m}$ of all non-zero polynomials $p \in \mathbf{F}[z]$ of degree at most $2m$, up to proportionality, will be identified with $\mathbf{FP}^{2m}$ (the coefficients of polynomials serving as homogeneous coordinates). Now we define the map $\phi : G_{\mathbf{F}} \to \mathbf{FP}^{2m}$ using the representation of $G_{\mathbf{F}}$ by $m \times (m+2)$ matrices K:

$$K \mapsto \phi(K) = \det \begin{pmatrix} E(z) \\ K \end{pmatrix} \in \mathrm{Poly}_{\mathbf{F}}^{2m}. \tag{5}$$

It is clear that ϕ is well defined: changing K to UK, $U \in GL(m, \mathbf{F})$, will result in multiplication of the polynomial $\phi(K)$ by $\det U$. Furthermore, this map ϕ, when expressed in terms of Plücker coordinates, coincides with the restriction to $G_{\mathbf{F}}$ of a projection of the form π_S as in (2), with some center which we will call S_0. We do not need the explicit equations of S_0, but they can be easily obtained by expanding the determinant in (5) with respect to the last m rows, and collecting the terms with the same powers of z.

We can interpret the polynomial $\phi(K)$ in (5) as a Wronskian determinant of a pair of polynomials. To see this, we first consider the "big cell" of the

Grassmannian $G_{\mathbf{F}}$, which is represented by the matrices K whose rightmost minor is different from zero. We can normalize K to make the rightmost $m \times m$ submatrix the unit matrix. If the remaining (leftmost) two columns of K are $(k_{i,j})$, $1 \leq i \leq m, j = 1, 2$, then

$$\phi(K) = f_{1,K} f'_{2,K} - f'_{1,K} f_{2,K} = W(f_{1,K}, f_{2,K}),$$

where

$$\begin{array}{rcl} f_{1,K}(z) & = & z^{m+1} - k_{1,1} z^{m-1} - \ldots - k_{m,1}, \quad \text{and} \\ f_{2,K}(z) & = & z^m - k_{1,2} z^{m-1} - \ldots - k_{m,2}. \end{array} \tag{6}$$

Now we consider all pairs of non-proportional polynomials of degree at most $m+1$, modulo the following equivalence relation: $(f_1, f_2) \sim (g_1, g_2)$ if $(f_1, f_2) = (g_1, g_2)V$, where $V \in GL(2, \mathbf{F})$. The equivalence classes parametrize the Grassmannian $G_{\mathbf{F}}$. Coefficients of two polynomials correspond to the coefficients of two linear forms which define a subspace of codimension 2, that is a point of $G_{\mathbf{F}}$. Then ϕ becomes the *Wronski map*, which assigns to the equivalence class of a pair of polynomials the Wronskian determinant of this pair, modulo proportionality.

Remark. Notice that the Wronski map sends the big cell X of the Grassmannian into the big cell Y of the projective space consisting of those polynomials whose degree is exactly $2m$. Moreover, it sends the complement $G_{\mathbf{F}} \backslash X$ into $\mathbf{FP}^{2m} \backslash Y$.

So, for our special choice of the center of projection S_0, the projection map restricted to the Grassmannian $G_{\mathbf{F}}$ becomes the Wronski map. This interpretation of the Wronski map as a projection comes from [6], where the following corollary from Theorem A was obtained

Theorem B *For a given set of $2m$ points in the complex plane there exist at most $u(m+1)$ classes of rational functions of degree $m+1$ with these critical points, with equality for a generic set of $2m$ points.*

Now we turn to the case $\mathbf{F} = \mathbf{R}$. We recall the definition of degree of a smooth map $f : X \to Y$ of real manifolds of equal dimensions. Suppose that X is orientable, and fix an orientation of X. We choose a regular value $y \in Y$ of f, which exists by Sard's theorem, and define

$$\deg f = \pm \sum_{x \in f^{-1}(y)} \operatorname{sgn} \det f'(x), \tag{7}$$

using local coordinates in X consistent with the orientation, and any local coordinate at y. The degree $\deg f$ is defined up to sign which depends on the choice of orientations. The absolute value of degree is independent of the choice of local coordinates, and, in the case of connected Y, of the regular value y.

Using the Remark above, we define the degree of the Wronski map[2] as the degree of its restriction to the big cell X of the Grassmannian.

The main result of this paper is

Theorem 1.1 *When m is odd, the degree of the Wronski map is*

$$\pm u((m+1)/2),$$

where u is the Catalan number defined in (3).

If (f_1, f_2) is a pair of coprime polynomials, then zeros of $W(f_1, f_2)$ coincide with finite critical points of f_2/f_1. Notice that $G_{\mathbf{R}} \subset G_{\mathbf{C}}$ can be represented by pairs of real polynomials, and to each such pair corresponds a real rational function. Thus Theorem 1 can be restated in terms of rational functions:

Corollary 1.2 *Let X be a set of $2m$ points in general position in $\overline{\mathbf{C}}$, symmetric with respect to $\mathbf{R}$. Then the number k of equivalence classes of real rational functions of degree $m+1$ whose critical sets coincide with X satisfies*

$$0 \leq k \leq u(m+1) \quad \textit{if } m \textit{ is even, and} \tag{8}$$

$$u((m+1)/2) \leq k \leq u(m+1) \quad \textit{if } m \textit{ is odd.} \tag{9}$$

Upper estimates in (8) and (9) follow from Theorem B. These upper estimates are best possible for every $m \geq 2$ (see [11] or remark after Proposition 2.3). For every even m the lower estimate in (8) is best possible as examples in [4] show (see also Example 2.5 below).

The lower estimate in (9) is also best possible:

Example 1.3 *For every odd m, there exist real polynomials $y \in \mathrm{Poly}_{\mathbf{R}}^{2m}$ which are regular values of the Wronski map ϕ in* (5)*, such that the cardinality of $\phi^{-1}(y)$ is $u((m+1)/2)$.*

[2] Topological degree can be defined for arbitrary projections (2) without using special properties of the Wronski map.

The proof of Theorem 1.1 occupies sections 2–3. In section 2 we state the relevant results about rational functions from [3], describe Examples 1.3 and 2.5, and prove Propositions 2.13-2.15 necessary for Theorem 1.1. In section 3 we compute the degree of the real Wronski map. This is done by computing the signs of Jacobian determinants in (7) with a special choice of the regular values $w \in \mathbf{RP}^{2m}$ which correspond to very degenerate rational functions. A crucial ingredient is the information that these Jacobian determinants are different from zero. This is provided by Proposition 2.15.

Finally, in section 4 we interpret Theorem 1.1 in terms of control theory.

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2 Rational functions with critical points on a circle

We fix an integer $d \geq 3$, which is related to the number m of the Introduction by

$$d = m + 1. \tag{10}$$

In [3] we established the following fact.

Proposition 2.1 *Given $2d-2$ points on a circle C, there exist precisely $u(d)$ classes of equivalence of rational functions of degree d with these critical points, and mapping C into circles.* □

Here and in what follows a "circle" means a circle on the Riemann sphere $\overline{\mathbf{C}} = \mathbf{CP}^1$. The number $u(d)$ was defined in (3) and the equivalence relation on rational functions in (1). From Theorem A and Proposition 2.1 we obtain

Corollary 2.2 *If q is a polynomial of degree $2d-2$, whose zeros are simple and belong to a circle, then the Wronski map $G_{\mathbf{C}} \to \mathbf{CP}^{2d-2}$ is unramified over q.*

Remark. The main purpose of this section is to extend Corollary 2.2 to some cases when q has multiple zeros (Proposition 2.15).

Proof of Corollary 2.2. Denote the roots of q by $x_1, \ldots, x_{2d-2}$. Applying Proposition 2.1 we obtain $u(d)$ classes of equivalence of rational functions of degree d whose critical points are $x_1, \ldots, x_{2d-2}$. If $f = f_2/f_1$ is such a function, we have $W(f_1, f_2) = cq$, where c is a constant. Thus q has $u(d)$

preimages under the Wronski map. On the other hand, by Theorem A, $u(d)$ is the maximal number of complex preimages under this map. It follows that ϕ is unramified over q. □

From Proposition 2.1 and Theorem A we derived in [3]

Proposition 2.3 *If all critical points of a rational function f belong to a circle C then $f(C)$ is contained in a circle.* □

Proposition 2.1 implies that the upper estimates (8) and (9) of Corollary 1.2 are best possible. An application of Proposition 2.3 is

Proposition 2.4 *Let $d \geq 4$ be an even integer, and $d-2$ real points*

$$x_{d-2} < x_{d-3} < \ldots < x_1 < 0$$

are given. Consider two sets of rational functions:

A: The set of all real rational functions g of degree $d/2$ with critical points at $x_1, \ldots, x_{d-2}$, and

B: The set of all real rational functions f of degree d with critical points $0, \infty$ and $\{\pm\sqrt{x_j} : 1 \leq j \leq d-2\}$.

Then there is a bijective correspondence between A and B, given by $f(z) = g(z^2)$. This correspondence respects the equivalence relation (1).

Proof. If $g \in A$ and $f(z) = g(z^2)$, then evidently $f \in B$.
Suppose now that $f \in B$. As f is real, we have

$$f(\overline{z}) = \overline{f(z)}. \tag{11}$$

Applying Proposition 2.3 with $C = i\mathbf{R}$, we conclude that $f(i\mathbf{R}) \subset C'$, where C' is a circle. By (11), C' is symmetric with respect to $\mathbf{R}$. Every such circle C' is either perpendicular to $\mathbf{R}$ or coincides with it. As 0 is a critical point of f, and $f(\mathbf{R}) \subset \mathbf{R}$, the second possibility occurs, and $f(i\mathbf{R}) \subset \mathbf{R}$. This implies that $f(-\overline{z}) = \overline{f(z)}$. Combining this with (11), we obtain $f(z) = \overline{f(-\overline{z})} = f(-z)$. Thus f is even, so $f(z) = g(z^2)$, where $g \in B$. □

Construction of Example 1.3. We take

$$y(z) = z \prod_{k=1}^{d-2} (z^2 + k^2), \tag{12}$$

or any other real polynomial y of degree $2d-3$, with all zeros on the imaginary axis and simple, and $y(0)=0$. Suppose that (f_1,f_2) is a pair of real polynomials such that

$$W(f_1,f_2)=y. \tag{13}$$

Then $f=f_2/f_1$ is a function of the set B. By Proposition 2.4, the number of equivalence classes of functions in B is the same as the number of equivalence classes of functions in A, and the last number is $u(d/2)$ by Proposition 2.1 with $C=\mathbf{R}$. Corollary 2.2 implies that the Wronski map is unramified over y. □

Besides Propositions 2.1 and 2.3 themselves we need some elements of their proof from [3], so we recall the relevant definitions and results here.

We fix an oriented circle $C\subset\overline{\mathbf{C}}$ until the end of this section.

If z_1 and z_2 are two points on C, the arc from z_1 to z_2, following orientation of C, will be denoted by (z_1,z_2). We also fix a point $v_0\in C$.

Let $R(d)$ be the set of all non-constant rational functions of degree at most d, with complex coefficients, and having the properties that all critical points belong to C, v_0 is a simple critical point, and there are at least 3 critical points. By Proposition 2.3, for $f\in R(d)$, $f(C)$ is a subset of a circle C'. Consider the full preimage $\Gamma(f)=f^{-1}(C')\subset\overline{\mathbf{C}}$. The *net of* f is defined as the pair $\gamma(f)=(\Gamma(f),v_0)$. The point v_0 and the oriented circle C will be called the *reference point* and the *reference circle* of the net $\gamma(f)$.

Evidently, equivalent functions have equal nets. The set $\Gamma(f)$ is a one-dimensional, real-algebraic subvariety of $\overline{\mathbf{C}}$, whose singular points coincide with the critical points of f. The set $\Gamma=\Gamma(f)$ defines a cellular decomposition of $\overline{\mathbf{C}}$, whose 2-dimensional cells (faces) are components of $\overline{\mathbf{C}}\backslash\Gamma$, 0-dimensional cells (vertices) are the singular points of Γ, and 1-dimensional cells are components of $\Gamma\backslash\{\text{vertices}\}$. The set Γ is the 1-skeleton of this decomposition. Each face of the cellular decomposition is mapped by f homeomorphically onto one of the two components of $\overline{\mathbf{C}}\backslash C'$. The net of a function defines the corresponding cellular decomposition uniquely, so we will permit ourselves such expressions as "a face of the net", "a vertex of the net" and so on. The following properties of $\gamma(f)=(\Gamma,v_0)$, $f\in R(d)$, are evident:

(i) Γ is symmetric with respect to C,

(ii) $C\subset\Gamma$,

(iii) all vertices of Γ belong to C, and there are at least 3 of them,

(iv) an even number ≥ 4 of edges meet at every vertex,

(v) v_0 is a vertex, and 4 edges meet at v_0.

A simple application of the nets is

Example 2.5 *If d is odd, there exists a real polynomial of degree $2d-2$, which is not a Wronskian determinant of any pair of real polynomials.*

Construction. Take $p(z) = z^{2d-2} - 1$ (or any other real polynomial of degree $2d-2$, with $2d-2$ roots on a circle centered on the real axis, two of these roots real). Suppose that (f_1, f_2) is a pair of real polynomials such that $W(f_1, f_2) = p$. Then $f = f_2/f_1$ is a real rational function of degree d, whose critical points coincide with the roots of unity of degree $2d-2$. By Proposition 2.3, f maps the unit circle into some circle C', and C' is symmetric with respect to the real axis. Then the net $\gamma(f) = (f^{-1}(C'), 1)$ is also symmetric with respect to the real axis, and it has vertices exactly at the roots of unity of degree $2d-2$. It is easy to see that a net with such properties cannot exist for odd d (for a formal proof, see [4]). □

In the rest of this section we will establish three propositions 2.13, 2.14 and 2.15, needed in the next section for the proof of Theorem 1.1.

Given an oriented circle C and a point $v_0 \in C$, a *net* $\gamma = (\Gamma, v_0)$ (with no reference to a function) is defined as a pair, consisting of the 1-skeleton Γ of a cellular decomposition of the Riemann sphere, having properties (i)-(iv), and the reference point v_0 satisfying (v). Two nets $\gamma_1 = (\Gamma_1, v_0)$ and $\gamma_2 = (\Gamma_2, v_0)$ (with the same oriented reference circle C) are called *equivalent*, if there exists a homeomorphism $h : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, preserving orientation of $\overline{\mathbf{C}}$ and of C, leaving the point v_0 fixed, commuting with reflection with respect to C, and having the property $h(\Gamma_1) = \Gamma_2$. The equivalence class of a net γ will be denoted by $[\gamma]$.

Every net has an even number of faces, one half of this number is called the *degree of a net.* For rational functions $f \in R(d)$ we have $\deg \gamma(f) = \deg f$. A net is called *degenerate* if some vertices have order greater than 4. Otherwise it is called *non-degenerate.* For a rational function $f \in R(d)$, the net $\gamma(f)$ is degenerate iff f has multiple critical points. In [3] we considered only non-degenerate nets.

For each net, we define certain distinguished elements in the following way. Let v_1 be the vertex following v_0 on C, according to the orientation of C. Let D be that component of $\overline{\mathbf{C}} \backslash C$, whose positively oriented boundary is C. There is a unique face G_0 in D, whose boundary contains at least 3 vertices, v_0 and v_1 among them. Let v_{-1} be the vertex preceding v_0 on ∂G_0. So when tracing ∂G_0 according to its positive orientation, we encounter v_{-1}, v_0, v_1 in this order. We also introduce two edges of γ on ∂G_0: $e_1 = [v_0, v_1]$ and $e_{-1} = [v_{-1}, v_0]$. Of these two edges, one belongs to C, another does not. For every net γ satisfying (i)-(v), there is a unique choice of *distinguished elements* $D, G_0, v_{-1}, v_0, v_1, e_1$ and e_{-1}.

The edges of γ in D are called *chords* of γ. For example, of the 4 edges incident to v_0, one is a chord, and two are arcs of C.

Now we recall the notion of a labeling of a non-degenerate net γ from [3]. Let E be the set of edges of γ, Q the set of faces of γ in D, and $s : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ the reflection with respect to C. A *labeling* of γ is a function $p : E \to \mathbf{R}_{\geq 0}$, satisfying the following conditions

$$p(s(e)) = p(e) \quad \text{for every} \quad e \in E, \tag{14}$$

$$\sum_{e \subset \partial G} p(e) = 2\pi \quad \text{for every} \quad G \in Q, \tag{15}$$

and

$$p(e_{-1}) = p(e_1) = 2\pi/3, \tag{16}$$

where e_{-1} and e_1 are the distinguished edges of γ.

The set of all labelings of γ is identified with a closed convex polytope $\overline{L}_\gamma$ in the affine subspace A of $\mathbf{R}^{4d-4}$ described by equations (14), (15), (16). The dimension of A and $\overline{L}_\gamma$ is $2d-5$. The polytope $\overline{L}_\gamma$ depends only on the equivalence class of γ.

A labeling is called *non-degenerate* if $p(e) > 0$ for all e, otherwise it is called *degenerate*. Non-degenerate labelings correspond to the interior L_γ of $\overline{L}_\gamma$ with respect to A.

The following construction was used in [3, section 3].

Suppose that p is a labeling of a non-degenerate net γ. Let $Z(p)$ be the union of closures of edges of γ whose labels are zero, and $D(p)$ the component of $\overline{\mathbf{C}} \backslash Z(p)$, containing G_0. Put $B = \overline{\mathbf{C}} \backslash D(p)$ and introduce the following equivalence relation on $\overline{\mathbf{C}}$: $x \sim y$ if x and y belong to the same component of B. Let $Y = \overline{\mathbf{C}}/\sim$ be the factor space, and $w : \overline{\mathbf{C}} \to Y$ the projection map. Since $D(p)$ is connected, every component of $\overline{\mathbf{C}} \backslash D(p)$ is contractible, hence Y is a topological sphere, so we can identify it with the Riemann sphere. The reflection $s : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is an involution which leaves every point of C fixed. Since every component of B contains a vertex, it intersects C. It follows that each component of B is symmetric with respect to C. So Y also has an involution, such that w splits the involutions. This means that the identification of Y with $\overline{\mathbf{C}}$ can be made in such a way that

$$w \circ s = s \circ w, \quad w(C) = C \quad w(v_0) = v_0,$$

and w preserves orientation of both $\overline{\mathbf{C}}$ and C.

The image $\gamma' = w(\gamma)$ of our net γ can be a degenerate net or a net of smaller degree than γ. We say that γ' is obtained by *collapse* of γ *according to* p. If the labeling p is non-degenerate, then $\gamma' = \gamma$.

Lemma 2.6 *Suppose that a degenerate net γ' of degree d was obtained by collapse of a non-degenerate net γ of the same degree d. Then γ' determines γ uniquely.*

Proof. As in [3], we associate to each net γ two rooted trees $S(\gamma)$ and $\hat{S}(\gamma)$ embedded into the closed disc $\overline{D}$. The first tree $S(\gamma)$ is the dual graph of the cellular decomposition of D defined by γ. The vertices q_G of $S(\gamma)$ correspond to faces G of γ in D, and two vertices of $S(\gamma)$ are connected by an edge τ_e of $S(\gamma)$ if the corresponding faces of γ have a common boundary edge e. The root q_0 of $S(\gamma)$ corresponds to the distinguished face G_0 of γ.

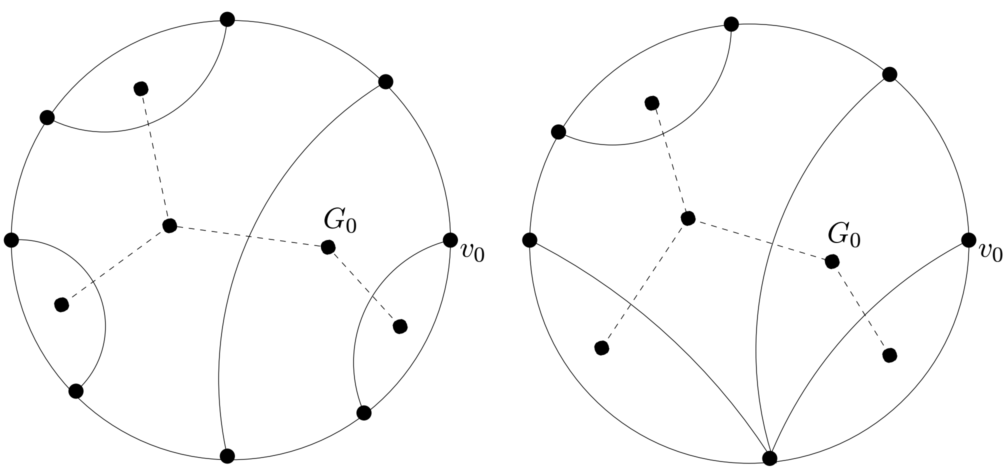

Fig. 2. Collapse of a net. The trees S are shown in dotted lines.

The other rooted tree $\hat{S}(\gamma)$ is obtained by the following extension of $S(\gamma)$: for each edge $e \subset C$ of γ, a vertex q_e and an edge τ_e are added to $S(\gamma)$, such that τ_e connects q_e to q_G, where G is the face of γ in D with $e \in \partial G$. The edge τ_1 of $\hat{S}(\gamma)$, corresponding to the edge e_1 of γ, is called the distinguished edge of $\hat{S}(\gamma)$. It may belong to $S(\gamma)$ or not.

It is easy to see that the triple $(S(\gamma), \hat{S}(\gamma), \tau_1)$, modulo orientation preserving homeomorphisms of $\overline{D}$, defines the equivalence class of the net γ completely. Indeed, the trees $S(\gamma) \subset \hat{S}(\gamma)$ define the cellular decomposition of $\overline{D}$, and τ_1 defines the edge e_1 of this decomposition. Of the two extremities of e_1, the one which precedes the other on C, is v_0.

For a non-degenerate net γ, the dual graph $S(\gamma)$ defines its extension $\hat{S}(\gamma)$ completely: for each vertex q of $S(\gamma)$ of order k, exactly k new edges incident to q are added, one in each space between the old edges.

The procedure of collapse of a non-degenerate net described above has a simple interpretation in terms of the tree $\hat{S}$: we first delete some edges, and then retain the component of the root, deleting all other components, if any. Our assumption that both γ and γ' are of the same degree d means that $S(\gamma) = S(\gamma')$. As γ is non-degenerate, $S(\gamma)$ uniquely defines $\hat{S}(\gamma)$. Since $v_0(\gamma')$ is of order 4, the distinguished edge $e_1(\gamma)$ does not disappear after collapse, so $\tau_1(\gamma) = \tau_1(\gamma')$, and this proves Lemma 2.6. □

Now we recall the main technical result from [3]. Fix a non-degenerate net γ of degree d, and denote its vertices by $v_0, v_1, \ldots, v_{2d-3}$, where the enumeration is consistent with the orientation of C. The vertex v_{-1} defined above coincides with some v_N, where N depends on γ and satisfies $3 \leq N \leq 2d-3$. We associate to γ the convex polytope of its labelings $\overline{L}_\gamma$, defined in (14)–(16), and another convex polytope $\overline{\Sigma}_\gamma$ of *critical sequences*, of the same dimension $2d-5$ as $\overline{L}_\gamma$. To define a critical sequence, we first fix three points z_{-1}, z_0, z_1 on C, enumerated according to the orientation of C. A critical sequence is a map c from the set V of vertices of γ to C, which preserves the non-strict cyclic order, and satisfies

$$c(v_{-1}) = z_{-1}, \quad c(v_0) = z_0, \quad c(v_1) = z_1. \tag{17}$$

Choosing the spherical lengths of the arcs $l_j = (c(v_j), c(v_{j+1})) \subset C$ as coordinates, we obtain an embedding of the space of critical sequences to the hyperoctant $\mathbf{R}^{2d-2}_{\geq 0}$. The image of this embedding is the polytope $\overline{\Sigma}_\gamma \subset \mathbf{R}^{2d-2}_{\geq 0}$, defined by three linear equations, resulting from (17).

The polytope $\overline{\Sigma}_\gamma$ depends only on the equivalence class of γ, and the choice of z_{-1}, z_0, z_1 in (17). A *critical sequence of a function* $f \in R(d)$ of degree d is defined as the sequence of vertices of $\gamma(f)$, starting from v_0, going on according to the orientation of C, where each vertex is repeated according to the multiplicity of the corresponding critical point of f.

In [3, section 3], for each non-degenerate net γ of degree d, we defined a map

$$F_\gamma : \overline{L}_\gamma \to R(d) \times \overline{\Sigma}_\gamma,$$

which depends only on the equivalence class of γ and the choice of z_{-1}, z_0, z_1 in (17). This map $F_\gamma = (\Psi_\gamma, \Phi_\gamma)$ has the following properties. For $p \in \overline{\Sigma}_\gamma$, the first component $f = \Psi_\gamma(p)$ is a rational function of the class $R(d)$, and the second component $c = \Phi_\gamma(p)$ is a critical sequence. If $\deg f = d$, then c is the critical sequence of f. If p is non-degenerate, then $\deg f = d$, and f has $2d-2$ critical points on C.

Lemma 2.7 [3, section 6]. *The map* $\Phi_\gamma : \overline{L}_\gamma \to \overline{\Sigma}_\gamma$ *is surjective.* □

Lemma 2.8 [3, section 4]. *If* $p_j \to p$ *is a convergent sequence in* $\overline{L}_\gamma$*, then* $\Phi_\gamma(p_j) \to \Phi_\gamma(p)$*, and*

$$\Psi_\gamma(p_j) \to \Psi_\gamma(p), \quad \textit{uniformly on compact subsets of} \quad \overline{\mathbf{C}} \backslash X, \qquad (18)$$

where X *is a finite set.* □

Comments. 1. In [3] we used (17) with a special choice of z_{-1}, z_0, z_1, the cubic roots of unity, but this choice is irrelevant and was made only for convenience.

2. It is equation (28) and Lemma 4 in [3], which imply (18).

3. When $\Psi_\gamma(p)$ is of degree d, the exceptional set X is empty, and convergence in (18) is uniform on $\overline{\mathbf{C}}$.

Lemma 2.9 *Let* γ *be a non-degenerate net of degree* d*, and* $p \in \overline{L}_\gamma$ *a labeling. Suppose that* $f = \Psi_\gamma(p)$ *is of degree* d*. Then* $\gamma(f)$ *is obtained by collapse of* γ *according to* p*. In particular,* $\gamma(f) \sim \gamma$ *if* p *is non-degenerate.* □

Lemma 2.9 follows from the explicit construction of Ψ_γ in [3, section 3]. Lemmas 2.8 and 2.9 imply the following. Suppose that $p_k \to p$, $f_k = \Psi_\gamma(p_k) \to \Psi_\gamma(p) = f$, and $c_k = \Phi_\gamma(p_k) \to \Phi_\gamma(p) = c$, where $\deg f = d$. Then for every edge $e = [v_i, v_j]$ of γ we have

$$c(v_i) = c(v_j) \quad \text{if and only if} \quad p(e) = 0. \qquad (19)$$

Furthermore, $\deg f = d$ if and only if $p(e) > 0$ for all chords e of γ. We also need a criterion in terms of the labeling p for $f = \Psi_\gamma(p)$ to be of degree $d-1$. A chord is called *extreme* if it connects two consecutive vertices on the unit circle.

Lemma 2.10 *Let γ be a non-degenerate net of degree d. Then the rational function $\Psi_\gamma(p)$ is of degree $d-1$ if and only if there is exactly one chord e of γ such that $p(e)=0$, and this chord is extreme.*

Proof. This follows from Lemma 2.9 and the explicit description of collapse. Let γ' be the result of collapse of γ according to p. Then γ has d faces in D and γ' has $d-1$ faces. Each chord separates D into two parts. One of these parts consists of $d-1$ faces if and only if the separating chord is extreme. □

We also need the following elementary

Lemma 2.11 *Let $f_j \to f \neq \text{const}$ be a sequence of rational functions converging uniformly on $\overline{\mathbf{C}}$ with respect to the spherical metric. Let ℓ_j be a sequence of fractional linear transformations, such that $\ell_j \circ f_j \to g$ uniformly on compact subsets of $\overline{\mathbf{C}}\backslash X$, where X is a finite set, and $g \neq \text{const}$. Then $\ell_j \to \ell$, uniformly in $\overline{\mathbf{C}}$, where ℓ is a fractional-linear transformation.*

Proof. Consider a disc $Y \subset \mathbf{C}$, where the convergence $g_j = \ell_j \circ f_j \to g$ is uniform, and let $Z = f(Y)$. As $f \neq \text{const}$, Z is open, and we can choose a disc $Z' \subset Z$, which contains no critical values of f and f_j. Let f^{-1} be a branch which sends Z' into Y. Then there are branches $f_j^{-1} \to f^{-1}$ on Z', and we have have $\ell_j = g_j \circ f_j^{-1}$ in Z'. Passing to the limit gives $\ell_j \to \ell$ in Z', where ℓ is a con-constant meromorphic function in Z'. This implies that the sequence ℓ_j is convergent to ℓ uniformly in $\overline{\mathbf{C}}$, and that ℓ is a fractional-linear transformation. □

Lemma 2.12 *Let $f \in R(d)$ be a function of degree d, with simple critical points. Then $f \sim \Psi_\gamma(p)$, where $\gamma = \gamma(f)$ and $p \in L_\gamma$. If f_1 and f_2 are two functions from $R(d)$ with the same simple critical points, and $\gamma(f_1) \sim \gamma(f_2)$, then $f_1 \sim f_2$.*

Proof. The total number of equivalence classes of rational functions of degree d sharing the critical set of f is at most $u(d)$ (by [6] or by Theorem A). On the other hand, there are $u(d)$ equivalence classes of non-degenerate nets of degree d [12, Exercise 6.19 n].

We claim that each equivalence class $[\gamma]$ of these nets gives a rational function $\Psi_\gamma(p_{f,\gamma})$ with the same critical points as f, and all these $u(d)$ functions represent different equivalence classes. Indeed, let $c_0,\ldots,c_{2d-3}$ be the sequence of critical points of f, enumerated according to the orientation of C, and $c_0 = v_0$. Given a non-degenerate net γ, we define $z_0 = c_0$, $z_1 = c_1$, and $z_{-1} = c_N$, where $N = N(\gamma)$ is the integer with the property $v_N = v_{-1}$, the distinguished vertex of γ. The choice of γ and z_{-1}, z_0, z_1 defines the map $F_\gamma = (\Psi_\gamma, \Phi_\gamma)$, where Φ_γ is surjective according to Lemma 2.7. So for each γ there exists a labeling $p_{\gamma,f}$ such that $\Psi_\gamma(p_{\gamma,f})$ has the same critical points as f. For different γ, these functions $\Psi_\gamma(p_{\gamma,f})$ are non-equivalent by Lemma 2.9. This proves our claim.

As there are $u(d)$ classes of nets of degree d, it follows that $f \sim \Psi_\gamma(p_{f,\gamma})$ for some γ and $p_{f,\gamma} \in L_\gamma$. By Lemma 2.9, $\gamma(f) \sim \gamma$, so the class $[\gamma]$ is uniquely defined by f. Furthermore, $p_{f,\gamma}$ is uniquely defined by f, otherwise there would be more than $u(d)$ equivalence classes of rational functions sharing the same critical set of f. □

To state the next three Propositions, it is convenient to define the following subclass $R_0(d) \subset R(d)$. It consists of rational functions f of degree exactly d, having at least three critical points, and at least two of them, $z_0 = v_0$ and z_1 are simple. Furthermore, we require that the arc $(z_0, z_1) \subset C$, traced from z_0 to z_1 according to the orientation of C, be free from critical points of f.

Proposition 2.13 *Every $f \in R_0(d)$ can be represented as $f = \ell \circ \Psi_\gamma(p)$, where γ is a non-degenerate net of degree d, $p \in \overline{L}_\gamma$, and ℓ is a fractional-linear transformation. Furthermore, $\gamma(f)$ is the result of collapse of γ according to p.*

Proof. If all critical points of f are simple, this follows from Lemma 2.12. The general case is proved by a perturbation argument.

We suppose, without loss of generality, that all critical points of f are finite. Let $f = f_2/f_1$ be a coprime representation, and $q = W(f_1, f_2)$. Then $\deg q = 2d - 2$. We choose a sequence $q_j \to q$, $j \to \infty$ of polynomials of degree $2d-2$, each with $2d-2$ distinct finite roots on C, each q_j having a simple root at z_k for $k = 0, 1$, and such that the arc (z_0, z_1) is free from the roots of all q_j.

Let us choose rational functions $f_j \in R_0(d)$, such that

$$f_j \to f, \quad \text{as} \quad j \to \infty, \quad \text{and} \quad \phi(f_j) = q_j, \tag{20}$$

where ϕ is the Wronski map (5). Such rational functions f_j always exist because ϕ is surjective.

We consider the nets $\gamma(f_j)$, using $v_0 = z_0$ as a common reference point for all nets. As there are only finitely many nets of degree d, we may assume, by choosing a subsequence, that $\gamma(f_j) \in [\gamma]$ for some fixed net γ. If v_1 is the distinguished vertex of $\gamma(f_j)$, then $v_1 = z_1$, which follows from our definitions of v_1 and of the class $R_0(d)$. Consider the distinguished vertex $v_{-1}(j)$ of $\gamma(f_j)$. By choosing a subsequence, we may assume that $v_{-1}(j) \to z_{-1}$, where z_{-1} is a point on a closed arc $[a, b] \subset C$, which is disjoint from the closed arc $[z_0, z_1]$. This point z_{-1} depends on γ. Let χ_j be the fractional-linear transformation which sends the triple (z_{-1}, z_0, z_1) to $(v_{-1}(j), z_0, z_1)$. This defines χ_j uniquely, and clearly $\chi_j \to \text{id}$ uniformly in $\overline{\mathbf{C}}$. Put $h_j = f_j \circ \chi_j$. Then $\gamma(h_j) \sim \gamma$, and the critical sequence of h_j satisfies (17). It follows from Lemma 2.12 that there exist labelings $p_j \in \overline{L}_\gamma$ and fractional-linear transformations ℓ_j, such that

$$f_j \circ \chi_j = h_j = \ell_j \circ (\Psi_\gamma(p_j)). \tag{21}$$

As $\overline{L}_\gamma$ is compact, and Ψ_γ is continuous (in the sense described in Lemma 2.8), we have $\Psi_\gamma(p_j) \to \Psi_\gamma(p)$, uniformly on compact subsets of $\overline{\mathbf{C}} \backslash X$, for some finite set X, and the limit is non-constant. By Lemma 2.11, $\ell_j \to \ell$, and thus

$$f = \ell \circ \Psi_\gamma(p). \tag{22}$$

This proves the first statement of Proposition 2.13. The second statement follows from Lemma 2.9. □

Proposition 2.14 *Suppose that f and g are two rational functions in $R_0(d)$, sharing their critical points on C. If $\gamma(f) \sim \gamma(g)$, then $f \sim g$.*

Proof. When critical points are simple, this Proposition follows from Lemma 2.12. If some of them are multiple, we run the same perturbation argument as in the proof of Proposition 2.13. Namely, we construct sequences q_j, f_j and g_j satisfying conditions similar to (20)

$$f_j \to f, \quad \phi(f_j) = q_j, \quad \text{and} \quad g_j \to g, \quad \phi(g_j) = q_j, \tag{23}$$

and $\gamma(f_j) \in [\gamma]$, $\gamma(g_j) \in [\beta]$, for some nets γ and β. As in the proof of (22), we derive

$$f \sim \Psi_\gamma(p_f), \quad \text{and} \quad g \sim \Psi_\beta(p_g). \tag{24}$$

It follows that γ and β can be collapsed to the same net $\gamma(f) \sim \gamma(g)$, so by Lemma 2.6 we conclude that $\gamma \sim \beta$. Now, as f_j and g_j have the same critical points and equivalent nets, we conclude from Lemma 2.12 that $f_j \sim g_j$, that is $f_j = \ell_j \circ g_j$, and passing to the limit, we obtain $f \sim g$. □

Proposition 2.15 *If $f \in R_0(d)$, then $\det \phi'(f) \neq 0$, where $\phi : G_{\mathbf{C}}(d-1, d+1) \to \mathbf{CP}^{2m}$ is the Wronski map* (5)*.*

Proof. In the case when all critical points are simple, Proposition 2.15 follows from Corollary 2.2. Otherwise, we use the same perturbation argument as in the proof of Propositions 2.13 and 2.14. We begin with construction of a sequence of polynomials q_j explained in the proof of Proposition 2.13.

Claim. There exists a neighborhood U of f in $G_{\mathbf{C}}$ which contains at most one preimage of any q_j under the Wronski map ϕ.

We prove this claim by contradiction. Assume that there exist two sequences

$$f_j \to f, \quad g_j \to f, \quad \phi(f_j) = \phi(g_j) = q_j, \quad \text{and} \quad f_j \not\sim g_j \tag{25}$$

Arguing exactly as in the proof of Proposition 2.14, we conclude that $\gamma(f_j) = \gamma(g_j)$. Then by Lemma 2.12, $f_j \sim g_j$, which contradicts (25). This contradiction proves the claim.

Now we recall that the complex Wronski map $\phi : G_{\mathbf{C}} \to \mathbf{CP}^{2m}$ is a finite regular map. So it has local degree $\deg_f(\phi)$ at the point $f \in G_{\mathbf{R}} \subset G_{\mathbf{C}}$, which means that there exist a neighborhood $U \subset \mathbf{CP}^{2m}$ of $q = \phi(f)$, and a neighborhood $U' \subset G_{\mathbf{C}}$ of f, such that every $r \in U$ has exactly $\deg_f(\phi)$ preimages in U' under ϕ, counting multiplicity. The multiplicity of all preimages of q_j under ϕ is one by Corollary 2.2. So our claim above shows that the local degree $\deg_f(\phi) = 1$, and thus by a well-known result (see, for example, [7, Ch. 0, 2]), $\det \phi'(f) \neq 0$. □

3 Computation of degree of the real Wronski map

In this section we work in the class $R_0(d)$ with the reference circle $C = \mathbf{R}$ and $v_0 = z_0 < z_1 < 0$. Consider the subclass $B(d) \subset R_0(d)$ of real rational functions f of degree d, whose critical points belong to the segment $[-1, 0] \subset$

$\mathbf{R}$, the leftmost critical point is v_0, and

$$f(z) = z + O(1), \quad z \to \infty, \quad \text{and} \quad f(0) = 0. \tag{26}$$

We first verify that every real rational function f whose critical points belong to $[-1, 0]$ can be normalized to satisfy (26). The arc $(\mathbf{R} \cup \infty) \backslash [-1, 0]$ is contained in one edge of the net $\gamma(f)$, so this arc is mapped into $\mathbf{R} \cup \infty$ injectively. So

$$f(0) \neq f(\infty), \tag{27}$$

and thus (26) can be achieved by post-composing f with a fractional-linear transformation.

It follows from (26) that f has a unique representation as $f = Q/P$, where P and Q are real polynomials of the form

$$\begin{array}{rcl} P(z) & = & z^{d-1} + a_{d-2}z^{d-2} + \ldots + a_1 z + a_0, \\ Q(z) & = & z^d + b_{d-1}z^{d-1} + \ldots + b_1 z. \end{array} \tag{28}$$

We use coefficients of these polynomials as coordinates in $B(d)$, and orient $B(d)$ by ordering these coordinates:

$$b_1, b_2, \ldots, b_{d-1}, a_0, a_1, \ldots, a_{d-2}. \tag{29}$$

The Wronskian determinant $W = W(P, Q)$ equals

$$\begin{array}{rcl} W(z) & = & a_0 b_1 + 2a_0 b_2 z + (3a_0 b_3 + a_1 b_2 - a_2 b_1) z^2 + \ldots \\ & & + \left((n+1)a_0 b_{n+1} + (n-1)a_1 b_n + \ldots + (1-n)a_n b_1\right) z^n \\ & & \ldots + (3a_{d-3} + a_{d-2}b_{d-1} - b_{d-2}) z^{2d-4} + 2a_{d-2} z^{2d-3} + z^{2d-2} \\ & & = z^{2d-2} + \displaystyle\sum_{n=0}^{2d-3} c_n z^n. \end{array}$$

Here

$$c_n = \sum_{j=0}^{n} (n + 1 - 2j) a_j b_{n-j+1}. \tag{30}$$

It is convenient to extend the finite sequences (a_j) and (b_j) to all integer values of j in the natural way: $a_{d-1} = 1$, $a_j = 0$ for $j \geq d$ and $j < 0$, $b_d = 1$, $b_j = 0$ for $j > d$ and $j \leq 0$. Then

$$W(z) = \sum_{i,j} (j - i) a_i b_j z^{i+j-1}, \tag{31}$$

where the summation in the last sum is over all pairs of integers (i, j).

Polynomials P and Q in (28) have no common factor, because their ratio is of degree d. This implies that all zeros of the Wronskian W belong to $[-1, 0]$, and thus its coefficients c_j are bounded for $f \in B(d)$. We claim that the coefficients a_j of P and b_j of Q are also bounded for $f \in B(d)$.

To prove this claim, consider a rational function g equivalent to f,

$$g = f - b_{d-1} = q/p, \quad \text{where } p = P \text{ and } q = Q - b_{d-1}P, \tag{32}$$

so that p and q are normalized as in (6) with $m = d - 1$. It follows from (28) that

$$g(z) = z + o(1), \quad z \to \infty. \tag{33}$$

The set of real rational functions of degree d satisfying (33), and with critical points on $[-1, 0]$ will be called $B'(d)$. Equation (32) establishes a homeomorphism $f \mapsto g$ from $B(d)$ to $B'(d)$.

According to Remark after (6), the Wronski map sends the big cell X of $G_{\mathbf{R}}$ into the big cell Y of $\mathrm{Poly}_{\mathbf{R}}^{2d-2}$, and the complement $G_{\mathbf{R}} \backslash X$ into the complement $\mathrm{Poly}_{\mathbf{R}}^{2d-2} \backslash Y$. As $W = \phi(g) = \phi(f)$ belongs to a compact subset of Y, for $g \in B'(d)$, and the restriction of the Wronski map $\phi : X \to Y$ is proper, we conclude that $B'(d)$ is contained in a compact subset of X, which means that the coefficients of p and q are bounded as $g \in B'(d)$.

To pass back from g to f, we need to show that $b_{d-1} = -g(0) = -q(0)/p(0)$ is bounded. Suppose that it is not. Then there is a sequence $g_j = q_j/p_j$, and $g_j(0) \to \infty$. As coefficients of p_j and q_j are bounded, we can choose a convergent subsequence $g_j \to g$. It is clear that $g \neq \text{const}$, and $g(0) = g(\infty) = \infty$. Furthermore, $g \in B(d')$ for some $d' \geq 1$, and we obtain a contradiction with (27). This contradiction proves that $g(0)$ is bounded for $g \in B'(d)$.

This proves our claim that functions P and Q in (28) have bounded coefficients for $f \in B(d)$.

We need the following elementary

Lemma 3.1 *Suppose that $a \in \mathbf{C}$ is a critical point of multiplicity μ of a rational function Q/P, and ν is the maximal integer such that $(z - a)^\nu$ divides both P and Q. Then the Wronskian determinant $W(P, Q)$ has zero of multiplicity $\mu + 2\nu$ at a.* □

We consider a function $f = Q/P \in B(d)$, where P and Q are as in (28), with $a_0 \neq 0$. Then zeros of $W = W(P, Q)$ coincide with the critical points of f. We list these zeros as

$$-1 \leq x_{2d-2} < x_{2d-3} < \ldots < x_k < x_{k-1} = x_{k-2} = \ldots = 0, \qquad (34)$$

where $k = k(f)$, $1 \leq k \leq 2d-4$. In particular, (34) says that 0 is a critical point of multiplicity $k-1$, if $k \geq 2$.

Now we perform on f the following operation. We move the critical point x_k to the right, to 0, while keeping all other points x_j fixed. We can do it in such a way that P and Q change continuously. More precisely, for $0 \leq t \leq 1$, we consider the family of polynomials

$$W_t(z) = (z - tx_k) \prod_{j \in \{1,\ldots,2d-2\} \setminus \{k\}} (z - x_j).$$

This uniquely defines a continuous family (P_t, Q_t) of polynomial pairs of the form (28), so that $W(P_t, Q_t) = W_t$. Indeed, the map $(P, Q) \mapsto W(P, Q)$ is finite and regular, so the curves can be lifted via this map. This lifting is unique because ϕ is unramified over W_t, $0 < t \leq 1$, by Proposition 2.15 (where $z_0 = x_{2d-2}$ and $z_1 = x_{2d-3}$ are used in the definition of the class $R_0(d)$).

Using Lemma 3.1, we conclude that while $0 < t \leq 1$, the pair remains irreducible, that is $f_t = Q_t/P_t$ has degree d. In particular, this implies that $a_0 \neq 0$ for $0 < t \leq 1$, and then, by assumptions (34) about zeros of W_t, 0 is a critical point of f_t of multiplicity $k-1$, that is f_t has zero of multiplicity k at 0:

$$b_1 = \ldots = b_{k-1} = 0, \quad b_k \neq 0 \quad \text{for} \quad 0 < t \leq 1. \qquad (35)$$

By (30), the first non-zero coefficient at W_t occurs at z^{k-1}, and this coefficient is $a_0 b_k$. All zeros of W_t are non-positive, so all its coefficients are non-negative, and we obtain

$$a_0 b_k > 0. \qquad (36)$$

Furthermore, Proposition 2.15 implies that $\det W'(P_t, Q_t) \neq 0$ for $0 < t \leq 1$, where W' stands for the Jacobi matrix of the Wronski map defined by (30), and thus this Jacobi determinant keeps constant sign for $0 < t \leq 1$.

When $t = 0$, one of the two cases may occur:

Case 1. The pair (P_0, Q_0) is irreducible. Then by Proposition 2.15, $\det W'(P_0, Q_0) \neq 0$, so the sign of this determinant is constant for all t, $0 \leq t \leq 1$.

Case 2. The pair (P_0, Q_0) has a common factor. By Lemma 3.1 and (34), this common factor has to be a power of z, because at all points other than 0, W_t can have only simple zeros. This means that

$$a_0 \to 0, \quad \text{as} \quad t \to 0, \tag{37}$$

which is a necessary and sufficient condition for the Case 2 to occur. When $t = 0$, W has a zero of order k at 0. Let λ_1 and λ_2 be the orders of zeros at 0 for P_0 and Q_0, respectively. Then P_0 and Q_0 have a common factor z^ν, where $\nu = \min\{\lambda_1, \lambda_2\}$. We have

$$\lambda_1 \geq \nu \geq 1 \quad \text{and} \quad \lambda_2 \geq k. \tag{38}$$

The first inequality holds because z is a common factor, second because of (35). The multiplicity of the critical point of $f_0 = Q_0/P_0$ at 0 is $\mu = \max\{\lambda_1, \lambda_2\} - \min\{\lambda_1, \lambda_2\} - 1$, so Lemma 3.1 and (38) imply

$$k = \max\{\lambda_1, \lambda_2\} - \min\{\lambda_1, \lambda_2\} - 1 + 2\min\{\lambda_1, \lambda_2\} = \lambda_1 + \lambda_2 - 1 \geq k.$$

This implies that both inequalities in (38) are in fact equalities, that is $\lambda_1 = \nu = 1$ and $\lambda_2 = k$. This means that

$$a_0 = 0, \quad a_1 \neq 0 \quad \text{and} \quad b_k \neq 0, \quad \text{for} \quad t = 0 \quad \text{in Case 2.} \tag{39}$$

So the common factor of (P_0, Q_0) is z. Let (P^*, Q^*) be the reduced pair of polynomials of degrees $(d-2, d-1)$, that is $P^*(z) = P_0(z)/z$ and $Q^*(z) = Q_0(z)/z$, and $W^* : B(d-1) \to \mathrm{Poly}^{2d-4}_{\mathbf{R}}$ the corresponding Wronski map. We prove the following:

$$\mathrm{sgn}\det(W^*)'(P^*, Q^*) = (-1)^{d-k}\mathrm{sgn}\det W'(P_t, Q_t), \quad t > 0. \tag{40}$$

We compare both determinants, using the explicit expression (30) for the coefficients of W and a similar expression for W^*. To describe submatrices of a matrix Δ, we use notation $\Delta\begin{pmatrix} i_k & \dots & i_l \\ j_r & \dots & j_s \end{pmatrix}$ for the submatrix formed by the elements of Δ in rows $(i_k, \dots, i_l)$ and columns $(j_r, \dots, j_s)$. The transposed[3] Jacobian matrix $\Delta = (W')^T$ has the following structure:

[3]It is transposed only for convenience of typesetting.

$$
\begin{array}{c|cccccccc}
 & c_0 & c_1 & c_2 & \dots & c_{k-1} & c_k & \dots \\
\hline
b_1 & a_0 & 0 & -a_2 & \dots & (2-k)a_{k-1} & (1-k)a_k & \dots \\
b_2 & 0 & 2a_0 & a_1 & \dots & (4-k)a_{k-2} & (3-k)a_{k-1} & \dots \\
b_3 & 0 & 0 & 3a_0 & \dots & (6-k)a_{k-3} & (5-k)a_{k-2} & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
b_k & 0 & 0 & 0 & \dots & ka_0 & (k-1)a_1 & \dots \\
b_{k+1} & 0 & 0 & 0 & \dots & 0 & (k+1)a_0 & \dots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
a_0 & b_1 & 2b_2 & 3b_3 & \dots & kb_k & (k+1)b_{k+1} & \dots \\
a_1 & 0 & 0 & b_2 & \dots & (k-2)b_{k-1} & (k-1)b_k & \dots \\
a_2 & 0 & 0 & -b_1 & \dots & (k-4)b_{k-2} & (k-3)b_{k-1} & \dots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots &
\end{array}
$$

The submatrix $\Delta\begin{pmatrix} b_1 & \dots & b_k \\ c_0 & \dots & c_{k-1} \end{pmatrix}$ is upper triangular, with $(a_0, 2a_0, \dots, ka_0)$ on the main diagonal. The submartix $\Delta\begin{pmatrix} a_0 & \dots & a_{d-2} \\ c_0 & \dots & c_{k-1} \end{pmatrix}$ has only one non-zero entry, which is $kb_k \neq 0$; this follows from (35) and (39). Expanding $\det\Delta$ consecutively with respect to columns $c_0, c_1, \dots, c_{k-1}$, and using (37), and (39), we obtain:

$$
\begin{aligned}
\det\Delta &= k! a_0^{k-1} b_k (-1)^{d-k} \det\Delta\begin{pmatrix} b_k, & \dots, & b_{d-1}, a_1, & \dots, & a_{d-2} \\ c_k, & \dots & \dots & \dots, & c_{2d-3} \end{pmatrix} \\
&+ k! a_0^k \det\Delta\begin{pmatrix} b_{k+1}, & \dots, & a_{d-2} \\ c_k, & \dots, & c_{2d-3} \end{pmatrix}. \qquad (41)
\end{aligned}
$$

This matrix $\Delta^* = (W^*)'$ is obtained from Δ by performing the following operations: a) crossing out the rows b_1 and a_0, and columns c_0 and c_1, and b) setting $a_0 = 0$. These rules of obtaining Δ^* follow from (30) or (31).

Consecutively expanding $\det\Delta^*$ with respect to columns $c_2, \dots, c_{k-2}$, we obtain

$$
\det\Delta^* = (k-2)! a_1^{k-2} \det\Delta^*\begin{pmatrix} b_k, & \dots, & a_{d-2} \\ c_k, & \dots & c_{2d-3} \end{pmatrix}
$$

$$= \quad (k-2)!a_1^{k-2} \det\Delta\begin{pmatrix} b_k, & \dots, & b_{d-1}, a_1, & \dots, & a_{d-2} \\ c_k, & \dots & \dots & \dots, & c_{2d-3} \end{pmatrix}. \quad (42)$$

Notice that the last determinant in (42) coincides with the determinant in the first summand in (41).

Applying to (P^*, Q^*) the same argument which proves (36), we obtain $a_1 b_k > 0$. Together with (36) this implies

$$a_0 a_1 > 0. \quad (43)$$

The reduced rational function $f^* = Q^*/P^*$ belongs to the class $R_0(d-1)$ defined before Proposition 2.13, with $C = \mathbf{R}$, $z_0 = x_{2d-3}$ and $z_1 = x_{2d-4}$. So we can apply Proposition 2.15 with d replaced by $d-1$. This gives

$$\det\Delta^* \neq 0. \quad (44)$$

It follows from (44) that the determinant in the first summand in (41) has non-zero limit as $t \to 0$. In addition, by (39), b_k has non-zero limit, while $a_0 \to 0$ as $t \to 0$. Thus, as $t \to 0$, the first summand in (41) is equivalent to ca_0^{k-1}, where $c \neq 0$, while the second summand is $O(a_0^k)$. So the sign of $\det\Delta$ for $0 < t < 1$ is the same as the sign of the first summand in (41), and using (36), (43) we conclude that (40) holds.

Now, using Lemma 2.10, we can tell Case 1 from Case 2 by looking at the net γ of f. This net is constructed using $C = \mathbf{R}$ and $v_0 = x_{2d-2}$, and has degree d. According to Proposition 2.13, for $0 < t \leq 1$, $f_t = \Psi_{\gamma'}(p_t)$, with a non-degenerate net γ' of degree d, and some labelings $p_t \in \overline{L}_{\gamma'}$. Passing to the limit as $t \to 0$, we obtain $f_0 = \Psi_{\gamma'}(p_0)$. By (19), for $t \in (0, 1]$, we have $p_t(e) = 0 \Leftrightarrow p(e) = 0$ for all edges e, except those connecting x_k and x_{k-1}. If there is only one such edge, it belongs to C, and $\deg f_0 = d$ by Lemma 2.10. Otherwise, there is an extreme chord between x_k and x_{k-1} and $\deg f_0 = d-1$. Thus Case 2 occurs if and only if there is an extreme chord between x_k and x_{k-1}.

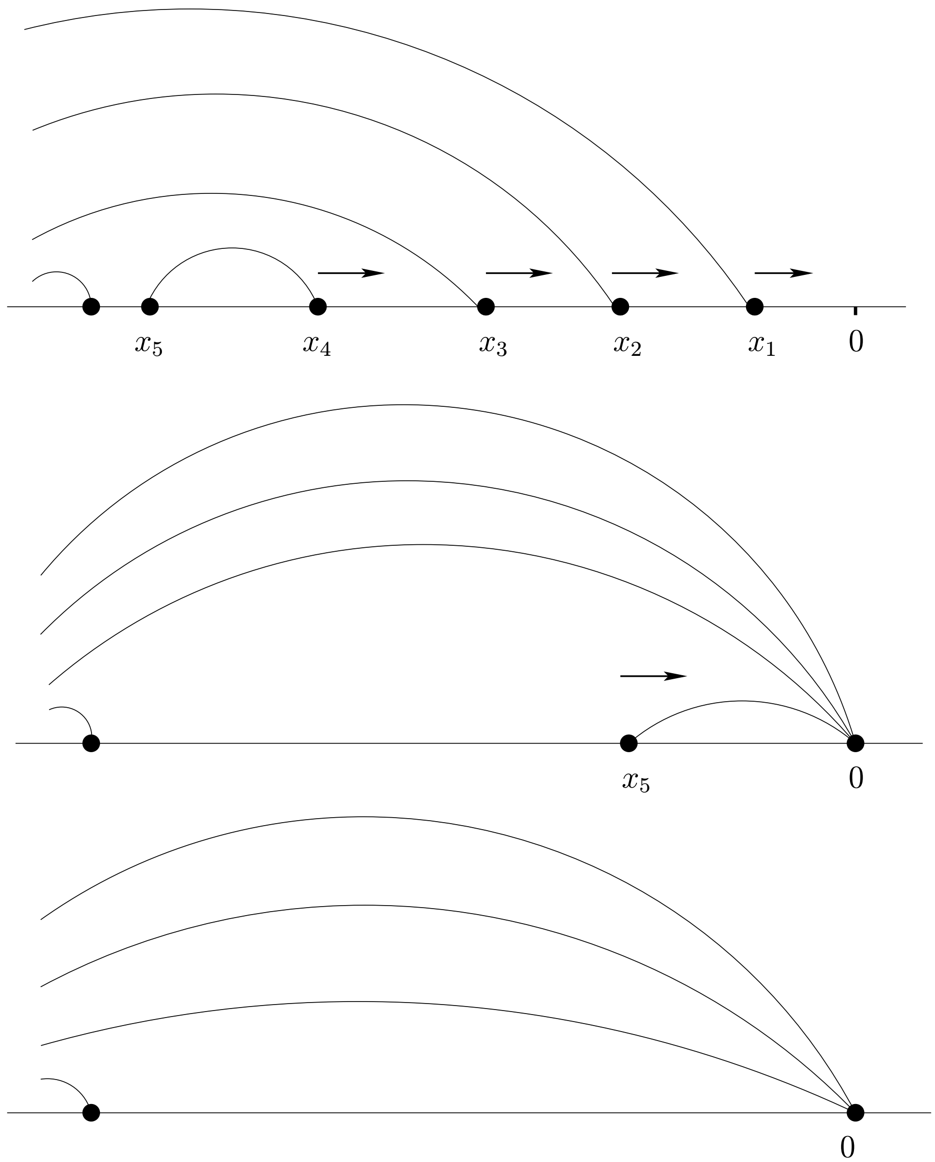

Fig. 3. Steps of collapse. Case 2 occurs on the 5-th step with $k = 5$.

Now we relate $\mathrm{sgn}\det\phi'(f)$ to the net of f. Suppose first that f is a real rational function $f = Q/P$, where P and Q are as in (28), and all critical points $x_{2d-2} < \ldots < x_1 < 0$ are real and simple. The net $\gamma = \gamma(f)$ is defined using the reference circle $C = \mathbf{R}$ oriented from left to right, and the reference point $v_0 = x_{2d-2}$. Let k be the smallest integer such that γ

contains an extreme chord between x_k and x_{k-1}. Then $2 \le k \le d$.

Consider another non-degenerate net γ_1, of degree $d-1$, which is obtained from γ by removing two vertices x_k and x_{k-1}, and all five edges incident to any of these two vertices, and replacing them by a single edge from x_{k+1} to x_{k-2}. Let f_1 be the real rational function of degree $d-1$ with critical points at the vertices of γ_1, and such that $\gamma(f_1) = \gamma_1$. Such function exists by Proposition 2.3, and it is unique by Proposition 2.14.

We claim that

$$\operatorname{sgn}\det \phi'(f) = (-1)^{k+d} \operatorname{sgn}\det (\phi^*)'(f_1). \tag{45}$$

To prove the claim, we shift one-by-one the critical points $x_1, \ldots, x_{k-2}$ of both functions f and f_1 to the right to 0. Case 1 occurs $k-2$ times for each function. This does not change the sign of the determinants in (45). Next we shift the critical point x_{k-1} of f to zero, and this still does not change the LHS of (45). Finally we move the critical pojnt x_k of f to 0, and this time Case 2 occurs, f becomes reducible, and we obtain a rational function f^* of degree $d-1$. The net $\gamma(f^*)$ is equivalent to γ_1, which follows from Lemma 2.9. Then by Proposition 2.14, $f^* = f_1$, and equation (40) implies (45).

To reformulate (45) in a convenient form, we associate to each non-degenerate net γ a sequence $(w_0, \ldots, w_{2d-3})$ of 0's and 1's, called a *Catalan sequence*, in the following way. Consider the chord starting at a vertex v_j, $0 \le j \le 2d-3$. Let the other end of this chord be v_i. If $i > j$, we put $w_j = 0$, and if $i < j$ we put $w_j = 1$. Then $\operatorname{inv}\gamma$ is defined as the number of inversions in the Catalan sequence, that is $\operatorname{inv}\gamma = \#\{(i,j) : i > j, w_i < w_j\}$. We claim that

$$\operatorname{sgn}\det \phi'(f) = (-1)^{\operatorname{inv}\gamma}, \tag{46}$$

Proof of (46). We use induction on d, starting from $d = 3$.

Consider a function $f = Q/P$ of degree 3, where

$$P(z) = z^2 + a_1 z + a_0, \quad \text{and} \quad Q(z) = z^3 + b_2 z^2 + b_1 z. \tag{47}$$

Then

$$\begin{aligned} W(z) &= a_0 b_1 + 2a_0 b_2 z + (3a_0 + a_1 b_2 - b_1) z^2 + 2a_1 z^3 + z^4 \\ &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + z^4, \end{aligned} .$$

Now we suppose that zeros of W are

$$x_4 < x_3 < x_2 < x_1 = 0.$$

This implies that $b_1 = 0,\ c_0 = 0$, but c_1, c_2 and c_3 are positive. In particular,

$$a_0 b_2 > 0, \quad \text{and} \quad a_1 > 0. \tag{48}$$

To write the Jacobian determinant $\Delta = \det \phi'(f)$, we order the coordinates as in (29). Then

$$\Delta = \begin{vmatrix} a_0 & 0 & -1 & 0 \\ 0 & 2a_0 & a_1 & 0 \\ 0 & 2b_2 & 3 & 0 \\ 0 & 0 & b_2 & 2 \end{vmatrix} = 4a_0(3a_0 - a_1 b_2).$$

When $x_2 \to 0$, and the limit function has degree 3, we have $b_2 \to 0$, while a_0^2 has positive limit, so $\Delta > 0$. Considering the net and its Catalan sequence, we conclude $\operatorname{inv} f = 0$, so (46) holds for our function f. Similarly, if $x_2 \to 0$, and the limit function has degree 2, we have $a_0 \to 0$, and by (48) $a_0 a_1 b_2 > 0$, while $a_1 b_2$ does not tend to zero, so $\Delta < 0$. Considering the net and its Catalan sequence, we conclude that $\operatorname{inv} f = 1$, so (46) holds. This establishes the base of our induction.

Now suppose that $\deg f = d$, and f_1 in (45) satisfies an equation, similar to (46):

$$\operatorname{sgn} \det(\phi^*)'(f_1) = (-1)^{\operatorname{inv} \gamma_1}, \quad \gamma_1 = \gamma(f_1). \tag{49}$$

Comparison of nets $\gamma(f)$ and $\gamma(f_1)$ shows that

$$\operatorname{inv} f = \operatorname{inv} f_1 + d - k. \tag{50}$$

Now (45), (49) and (50) imply (46). □

To complete the proof of Theorem 1.1 we use the following result from [5, (2.6)].

Lemma 3.2 *For even d, we have*

$$\sum_{\gamma} (-1)^{\operatorname{inv} \gamma} = u(d/2),$$

where sum is taken over all $u(d)$ non-degenerate nets of degree d. □

Remarks. One can show directly, using (46), that for all $u(d/2)$ real rational functions f of degree d from Example 1.3 the sign of $\det \phi'(f)$ is the same. This gives an alternative proof of Theorem 1.1, not using Lemma 3.2.

4 A related problem of control theory

Suppose that a triple of real matrices $S = (A, B, C)$ of sizes $n \times n$, $n \times m$ and $p \times n$ is given. This triple S defines a *linear system*

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{aligned} \tag{51}$$

Here x, u and y are functions of time (a real variable) taking their values in $\mathbf{R}^n$, $\mathbf{R}^m$ and $\mathbf{R}^p$, respectively. The values of these functions at a point $t \in \mathbf{R}$ are interpreted as the state, input and output of our system at the moment t.

Behavior of system (51) is completely determined by the *transfer function* $z \mapsto C(zI - A)^{-1}B$, which is a function of a complex variable z with values in the set of $p \times m$ matrices. One wishes to control such system (51) by arranging a feedback, which means sending the output to the input via an $m \times p$ matrix K, which is called a *gain* matrix:

$$u = Ky. \tag{52}$$

Elimination of u and y from (51), (52) gives the *closed loop system*

$$\dot{x} = (A - BKC)x,$$

whose transfer function has poles at the zeros of the polynomial

$$\psi_K(z) = \det(zI - A - BKC). \tag{53}$$

The map $K \mapsto \psi_K$ is called the *pole placement map*, and the problem of pole assignment is: given a system S, and a set $\{z_1, \ldots, z_n\}$, symmetric with respect to $\mathbf{R}$, to find real K so that the zeros of ψ_K are $\{z_1, \ldots, z_n\}$. We refer to the survey [2] for the results on the pole assignment prior to 1980.

We say that for given (m, n, p) the pole placement map is *generically surjective* if there is an open dense set U of triples $S = (A, B, C)$ such that for $S \in U$ the pole placement map is surjective. Dimensions count shows that the condition $n \leq mp$ is a necessary condition for generic surjectivity. If complex gain matrices are permitted, this condition is also sufficient, see, for example [2]. It was proved by X. Wang [13] that for $n < mp$ the real pole placement map is generically surjective (see also [14, 9]. We consider the real pole placement map with $n = mp$. We also assume that our system is given in *minimal representation* [1] which means that its open loop transfer function $C(zI-A)^{-1}B$ cannot be represented in a similar form with a matrix A of smaller size.

To understand the structure of the pole placement map, we use a coprime factorization of the open loop transfer function [1, Assertion 22.6]:

$$C(zI-A)^{-1}B = D(z)^{-1}N(z), \quad \det D(z) = \det(zI-A), \tag{54}$$

where D and N are polynomial matrix-functions of sizes $p \times p$ and $p \times m$, respectively. This factorization associates to our linear system a polynomial matrix $[D(z), N(z)]$ of size $p \times (m+p)$ with the property that the leftmost $p \times p$ minor $D(z)$ has degree $n = mp$, while all other minors have smaller degree, and minors of order p have no common zeros. In the opposite direction, for a given polynomial matrix $[D(z), N(z)]$ with these properties, there exists a linear system (A, B, C) such that (54) holds.

Using this (54) and the identity $\det(I - PQ) = \det(I - QP)$, which is true for all rectangular matrices of appropriate dimensions, we write

$$\begin{aligned}
\psi_K(z) &= \det(zI - A - BKC) = \det(zI - A)\det(I - (zI - A)^{-1}BKC) \\
&= \det(zI - A)\det(I - C(zI - A)^{-1}BK) \\
&= \det D(z)\det(I - D(z)^{-1}N(z)K) = \det(D(z) - N(z)K).
\end{aligned}$$

This can be rewritten as

$$\psi_K(z) = \begin{vmatrix} D(z) & N(z) \\ K & I \end{vmatrix}. \tag{55}$$

In the last determinant, the first p rows depend only on the given system, and the last m rows on the gain matrix. The maximal degree of the $p \times p$ minors of the first p rows of this determinant is called the McMillan degree of the system S, and it is equal to mp for a generic system with $n = mp$. Now we permit arbitrary $m \times (m+p)$ matrices $\hat{K}$ of maximal rank as the last m rows of the determinant in (55). A linear system represented by $[D(z)\,N(z)]$ is called *non-degenerate* if $\psi_{\hat{K}} \neq 0$ for all such $\hat{K}$. This property is generic, that is it holds for an open dense set of systems. Thus for non-degenerate systems the pole placement map extends to

$$\phi_S : G_{\mathbf{R}}(m, m+p) \to \mathbf{RP}^{mp}, \quad \phi_S(\hat{K}) = [\psi_K], \tag{56}$$

where [.] means the class of proportionality of a polynomial, which is identified with a point in $\mathbf{RP}^{mp}$, using the coefficients of a polynomial as homogeneous coordinates. As we have seen in the Introduction, the map (56) is well defined. Applying Laplace's expansion along the first p rows to the determinant in (55), we conclude that the map ϕ_S, when expressed in Plücker coordinates, is nothing but a projection of the Grassmann variety $G_{\mathbf{R}}(m, m+p)$ into $\mathbf{RP}^{mp}$ from some center S. This interpretation of the pole placement map as a projection comes from [14].

We notice that projections associated with linear systems share the property of the Wronski map stated in Remark 1 in the Introduction: $\phi_S^{-1}(Y) = X$, where X is the big cell of the Grassmann variety and Y is the big cell of $\mathbf{RP}^{mp}$. So the maps (56) have well-defined degree. As S varies, the degree can change only when S intersects the Grassmann variety, that is only when a system becomes degenerate. As the Wronski map corresponds to a non-degenerate system, we obtain

Corollary 4.1 *For every odd integer m there exists an open set U of systems with m inputs, 2 outputs and state of dimension $2m$ such that for systems in U the real pole placement map is surjective. Furthermore, the pole placement problem for such systems has at least $u((m+1)/2)$ real solutions for any generic set of $2m$ poles symmetric with respect to the real line.*

□

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