Degrees of real Wronski maps

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Abstract

We study the map which sends vectors of polynomials into their Wronski determinants. This defines a projection map of a Grassmann variety which we call a Wronski map. Our main result is computation of degrees of the *real* Wronski maps. Connections with real algebraic geometry and control theory are described.

1 Introduction

We study the map W which sends vectors of polynomials $(f_1, \ldots, f_p)$ to their Wronski determinants:

$$W(f_1, \ldots, f_p) = \left| \begin{array}{ccc} f_1 & \ldots & f_p \\ f_1' & \ldots & f_p' \\ \ldots & \ldots & \ldots \\ f_1^{p-1} & \ldots & f_p^{p-1} \end{array} \right|. \tag{1}$$

Besides an intrinsic interest, this map is related to several questions of algebraic geometry, combinatorics and control theory as we describe below.

The following properties of the Wronski determinant are well-known and easy to prove:
1. $W(f_1, \ldots, f_p) = 0$ if and only if $f_1, \ldots, f_p$ are linearly dependent.
2. Multiplication of $(f_1, \ldots, f_p)$ by a constant matrix A of size $p \times p$ results in multiplication of $W(f_1 \ldots, f_p)$ by $\det A$.

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These properties suggest that our map W should be considered as a map from a Grassmannian to a projective space. We recall the relevant definitions.

Let $\mathbf{F}$ be one of the fields $\mathbf{R}$ (real numbers) or $\mathbf{C}$ (complex numbers). For positive integers m and p we denote by $G_{\mathbf{F}} = G_{\mathbf{F}}(m, m+p)$ the Grassmannian, that is the set of all linear subspaces of dimension m in $\mathbf{F}^{m+p}$. Such subspaces can be described as row spaces of $m \times (m+p)$ matrices K of maximal rank. Two such matrices K_1 and K_2 define the same element of $G_{\mathbf{F}}$ if $K_1 = UK_2$, where $U \in GL(m, \mathbf{F})$. It is easy to see that $G_{\mathbf{F}}(m, m+p)$ is an algebraic manifold over $\mathbf{F}$ of dimension mp. We have $G_{\mathbf{F}}(1, m+p) = \mathbf{FP}^{m+p-1}$, the projective space over $\mathbf{F}$ of dimension $m+p-1$.

We may identify polynomials of degree at most $m+p-1$ in the domain of the map W in (1) with vectors in $\mathbf{F}^{m+p}$ using coefficients as coordinates, and similarly polynomials in the range of W with vectors in $\mathbf{F}^{mp+1}$. Then, in view of the properties 1 and 2 of the Wronski determinant, equation (1) will define a map $G_{\mathbf{F}}(p, m+p) \to \mathbf{FP}^{mp}$. Alternatively, we can also identify polynomials of degree at most $m+p-1$ with linear forms on $\mathbf{F}^{m+p}$. Then p linearly independent forms define a subspace of dimension m in $\mathbf{F}^{m+p}$, and we obtain a map

$$\phi : G_{\mathbf{F}}(m, m+p) \to \mathbf{FP}^{mp}, \tag{2}$$

which will be called a *Wronski map.* To understand the nature of this map, we use the Plücker embedding of the Grassmannian.

The Plücker coordinates of a point in $G_{\mathbf{F}}(m, m+p)$ represented by a matrix K are the full size minors of K. This defines an embedding of $G_{\mathbf{F}}(m, m+p)$ to $\mathbf{FP}^N$, $N = \begin{pmatrix} m+p \\ m \end{pmatrix} - 1$. We usually identify $G_{\mathbf{F}}$ with its image under this embedding, which is called a *Grassmann variety.* It is a smooth algebraic variety in $\mathbf{FP}^N$.

Let $S \subset \mathbf{FP}^N$ be a projective subspace disjoint from $G_{\mathbf{F}}$, and $\dim_{\mathbf{F}} S = N - \dim G_{\mathbf{F}} - 1$. We consider the *central projection* $\pi_S : \mathbf{FP}^N \backslash S \to \mathbf{FP}^{mp}$, and its restriction to $G_{\mathbf{F}}$,

$$\phi_S = \pi_S|_{G_{\mathbf{F}}} : G_{\mathbf{F}} \to \mathbf{FP}^{mp}. \tag{3}$$

Then ϕ_S is a finite regular map of projective varieties. When $\mathbf{F} = \mathbf{C}$ this map has a degree, which can be defined in this case as the number of preimages of a generic point and is independent of S. This degree was computed by Schubert in 1886 (see [12, 9, 10] for modern treatment).

Theorem A *When* $\mathbf{F}=\mathbf{C}$, *the degree of* ϕ_S *is*

$$d(m,p)=\frac{1!2!\dots(p-1)!\,(mp)!}{m!(m+1)!\dots(m+p-1)!}. \tag{4}$$

Projective duality implies that $d(m,p)=d(p,m)$. Here are some values of $d(m,p)$

$m=$	2	3	4	5	6	7	8
$p=2$	2	5	14	42	132	429	1430
$p=3$		42	462	6006	87516	1385670	23371634
$p=4$			24024	1662804	140229804	...	...
$p=5$				701149020	...	...	....

In particular,

$$d(m,2)=\frac{1}{m+1}\binom{2m}{m}, \quad \text{the } m\text{-th Catalan number.}$$

The numbers $d(m,p)$ have the following combinatorial interpretation: they count the Standard Young Tableaux (SYT) of rectangular shape $p\times m$.

To see that the Wronski map is a projection (3) we choose a center S_0 in the following way. Consider the $p\times(m+p)$ matrix of polynomials

$$E(z)=\begin{pmatrix} F(z)\\ F'(z)\\ \dots\\ F^{(p-1)}(z)\end{pmatrix}, \tag{5}$$

where $F(z)=(z^{m+p-1},\, z^{m+p-2}\,\dots,z,\,1)$. For a fixed z, the row space of this matrix represents the osculating $(p-1)$-subspace to the rational normal curve $F:\mathbf{FP}^1\to\mathbf{FP}^{m+p-1}$ at the point $F(z)$. The space $\mathrm{Poly}_{\mathbf{F}}^{mp}$ of all non-zero polynomials $q\in\mathbf{F}[z]$ of degree at most mp, up to proportionality, will be identified with $\mathbf{FP}^{mp}$ (coefficients of polynomials serving as homogeneous coordinates).

We claim that the Wrosnki map (2) $\phi: G_{\mathbf{F}}\to\mathbf{FP}^{mp}$ can be defined by the formula

$$K\mapsto\phi(K)=\det\begin{pmatrix}E(z)\\ K\end{pmatrix}\in\mathrm{Poly}_{\mathbf{F}}^{mp}, \tag{6}$$

where K is a matrix of size $m\times(m+p)$ representing a point in the Grassmannian $G_{\mathbf{F}}$. First of all it is clear that (6) indeed defines a map $G_{\mathbf{F}}\to\mathbf{FP}^{mp}$: changing K to UK, $U\in GL(m,\mathbf{F})$, will result in multiplication of the polynomial $\phi(K)$ by $\det U$. Furthermore, this map (6), when expressed in terms of Plücker coordinates, coincides with the restriction to $G_{\mathbf{F}}$ of a projection of the form π_S as in (3), with some center which we call S_0. We do not need the explicit equations of S_0, but they can be obtained by expanding the determinant in (6) with respect to the last m rows, and collecting the terms with equal powers of z.

Now we verify that polynomial $\phi(K)$ in (6) is a Wronski determinant. To see this, it is enough to consider the "big cell" X of the Grassmannian $G_{\mathbf{F}}$, which is represented by the matrices K whose rightmost minor is different from zero. We can normalize K to make the rightmost $m\times m$ submatrix the unit matrix. If the remaining (leftmost) p columns of K are $(k_{i,j})$, $1\leq i\leq m,\ 1\leq j\leq p$, then

$$\phi(K)=W(f_{1,K},\ldots,f_{p,K}),$$

where

$$\begin{array}{lcl} f_{1,K}(z) & = & z^{m+p-1}-k_{1,1}z^{m-1}-\ldots-k_{m,1},\\ f_{2,K}(z) & = & z^{m+p-2}-k_{1,2}z^{m-1}-\ldots-k_{m,2},\\ \ldots & & \ldots\\ f_{p,K}(z) & = & z^{m}-k_{1,p}z^{m-1}-\ldots-k_{m,p}. \end{array} \tag{7}$$

Coefficients of these polynomials correspond to p linear forms that define the row space of the matrix $K=[(k_{ij}),I]$. This proves our claim that (6) coincides with the Wronski map.

For $p=2$, this interpretation of the Wronski map as a projection is due to L. Goldberg [8]. Her notation for Catalan numbers is different from our present notation.

In this paper we study the *real* map ϕ, that is we set $\mathbf{F}=\mathbf{R}$. One motivation of this study is the following conjecture due to B. and M. Shapiro: *If $w\in\mathrm{Poly}_{\mathbf{R}}^{mp}$ is a polynomial all of whose roots are real, then the full preimage $\phi^{-1}(w)$ of this polynomial consists of real points.* In [3] we proved this conjecture in the first non-trivial case $\min\{m,p\}=2$. On the other hand, when m and p are both even, there are polynomials $w\in\mathrm{Poly}_{\mathbf{R}}^{mp}$ which do not have real preimages under the Wronski map. So it was natural to ask the question, whether for some m and p one can give a lower estimate for the number of real preimages. To asnwer this question, we compute in this paper the topological degree of the real Wronski maps.

Notice the following important property of ϕ: it sends the big cell X of the Grassmannian into the big cell Y of the projective space consisting of those polynomials whose degree is exactly mp. Moreover, it sends the complement $G_{\mathbf{F}}\backslash X$ into $\mathbf{FP}^{mp}\backslash Y$. When $\mathbf{F}=\mathbf{R}$ these cells X and Y can be identified with $\mathbf{R}^{mp}$, in particular they are orientable, and the restriction of ϕ to X is a smooth map

$$\phi : X \to Y, \quad \phi(\partial X) \subset \partial Y. \tag{8}$$

To define the *degree* of such map (see, for example [14]), we fix some orientations on X and Y. Then choose a regular value $y \in Y$ of ϕ, which exists by Sard's theorem, and define

$$\deg \phi = \pm \sum_{x \in \phi^{-1}(y)} \operatorname{sgn} \det \phi'(x), \tag{9}$$

using local coordinates in X consistent with the chosen orientation of X, and any local coordinate at y. The degree $\deg f$ changes sign if one changes one of the orientations of X or Y; it is independent of the choice of local coordinates within the class defined by the chosen orientation of X, and of the regular value y. In Section 3 we will discuss a more general definition of degree which does not use special properties of the Wronski map and applies to all equidimensional projections of real Grassmann varieties.

To state the main result of this paper, we need a definition. Consider the sequences $\sigma = (\sigma_j)$ of length mp whose entries are elements of the set $\{1, \ldots, p\}$, and each element occurs exactly m times. Suppose that the following additional condition is satisfied: for every $n \in [1, mp]$ and every pair $i < k$ from $\{1, \ldots, p\}$,

$$\#\{j \in [1, n] : \sigma_j = i\} \geq \#\{j \in [1, n] : \sigma_j = k\}. \tag{10}$$

Such sequences are called *ballot sequences* or *lattice permutations* [13]. For given m and p, the set of all ballot sequences is denoted by $\Sigma_{m,p}$. There is a natural correspondence between $\Sigma_{m,p}$ and the set of the standard Young tableaux of rectangular shape $p \times m$ [17, Proposition 7.10.3]: we fill the shape with integers from 1 to mp putting one integer in each cell; if $\sigma_j = i$ we put the integer j to the leftmost unoccupied place in the row i. (As usual, the row number increases downwards).

Frobenius and MacMahon independently found that the cardinality of $\Sigma_{m,p}$ is $d(m, p)$, the same number as in (4), see for example, [13, Sect. III, Ch. V, 103] or [17, Proposition 7.21.6]. Of course, the coincidence of these numbers is not accidental [7, 17].

Let $\sigma \in \Sigma_{m,p}$, $\sigma = (\sigma_j)$. A pair (σ_j, σ_k) is called an *inversion* if $j < k$ and $\sigma_j > \sigma_k$. In terms of the SYT, and inversion occurs each time when for a pair of integers the greater integer of the pair stands in higher row than the smaller one. The total number of inversions in σ is denoted by $\operatorname{inv}\sigma$. Now we define

$$I(m,p) = \left| \sum_{\sigma \in \Sigma_{m,p}} (-1)^{\operatorname{inv}\sigma} \right|. \tag{11}$$

It is clear that $I(m,p) = I(p,m)$ because a pair of entries in a SYT is an inversion if and only if the same pair in the transposed SYT is not an inversion. This permits us to restrict to the case

$$m \geq p \geq 2 \tag{12}$$

in the computation of the numbers $I(m,p)$. Recently, D. White [19] proved that $I(m,p) = 0$ iff $m+p$ is even. For odd $m+p$ satisfying (12), he found that $I(m,p)$ coincides with the number of shifted standard Young tableaux (SSYT) of shape

$$\left(\frac{m+p-1}{2}, \frac{m+p-3}{2}, \ldots, \frac{m-p+3}{2}, \frac{m-p+1}{2} \right).$$

An explicit formula for the number of SSYT (see, for example, [11, Proposition 10.4]) gives $I(m,p) =$

$$\frac{1!2!\cdots(p-1)!(m-1)!(m-2)!\cdots(m-p+1)!(mp/2)!}{(m-p+2)!(m-p+4)!\cdots(m+p-2)!\left(\frac{m-p+1}{2}\right)!\left(\frac{m-p+3}{2}\right)!\cdots\left(\frac{m+p-1}{2}\right)!},$$

when $m+p$ is odd. SSYT appear in Schur's theory of projective representations of symmetric groups, see, for example, [11]. Here are some values of $I(m,p)$:

$m=$	3	4	5	6	7	8	9	10	11	12
$p=2$	1	0	2	0	5	0	14	0	42	0
$p=3$	0	2	0	12	0	110	0	1274	0	17136
$p=4$		0	12	0	286	0	12376	0	759696	...
$p=5$			0	286	0	33592	0	8320480	0	...

When $p = 2$ and m is odd, $I(m,2) = d((m-1)/2, 2)$, a Catalan number. The main result of this paper is

Theorem 1 *The degree of the real Wronski map* (6) *is* $\pm I(m,p)$.

Corollary 2 *If* $m+p$ *is odd then the real Wronski map* (6) *is surjective; a generic point* $y \in \mathbf{RP}^{mp}$ *has at least* $I(m,p)$ *real preimages.* $\square$

It follows from Theorem A, that for all $y \in \mathrm{Poly}_{\mathbf{R}}^{mp}$,

$$\mathrm{card}\, \phi^{-1}(y) \cap G_{\mathbf{R}}(m, m+p) \leq d(m,p). \tag{13}$$

This estimate is best possible for every m and p, see [16], or the remark at the end of Section 2.

If both p and m are even, then Theorem 1 gives $\deg G_{\mathbf{R}}(m, m+p) = 0$, and in fact in this case the preimage in Corollary 2 may be empty, as examples in [5] show.

The lower bound in Corollary 2 with $p = 2$ is best possible, as the following example given in [4] shows:

Example 3 *For* $p = 2$ *and every odd* m*, there exist regular values* $y \in \mathbf{RP}^{2m}$ *such that the cardinality of* $\phi^{-1}(y)$ *is* $I(m,2) = d((m-1)/2, 2)$. $\square$

To each p-vector $(f_1, \ldots, f_p)$ of linearly independent polynomials one can associate a rational curve $f = (f_1 : \ldots : f_p)$ in $\mathbf{FP}^{p-1}$, whose image is not contained in any hyperplane. The following *equivalence relation* on the set of rational curves corresponds to the equivalence relation on the p-vectors of polynomials:

$$f \sim g \quad \text{if} \quad f = \ell \circ g, \quad \text{where } \ell \text{ is an automorphism of} \quad \mathbf{FP}^{p-1}. \tag{14}$$

If $(f_1, \ldots, f_p)$ is a coprime p-vector of polynomials, then the roots of $W(f_1, \ldots, f_p)$ coincide with finite inflection points of the curve f. Notice that $G_{\mathbf{R}} \subset G_{\mathbf{C}}$ can be represented by p-vectors of real polynomials, and to each such p-vector corresponds a real curve f. When $p = 2$, $f = f_2/f_1$ is a rational function. If the pair (f_1, f_2) is coprime, roots of $W(f_1, f_2)$ are the finite critical points of f. Thus our Theorem 1 has the following

Corollary 4 *Let* X *be a set of* mp *points in general position in* $\overline{\mathbf{C}}$*, symmetric with respect to* $\mathbf{R}$*. Then the number* k *of equivalence classes of real* *rational curves in* $\mathbf{RP}^{p-1}$ *of degree* $m+p-1$ *whose sets of inflection points coincide with* X *satisfies* $k \geq I(m,p)$*. In particular, for* $p=2$*, this number* k *satisfies*

$$0 \leq k \leq d(m,2), \quad \textit{if } m \textit{ is even, and} \tag{15}$$

$$d((m-1)/2,2) \leq k \leq d(m,2), \quad \textit{if } m \textit{ is odd.} \tag{16}$$

Examples in [4] show that for every m, the lower estimates in (15) and (16) are best possible. So when m is odd, the Wronski map $\phi : G_{\mathbf{R}}(m,m+2) \to \mathbf{RP}^{2m}$ is surjective, while for even m it is not. □

We prove Theorem 1 in Section 2. In Section 3 we discuss the definition of degree for arbitrary projections of real Grassmann varieties and interpret Theorem 1 in terms of control theory.

For the case $p=2$, the results of this paper were obtained in [4], with a different method based on [3]. We thank S. Fomin, Ch. Krattenthaler, F. Sottile and R. Stanley for helpful suggestions.

2 Computation of degree

In this section we prove Theorem 1. We fix integers $m,p \geq 2$. Consider vectors of integers $\mathbf{k} = (k_1,\ldots,k_p)$ satisfying

$$0 \leq k_1 < k_2 < \ldots < k_p < m+p,$$

and vectors of real polynomials $\mathbf{q} = (q_1,\ldots,q_p)$ of the form

$$\begin{array}{lcl} q_1(z) & = & z^m + a_{1,m-1}z^{m-1} + \ldots + a_{1,k_1}z^{k_1}, \\ q_2(z) & = & z^{m+1} + a_{2,m}z^m + \ldots + a_{2,k_2}z^{k_2}, \\ \ldots & & \ldots \\ q_p(z) & = & z^{m+p-1} + a_{p,m+p-2}z^{m+p-2} + \ldots + a_{p,k_p}z^{k_p}. \end{array} \tag{17}$$

Suppose that all coefficients a_{ij}, $k_i \leq j \leq m+i-2$, $1 \leq i \leq p$, are positive, all roots of the Wronskian $W_{\mathbf{q}} = W(q_1,\ldots,q_p)$ belong to the semi-open interval $(-1,0] \subset \mathbf{R}$, and those roots on the open interval $(-1,0)$ are simple. The set of all such polynomial vectors $\mathbf{q}$ will be denoted by $b(\mathbf{k})$. The greatest common factor of $\{q_1,\ldots,q_p\}$ is z^{k_1}.

It is easy to see that $b(\mathbf{k})$ parametrizes a subset of the big cell of the Grassmannian $G_{\mathbf{R}}(m,m+p)$: the representation of a point of $G_{\mathbf{R}}(m,m+p)$

by a vector from $b(\mathbf{k})$ is unique. Setting $k_i = i-1,\ 1 \le i \le p$, we obtain an open subset $b(0,1\ldots,p-1) \subset G_{\mathbf{R}}(m,m+p)$. We define

$$k = k_1 + (k_2 - 1) + \ldots + (k_p - p + 1) \ge 0.$$

It is easy to see that k is the multiplicity of the root of $W_{\mathbf{q}}$ at 0 for $\mathbf{q} \in b(\mathbf{k})$. Using coefficients of $\mathbf{q}$ as coordinates, we can identify $b(\mathbf{k})$ with a subset of $\mathbf{R}^{mp-k}$, and introduce an orientation by ordering these coefficients:

$$a_{1,k_1}, \ldots, a_{1,m-1}, a_{2,k_2}, \ldots, a_{2,m}, \ldots, a_{p,m+p-2}. \tag{18}$$

In this sequence, coefficients of q_j precede coefficients of q_k for $j < k$, and coefficients of one polynomial q_j are ordered according to their second subscript. It is useful to place these coefficients into a Young diagram Y with p rows, such that coefficients of q_i are in the i-th row, their second subscript decreasing left to right. For $\mathbf{q} \in b(\mathbf{k})$, we denote the negative roots of the Wronskian $W = W_{\mathbf{q}}$ by

$$-x_{mp-k} < -x_{mp-k-1} < \ldots < -x_1. \tag{19}$$

In addition to these, there is a root of multiplicity k at 0. We denote by $\Delta_{\mathbf{q}}$ the Jacobi matrix of the map $b(\mathbf{k}) \to \mathrm{Poly}_{\mathbf{R}}^{mp-k}$, $\mathbf{q} \mapsto W_{\mathbf{q}}$, using coordinates (18) in the domain and $\mathbf{x} = (x_1, x_2, \ldots, x_{mp-k})$ in the range, where $-x_j$ are the negative roots of $W_{\mathbf{q}}$ as in (19). So the i-th row of this matrix $\Delta_{\mathbf{q}}$ corresponds to x_i, and the j-th column to the j-th term of the sequence (18). When $k = 0$, so that $\mathbf{k} = (0,1,\ldots,p-1)$, and $b(\mathbf{k})$ is an open subset of $G_{\mathbf{R}}(m,m+p)$, we have $\Delta_{\mathbf{q}} = \phi'(\mathbf{q})$, the derivative of the Wronski map with respect to the chosen coordinates.

For example, $b(m-1, m+1, m+2\ldots, m+p-1)$ consists of vectors

$$q_1(z) = z^m + a_{1,m-1}z^{m-1}, \quad q_2(z) = z^{m+1}, \ldots, q_p(z) = z^{m+p-1}, \tag{20}$$

the Wronskian is

$$W(z) = (p-2)!z^{mp-1}\left((p-1)z + p!a_{1,m-1}\right),$$

and its only negative root is $-p!a_{1,m-1}/(p-1)$. So

$$\det \Delta_{\mathbf{q}} = p!/(p-1) > 0 \quad \text{for} \quad \mathbf{q} \in b(m-1, m+1, \ldots, m+p-1). \tag{21}$$

This example will be later used as a base of induction.

We denote by E the set of all increasing homeomorphisms $\epsilon : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$, $\epsilon(t) < t$ for $t > 0$. Let $n \geq 1$ be an integer, and $\epsilon \in E$. A *thorn* $T(n, \epsilon)$ in $\mathbf{R}^n$ is defined as

$$\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n_{>0} : \ x_j < \epsilon(x_{j+1}),\ 1 \leq j \leq n-1,\ x_n < \epsilon(1)\}. \quad (22)$$

Notice that this definition depends on the ordering of coordinates in $\mathbf{R}^n$. We always assume that this ordering corresponds to the increasing order of subscripts.

Lemma 5 *Intersection of any finite set of thorns in $\mathbf{R}^n$ is a thorn in $\mathbf{R}^n$.*

Proof. Take the minimum of their defining functions ϵ. $\square$

Lemma 6 *Let $T = T(n, \epsilon)$ be a thorn in $\mathbf{R}^n = \{(x_1, \ldots, x_n)\}$, and U its neighborhood in $\mathbf{R}^{n+1} = \{(x_0, x_1, \ldots, x_n)\}$. Then $U^+ = U \cap \mathbf{R}^{n+1}_{>0}$ contains a thorn $T(n+1, \epsilon_1)$.*

Proof. There exists a continuous function $\delta : T \to \mathbf{R}_{>0}$, such that U^+ contains the set $\{(x_0, \mathbf{x}) : \mathbf{x} \in T, 0 < x_0 < \delta(\mathbf{x})\}$. Let $\delta_0(t)$ be the minimum of δ on the compact subset $\{\mathbf{x} \in \overline{T(n, \epsilon/2)} : x_1 \geq t\}$ of T. Then there exists $\epsilon_0 \in E$ with the property $\epsilon_0 < \delta_0$. If we define $\epsilon_1 = \min\{\epsilon/2, \epsilon_0\}$, then $T(n+1, \epsilon_1) \subset U^+$. $\square$

Lemma 7 *Let $T = T(n+1, \epsilon)$ be a thorn in $\mathbf{R}^{n+1}$, and $h : T \to \mathbf{R}^{n+1}_{>0}$, $(x_0, \mathbf{x}) \mapsto (y_0(x_0, \mathbf{x}), \mathbf{y}(x_0, \mathbf{x}))$, a continuous map with the properties: for every $\mathbf{x}$ such that $(x_0, \mathbf{x}) \in T$ for some $x_0 > 0$, the function $x_0 \mapsto y_0(x_0, \mathbf{x})$ is increasing, and $\lim_{x_0 \to 0} \mathbf{y}(x_0, \mathbf{x}) = \mathbf{x}$. Then the image $h(T)$ contains a thorn.*

Proof. We consider the region $D \in \mathbf{R}^{n+1}$ consisting of T, its reflection T' in the hyperplane $x_0 = 0$ and the interior with respect to this hyperplane of the common boundary of T and T'. The map h extends to T' by symmetry: $h(-x_0, \mathbf{x}) = -h(x_0, \mathbf{x}), (x_0, \mathbf{x}) \in T$, and then to the whole D by continuity. It is easy to see that the image of the extended map contains a neighborhood U of the intersection of D with the hyperplane $x_0 = 0$. This intersection is a thorn T_1 in $\mathbf{R}^n = \{(x_0, \mathbf{x}) \in \mathbf{R}^{n+1} : x_0 = 0\}$. Applying Lemma 6 to this thorn T_1, we conclude that U^+ contains a thorn. $\square$

Given an increasing homeomorphism $\epsilon \in E$, we define $w(k, \epsilon) \subset \mathrm{Poly}^{mp}_{\mathbf{R}}$ as the set of all real monic polynomials of degree mp with $mp - k$ negative roots as in (19), these roots satisfying (22) with $n = mp - k$, and a root of multiplicity k at 0. Thus $w(k, \epsilon)$ is parametrized by a thorn $T(mp-k, \epsilon)$ in $\mathbf{R}^{mp-k}$.

Starting with $b(m-1, m+1, \ldots, m+p-1)$, we will generate subsets of $b(\mathbf{k})$ by performing the following operations F^i, $1 \leq i \leq p$, whenever they are defined. Suppose that for some $i \in \{1, \ldots, p\}$ and some multiindex $\mathbf{k}$, the following condition is satisfied:

$$i > 1 \quad \text{and} \quad k_i > k_{i-1} + 1, \quad \text{or} \quad i = 1 \quad \text{and} \quad k_1 > 0. \tag{23}$$

Notice that for given $\mathbf{k}$, this condition is satisfied with *some* $i \in \{1, \ldots, p\}$ iff $k > 0$. If (23) holds, we define a family of operators $F^i : b(\mathbf{k}) \to b(\mathbf{k} - \mathbf{e}_i)$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbf{R}^p$, by

$$\mathbf{q} \mapsto F^i_a(\mathbf{q}) = (\mathbf{q} + a z^{k_i - 1} \mathbf{e}_i), \tag{24}$$

where $a > 0$ is a small parameter, whose range may depend on $\mathbf{q}$. Thus an operation F^i leaves all polynomials in $\mathbf{q}$, except q_i, unchanged. The following Proposition shows, among other things, that F^i are well defined if the range of a is appropriately restricted.

Proposition 8 *Suppose that for some $\epsilon \in E$, and $\mathbf{k}$ and i satisfying* (23)*, a set $U \subset b(\mathbf{k})$ is given, such that the map $\mathbf{q} \mapsto W_{\mathbf{q}} : U \to w(k, \epsilon)$ is surjective, and*

$$\det \Delta_{\mathbf{q}} \neq 0 \quad \textit{for} \quad \mathbf{q} \in U. \tag{25}$$

Then there exist $\epsilon^* \in E$ and a set $U^* \subset b(\mathbf{k}^*)$, where $\mathbf{k}^* = \mathbf{k} - \mathbf{e}_i$, with the following properties. Every $\mathbf{q}^* \in U^*$ has the form $F^i_a(\mathbf{q})$ where F^i_a is defined in (24)*, $\mathbf{q} \in U$, and $a > 0$;*

$$\textit{the map} \quad \mathbf{q}^* \mapsto W_{\mathbf{q}^*} : U^* \to w(k-1, \epsilon^*) \quad \textit{is surjective,} \tag{26}$$

and $\det \Delta_{\mathbf{q}^*} \neq 0$ for $\mathbf{q}^* \in U^*$. Moreover,

$$\operatorname{sgn} \det \Delta_{\mathbf{q}^*} = (-1)^{\chi(\mathbf{k}, i)} \operatorname{sgn} \det \Delta_{\mathbf{q}}, \tag{27}$$

for every $\mathbf{q}^* \in U^*$ and every $\mathbf{q} \in U$, where $\chi(\mathbf{k}, i)$ is the number of terms in the sequence (18) *whose first subscript is less than i. In other words, $\chi(\mathbf{k}, i)$ is the total number of cells in the rows 1 to $i-1$ in the Young diagram Y described after* (18).

Proof. Let us fix $\mathbf{q} \in U$, and put $W = W_{\mathbf{q}}$. As $W \in w(k,\epsilon)$, we have $\operatorname{ord} W = k$, where ord denotes the multiplicity of a root at 0. Let cz^k be the term of the smallest degree in $W(z)$. Then $c > 0$, because all roots of W are non-positive. In fact,

$$c = \prod_{j>l}(k_j - k_l)\prod_{j} a_{j,k_j} > 0. \tag{28}$$

We define $W^* = W_{\mathbf{q}^*}$, where $\mathbf{q}^* = F_a^i(\mathbf{q})$. Then $\operatorname{ord} W^* = k-1$ and the term of the smallest degree in $W^*(z)$ is $c^* z^{k-1}$, where

$$c^* = a\prod_{j>l}(k_j^* - k_l^*)\prod_{j\neq i} a_{j,k_j} > 0. \tag{29}$$

We conclude that when a is small enough (depending on $\mathbf{q}$), the Wronskian W^* has one simple root in a neighborhood of each negative root of W, and in addition, one simple negative root close to zero, and a root of multiplicity $k-1$ at 0. To make this more precise, we denote the negative roots of W and W^* by

$$-x_n < \ldots < -x_1 \quad \text{and} \quad -y_n < \ldots < -y_1 < -y_0, \tag{30}$$

where $n = 2m - k$, and $y_j = y_j(a)$. We have

$$y_j(0) = x_j, \quad \text{for} \quad 1 \leq j \leq n, \quad \text{and} \quad y_0(0) = 0. \tag{31}$$

Furthermore, if a is small enough (depending on $\mathbf{q}$)

$$a \mapsto y_0(a) \quad \text{is increasing and continuous.} \tag{32}$$

The set $w(k,\epsilon)$ is parametrized by a thorn $T = T(n,\epsilon)$, where $\mathbf{x} = (x_1,\ldots,x_n)$, and $n = mp - k$. There exists a continuous function $\delta_0 : T \to \mathbf{R}_{>0}$, such that

$$\mathbf{q}^* \in b(\mathbf{k}^*), \quad \text{for} \quad a \in (0, \delta_0(\mathbf{x})), \quad \mathbf{x} \in T. \tag{33}$$

Now we are going to compare $\det \Delta_{\mathbf{q}}$ with $\det \Delta_{\mathbf{q}^*}$. For this purpose we investigate the asymptotic behavior of the 'new root' $y_0(a)$ of W^*, as $a \to 0$. Comparison of the terms of the lowest degrees in $W(z)$ and $W^*(z)$, (28) and (29) show that

$$-y_0(a) = -c^*/c + o(a) = -c_{\mathbf{k}} a / a_{i,k_i} + o(a), \quad a \to 0, \tag{34}$$

where $c_{\mathbf{k}} > 0$ depends only on the multiindex $\mathbf{k}$.

The Jacobi matrix $\Delta^* = \Delta_{\mathbf{q}^*}$ is obtained from the Jacobi matrix $\Delta = \Delta_{\mathbf{q}}$ by adding the top row, corresponding to y_0, and a column, corresponding to $a_{i,k_i-1} = a$. The position of the added column is

$$1 + \chi(\mathbf{k}, i),$$

where $\chi(\mathbf{k}, i)$ is the number of terms of the sequence (18) whose first subscript is less than i.

According to (34), the intersection of the added row with the added column contains the only essential element of this row:

$$\partial y_0 / \partial a = c_{\mathbf{k}} / a_{i,k_i} + o(1), \quad a \to 0.$$

The rest of the elements of the first row of Δ^* are $o(1)$ as $a \to 0$. Expanding Δ^* with respect to its first row, we obtain

$$\det \Delta^* = (-1)^{\chi(\mathbf{k},i)} \, (c_{\mathbf{k}} / a_{i,k_i}) \det \Delta + o(1), \quad a \to 0.$$

Now it follows from our assumption (25) that for sufficiently small a, $\Delta^* \neq 0$. Moreover, (27) holds, if a is sufficiently small. More precisely, for every $\mathbf{q} \in U$ there exists $\delta_1(\mathbf{q}) > 0$ such that for $0 < a < \delta_1(\mathbf{q})$ we have $\det \Delta^* \neq 0$, and (27). Taking $\delta = \min\{\delta_0, \delta_1\}$, where δ_0 was defined in (33), we obtain the set

$$U^* = \{\mathbf{q}^* = F_a(\mathbf{q}_{\mathbf{x}}) : \mathbf{x} \in T, a \in (0, \delta(\mathbf{x}))\} \subset b(\mathbf{k}^*), \tag{35}$$

and this set U^* satisfies (27). Here $\mathbf{q}_{\mathbf{x}} \in U$ is some preimage of $W_{\mathbf{x}} \in w(k, \epsilon) \cong T$ under the map $\mathbf{q} \to W_{\mathbf{q}}$. Such preimage exists by assumption of Proposition 8 that the map $\mathbf{q} \mapsto W_{\mathbf{q}}$, $U \to w(k, \epsilon)$ is surjective. It remains to achieve (26) by modifying the thorn T. This we do in two steps. First we apply Lemma 6 to the half-neighborhood (35) of T, with $x_0 = a$, to obtain a thorn $T_1(n+1, \epsilon_1)$ in $\mathbf{R}^{n+1}$. Then we apply Lemma 7 to the map $h : T_1 \to \mathbf{R}^{n+1}_{>0}$, defined by $y_j = y_j(x_0, \mathbf{x})$, where y_j are as in (30), and $x_0 = a$. This map h satisfies all conditions of Lemma 7 in view of (31) and (32). This proves (26). $\square$

Conclusion of the proof of Theorem 1. We begin with a brief outline of our argument. For each $\mathbf{k}$ and i satisfying (23), equation (24) defines an operator depending on parameter a: $F_a^i : b(\mathbf{k}) \to b(\mathbf{k} - \mathbf{e}_i)$. Starting from a subset of $b(m-1, m+1, \ldots, m+p-1)$, we will consecutively apply operators F^i in all possible sequences allowed by (23). In the end we will obtain a set of polynomial p-vectors in $b(0,1,\ldots,p-1)$, which will contain the full preimage of a point under the Wronski map. Equations (27) will permit to control the sign of the Jacobian determinant of the Wronski map at all points of this preimage.

Now we give the details. Consider the set of all finite (non-empty) sequences $\sigma=(\sigma_j)$, where $\sigma_j\in\{1,\ldots,p\},\ j\in\mathbf{N}$, satisfying (10). For every such sequence we define

$$\mathbf{k}(\sigma)=(k_1,\ldots,k_p),\quad \text{where}\quad k_i=m+i-1-\#\{j:\sigma_j=i\},$$

and

$$k(\sigma)=\sum_{i=1}^{p}k_i(\sigma)-i+1.$$

Let $\Sigma=\Sigma(m,p)$ be the set of all sequences σ satisfying (10) and $k(\sigma)\geq 0$. Notice that for $\mathbf{k}=\mathbf{k}(\sigma)$, condition (23) holds with some $i\in\{1,\ldots,p\}$ if and only if $k(\sigma)>0$.

To each sequence $\sigma\in\Sigma$ we put into correspondence an open set $U_\sigma\subset b(\mathbf{k}(\sigma))$ in the following way. For $\sigma=(1)$, we set

$$U_{(1)}=\{\mathbf{q}\in b(m-1,m+1,\ldots,p-1):W_{\mathbf{q}}\in w(1,\epsilon_0)\},$$

where $\epsilon_0(x)=x$. Then $U_{(1)}$ consists of the polynomial vectors of the form (20) with and $a_{1,m-1}\in(0,(p-1)/p!)$.

Applying operations F^i to $U_{(1)}$ means that we use Proposition 8 with $U=U_{(1)}$, and $\mathbf{k}=(m-1,m+1,\ldots,p-1)$. We obtain from this Proposition the sets U^*, which we call $U_{(1,i)}$. In fact, Proposition 8 can we applied in this situation only with $i=1$ or $i=2$. Then we apply operations $F^j,\ j\in\{1,\ldots,p\}$ to $U_{(1,i)}$, whenever permitted by (23) and so on.

In general, suppose U_σ is already constructed. If $\mathbf{k}(\sigma)$ and i satisfy (23), we apply operation F^i to U_σ. This means that we use Proposition 8 with $U=U_\sigma$, $\mathbf{k}=\mathbf{k}(\sigma)$ and this i. The resulting U^* is called $U_{(\sigma,i)}\subset b(\mathbf{k}(\sigma)-\mathbf{e}_i)$.

Every sequence $\sigma\in\Sigma$ encodes an admissible sequence of applications of operations F^i. If $\sigma=(\sigma_j)$, then $\sigma_j=i$ indicates that F^i was applied on the j-th step. Conditions (10) and $k(\sigma)>0$ imply (23) with some i, so that an operation F^i is applicable. Every operation decreases $k(\sigma)$ by 1, so the procedure stops when $k(\sigma)=0$.

Proposition 8 implies that for each $\sigma \in \Sigma$ with $k(\sigma) \geq 0$, there exists $\epsilon_\sigma \in E$ such that

$$W : U_\sigma \to w(k(\sigma), \epsilon_\sigma) \tag{36}$$

is surjective and unramified.

Observe that we can always replace ϵ^* in Proposition 8 by a smaller function from the set E. We use this observation to arrange that the coefficient, added to polynomials in $\mathbf{q}$ on each step, is strictly smaller than all coefficients added on the previous steps. This implies that for each $\mathbf{q} \in U_\sigma$, all coefficients are strictly ordered, and the sequence σ can be recovered from this order. More precisely, let $k = k(\sigma)$, and $c_1 > c_2 > \ldots > c_{2m-k} > 0$ be the ordering of the sequence of coefficients of $q_1 \ldots, q_p$. Then $\sigma_j = i$ if $c_j = a_{i,l}$ with some l. In other words, enumerating the cells of the Young diagram Y defined after (18) in the order of decrease of their entries gives a standard Young tableau. The sequence σ can be recovered from this tableau in a unique way.

We recall that the number of inversions $\operatorname{inv}\sigma$ was defined in the introduction, just before the equation (11). We claim that for every $\sigma \in \Sigma$,

$$\operatorname{sgn}\,\det \Delta_{\mathbf{q}} = \mu(\sigma)(-1)^{\operatorname{inv}\sigma} \quad \text{if} \quad \mathbf{q} \in U_\sigma, \tag{37}$$

where $\mu(\sigma) = \pm 1$ depends only on the length of σ. Indeed, by (27), on each step the sign of $\det \Delta$ is multiplied by $(-1)^\chi$, where $\chi = \chi(\mathbf{k}(\sigma), i)$ is the number of terms of σ which are less than i. This proves (37).

Now we consider the subset

$$\Sigma_{m,p} = \{\sigma \in \Sigma_m : k(\sigma) = 0\}.$$

It consists of ballot sequences, as defined in the introduction. The set $\Sigma_{m,p}$ corresponds to rectangular standard Young tableaux of the shape $p \times m$. The number of such tableaux is $d(m,p)$ (see, for example, [17, Proposition 7.21.6]). Sequences $\sigma \in \Sigma_{m,p}$ generate $d(m,p)$ open sets $U_\sigma \subset b(0, 1, \ldots, p-1)$ with the property that the maps (36) are surjective and unramified. Using Lemma 5, we restrict these maps so that they have a common range $w(0, \epsilon)$ with some $\epsilon \in E$.

As all maps (36) are surjective, every point from this common range has at least one preimage under the Wronski map in each $U_\sigma, \sigma \in \Sigma_{m,p}$. All these $d(m,p)$ preimages are different as elements of $b(0, 1, \ldots, p-1)$, because the sequence σ can be recovered from the ordered sequence of coefficients of $\mathbf{q} \in U_\sigma$. Furthermore, all these $d(m,p)$ polynomial vectors represent different points in the Grassmannian $G_{\mathbf{R}}(m, m+p)$, because to each point in $b(0,1,\ldots,p-1)$ corresponds only one point of $G_{\mathbf{R}}(m, m+p)$. Thus we found $d(m,p)$ different preimages of a point under the Wronski map. On the other hand, the complex Wronski map has degree $d(m,p)$ by Theorem A, so we found all preimages of the real or complex Wronski map. Equation (37) gives the signs of Jacobian determinants at these points, so the degree of the Wronski map is given by (11) $\square$

Remark. In the process of this proof, we constructed a point in $\mathbf{RP}^{mp}$ which has $d(m,p)$ distinct real preimages under the Wronski map. This proves the fact earlier established by Sottile [16], that the upper estimate $d(m,p)$ given by (13), is best possible for every m and p.

3 Additional comments

1. Let us show how to define topological degree (an unsigned integer) for arbitrary projections of real Grassmann varieties as in (3) with $\mathbf{F} = \mathbf{R}$.

Let $f : X \to Y$ be a smooth map of compact, connected real manifolds of equal dimensions. If X is orientable, the degree $\deg f$ can be defined by formula (9). If X is orientable but Y is not then $\deg f = 0$.

Now we suppose that both X and Y are non-orientable and consider canonical orientable 2-to-1 coverings $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$, which are called the *spaces of orientations* of X and Y [1, 10.2]. The set $\widetilde{X}$ consists of pairs (x, O) where $x \in X$ and O is one of the two orientations of the tangent space T_x. There is a unique structure of smooth manifold on X which makes the map $\widetilde{X} \to X$, $(x, O) \mapsto x$ a covering, and O depends continuously on x. The group of the covering $\widetilde{X} \to X$ is $\{\pm 1\}$. Notice that the spaces $\widetilde{X}$ and $\widetilde{Y}$ have canonical orientations. A map $f : X \to Y$ is called *orientable* if there exists a lifting $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$, which commutes with the action of $\{\pm 1\}$, [1, (10.2.5)]. A different but equivalent definition of an orientable map is given in [15, §5].

For an orientable map, we define $\deg f := \pm \deg \widetilde{f}$. Under our assumption that X and Y are connected, this degree is defined up to sign, which depends on the choice of the lifting. Though Y is connected, $\widetilde{Y}$ may consist of one or two components, but the degree is independent of the choice of the regular value $y \in \widetilde{Y}$.

Suppose that there exists a regular value $y \in Y$ and an affine chart $U \subset X$, so that $f^{-1}(y) \subset U$. Then we can compute the sum (9) using coordinates in U. The orientability of f ensures that this sum is independent of the choice of the chart U and coincides with $\deg f$.

To apply this construction to projections of Grassmann varieties

$$\phi_S : G_{\mathbf{R}}(m, m+p) \to \mathbf{RP}^{mp},$$

we recall that $\mathbf{RP}^{mp}$ is orientable iff mp is odd, and $G_{\mathbf{R}}(m, m+p)$ is orientable iff $m+p$ is even (see, for example, [6, Ch. 3 §2]). So in the case that $m+p$ is even, the degree of ϕ_S is defined in the usual sense, as in (9).

To deal with the case when $m+p$ is odd, we first identify the space of orientations of a Grassmannian $G_{\mathbf{R}}(m, n)$, with odd n. Consider the "upper Grassmannian" $G^+_{\mathbf{R}}(m, n)$, which consists of all oriented m-subspaces in $\mathbf{R}^n$. It can be also described as the set of all $m \times n$ matrices K of maximal rank, modulo the following equivalence relation: $K' \sim K$ if $K' = UK$, where $\det U > 0$. We have the natural 2-to-1 covering $G^+_{\mathbf{R}}(m, n) \to G_{\mathbf{R}}(m, n)$, which assigns to the class of K in $G^+_{\mathbf{R}}(m, n)$ the class of the same K in $G_{\mathbf{R}}(m, n)$. We also have $G^+_{\mathbf{R}}(1, n) = (\mathbf{RP}^{n-1})^+$, a sphere of dimension $n-1$.

Every upper Grassmannian is orientable and has canonical orientation. To see this, we consider the tangent space $T_x = T_x(G^+_{\mathbf{R}}(m, n))$ which is the product of m copies of a subspace $y \cong \mathbf{R}^{n-m}$, complementary to x. Orientation of x induces a unique orientation of y, such that $x \oplus y \cong \mathbf{R}^n$ has the standard orientation. This defines an orientation on each tangent space T_x which varies continuously with x. So we have a canonical orientation of $G^+_{\mathbf{R}}(m, n)$.

We claim that for odd n, the coverings

$$G^+_{\mathbf{R}}(m, n) \to G_{\mathbf{R}}(m, n) \text{ and } \widetilde{G}_{\mathbf{R}}(m, n) \to G_{\mathbf{R}}(m, n) \quad \text{are isomorphic.} \tag{38}$$

Indeed, for $x \in G_{\mathbf{R}}(m, n)$, orientation of $x \subset \mathbf{R}^n$ defines an orientation of $T_x(G_{\mathbf{R}}(m, n))$, as explained above. One can easily show that (in the case of odd n) changing the orientation of x changes the orientation of $T_x(G_{\mathbf{R}}(m, n))$. This proves (38).

We recall that a projection map $\pi_S : \mathbf{RP}^N \backslash S \to \mathbf{RP}^k$ can be described in homogeneous coordinates as

$$y = Ax, \tag{39}$$

where A is a $(k+1) \times (N+1)$ matrix of maximal rank, and x, y are column vectors of homogeneous coordinates. The null space of A represents the center of projection $S = S(A)$ (where the map is undefined). Two matrices define the same projection if they are proportional. A change of homogeneous coordinates in $\mathbf{RP}^N$ or in the target space $\mathbf{RP}^k$ results in multiplication of A by a non-degenerate matrix from the right or left, respectively.

Proposition 9 *Let n be an odd integer, $G_{\mathbf{R}}(m,n) \subset \mathbf{RP}^N$ a Grassmann variety, and $\phi : G_{\mathbf{R}}(m,n) \to \mathbf{RP}^{m(n-m)}$ a central projection. Then ϕ is orientable.*

Proof. The Plücker embedding $\mathrm{Pl} : G_{\mathbf{R}}(m,n) \to \mathbf{RP}^N$ lifts to $G^+_{\mathbf{R}}(m,n) \to (\mathbf{RP}^N)^+$, which is defined by the same rule as Pl. Using (38), we identify $\widetilde{G}_{\mathbf{R}}(m,n)$ with $G^+_{\mathbf{R}}(m,n)$, and obtain the lifting

$$\widetilde{\mathrm{Pl}} : \widetilde{G}_{\mathbf{R}}(m,n) \to (\mathbf{RP}^N)^+ \tag{40}$$

of Pl. Suppose now that a projection π_S is defined by (39) where A is an $(m(n-m)+1) \times (N+1)$ matrix. Then the same equation (39) defines the lifting

$$\pi^+ : (\mathbf{RP}^N)^+ \backslash S^+ \to (\mathbf{RP}^{m(n-m)})^+ \cong \widetilde{\mathbf{RP}}^{m(n-m)}, \tag{41}$$

where S^+ is the preimage of S under the covering $(\mathbf{RP}^N)^+ \to \mathbf{RP}^N$, and the last isomorphism holds because $m(n-m)$ is even. Composition of the maps (40) and (41) is the desired lifting of ϕ_S, which is evidently compatible with the action of $\{\pm 1\}$. The existence of such a lifting proves that ϕ_S is orientable. □

Thus equidimensional projections of real Grassmann varieties always have well-defined degrees. It is clear that when the center of projection varies continuously, the degree does not change until the center S intersects the Grassmann variety. These exceptional centers form a subvariety Z of codimension 1 in the Grassmannian $G_{\mathbf{R}}(N+1, N-mp)$ of all centers. So the degree is constant on every component of $G_{\mathbf{R}}(N+1, N-mp) \backslash Z$, in particular, all projections ϕ_S whose centers S belong to the same component of $G_{\mathbf{R}}(N+1, N-mp) \backslash Z$ as the center S_0 of the Wronski map have the same degree $\pm I(m,p)$.

2. Now we restate our results in terms of control theory by static output feedback. Suppose that a triple of real matrices $\Sigma = (A, B, C)$ of sizes

$n \times n$, $n \times m$ and $p \times n$ is given. This triple Σ defines a *linear system*

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{aligned} \tag{42}$$

Here x, u and y are functions of time (a real variable) taking their values in $\mathbf{R}^n$, $\mathbf{R}^m$ and $\mathbf{R}^p$, respectively. The values of these functions at a point $t \in \mathbf{R}$ are interpreted as the state, input and output of our system at the moment t.

Behavior of the system (42) is completely determined by its *transfer function* $z \mapsto C(zI - A)^{-1}B$, which is a function of a complex variable z with values in the set of $p \times m$ matrices. One wishes to control a given system (42) by arranging a feedback, which means sending the output to the input via an $m \times p$ matrix K, called a *gain* matrix:

$$u = Ky. \tag{43}$$

Elimination of u and y from (42), (43) gives the *closed loop system*

$$\dot{x} = (A - BKC)x,$$

whose transfer function has poles at the zeros of the polynomial

$$\psi_K(z) = \det(zI - A - BKC). \tag{44}$$

The map $K \mapsto \psi_K \in \mathrm{Poly}^n_{\mathbf{R}}$ is called the *pole placement map*, and the problem of pole assignment is: *given a system* Σ*, and a set* $\{z_1, \ldots, z_n\}$*, symmetric with respect to* $\mathbf{R}$*, to find a real gain matrix* K*, such that the zeros of* ψ_K *are* $\{z_1, \ldots, z_n\}$. Thus for a fixed system Σ, arbitrary pole assignment is possible iff the pole placement map is surjective.

When $n > mp$, X. Wang [18] proved that for generic Σ, the pole placement map is surjective. Here we consider the case $n = mp$. We also assume that n is the smallest possible size of a matrix A in the representation $C(zI-A)^{-1}B$ of the transfer function. Systems with this property are called "controllable and observable", and and they form an open dense subset of the set of all systems with fixed (m, n, p). To understand the structure of the pole placement map, we use a coprime factorization of the open loop transfer function of a generic system Σ (see, for example, [2, Assertion 22.6]):

(45)

$$C(zI-A)^{-1}B = D(z)^{-1}N(z), \quad \det D(z) = \det(zI-A),$$

where D and N are polynomial matrix-functions of sizes $p \times p$ and $p \times m$, respectively. The polynomial matrix $[D(z), N(z)]$ has the following properties: its full size minors have no common zeros, and exactly one of these minors, $\det D(z)$, has degree n while all other minors have strictly smaller degree. Every $p \times (m+p)$ polynomial matrix with these properties is related to a linear system of the form (42) via equations (45).

Using the factorization (45) and the identity $\det(I-PQ) = \det(I-QP)$, which is true for all rectangular matrices of appropriate dimensions, we write

$$\begin{aligned}
\psi_K(z) &= \det(zI-A-BKC) = \det(zI-A)\det(I-(zI-A)^{-1}BKC) \\
&= \det(zI-A)\det(I-C(zI-A)^{-1}BK) \\
&= \det D(z)\det(I-D(z)^{-1}N(z)K) = \det(D(z)-N(z)K).
\end{aligned}$$

This can be rewritten as

$$\psi_K(z) = \begin{vmatrix} D(z) & N(z) \\ K & I \end{vmatrix} \in \mathrm{Poly}^{mp}_{\mathbf{R}}. \tag{46}$$

In the last determinant, the first p rows depend only on the given system, and the last m rows on the gain matrix. Permitting arbitrary $m \times (m+p)$ matrices $\hat{K}$ of maximal rank as the last m rows of the determinant in (46) we extend the pole placement map to

$$\phi_\Sigma : G_{\mathbf{R}}(m, m+p) \to \mathbf{RP}^{mp}, \quad \phi_\Sigma(\hat{K}) = [\psi_K], \tag{47}$$

where $[.]$ means the class of proportionality of a polynomial, which is identified with a point in $\mathbf{RP}^{mp}$, using the coefficients of a polynomial as homogeneous coordinates. The map (47) is defined if for every matrix $\hat{K}$ of rank m in the last m rows of (46) the determinant in (46) does not vanish identically. Systems Σ with this property are called *non-degenerate*, and they form an open dense subset in the set of all systems with given (m,p) and $n = mp$. Applying Laplace's expansion along the first p rows to the determinant in (46), we conclude that the map ϕ_Σ, when expressed in Plücker coordinates, is nothing but a projection of the Grassmann variety $G_{\mathbf{R}}(m, m+p)$ into $\mathbf{RP}^{mp}$ from some center depending on Σ. This interpretation of the pole placement map as a projection comes from [18]. Now we notice that all projections arising from linear systems as in (46) have the property that they send the big cell X of $G_{\mathbf{R}}(m, m+p)$ represented by matrices $\hat{K}$ of the form $[K, I]$

into the big cell Y of $\mathrm{Poly}_{\mathbf{R}}^{mp}$ consisting of polynomials of exact degree mp. Furthermore, $G_{\mathbf{R}}(m,m+p)\backslash X$ corresponds to $\mathrm{Poly}_{\mathbf{R}}^{mp}\backslash Y$ inder such projections. Our arguments in the first part of this section imply that the real pole placement maps of a non-degenerate system has a well-defined degree[1]. As the center of projection $S=S(\Sigma)$ varies continuously, this degree remains constant as long as S does not intersect the Grassmann variety. Degenerate systems are precisely those for which S intersects the Grassmann variety.

Comparing (6) with (47) we conclude that the Wronski map is a pole placement map for some special linear system. So our Corollary 2 implies

Corollary 10 *For every m and p such that $m+p$ is odd there is an open set U of linear systems with m inputs, p outputs and state of dimension mp, such that for systems in U the real pole placement map is surjective. Furthermore, the pole placement problem for systems in U has at least $I(m,p)$ real solutions for any generic set of mp poles symmetric with respect to the real line.* □

References

[1] N. Bourbaki, Variétés différentielles et analytiques. Fasc. de résultats, Paragraphes 8 à 15, Hermann, Paris, 1971.

[2] D. Delchamps, State space and input-output linear systems, Springer, NY, 1988.

[3] A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, Ann. Math., 155 (2002) 109–129..

[4] A. Eremenko and A. Gabrielov, The Wronski map and Grassmannians of real codimension 2-subspaces, Computational Methods and Function Theory, to appear in vol. 1.

[5] A. Eremenko and A. Gabrielov, Pole placement by static output feedback for generic linear systems, to appear in SIAM Journal on Control.

[1]Projections arising from the linear systems map a fixed big cell X of the Grassmannian to a fixed big cell Y of the projective space, and also send ∂X to ∂Y. This permits to define the degrees of these projections as we did it for the Wronski map in the Introduction

[6] D. Fuks and V. Rokhlin, Beginner's course in topology. Geometric chapters. (Transl. from Russian) Springer-Verlag, Berlin-New York, 1984.

[7] W. Fulton, Young Tableaux, Cambridge UP, 1997.

[8] L. Goldberg, Catalan numbers and ramified coverings of the sphere, Adv. Math., 85 (1991) 129-144.

[9] Ph. Griffiths and J. Harris, Principles of algebraic geometry, Willey, NY, 1978.

[10] W. Hodge and D. Pedoe, Methods of algebraic geometry, v. 2, Cambridge UP, 1953.

[11] P. Hoffman and J. Humphreys, Projective representations of the symmetric groups, Clarendon Press, Oxford, 1992.

[12] S. Kleiman, Problem 15. Rigorous foundation of Schubert's enumerative calculus, in: F. Browder, ed., Mathematical development arising from Hilbert problems, Proc. Symp. Pure Math., v. 28, AMS, Providence, 1976.

[13] P. MacMahon, Combinatory Analysis, 2 vols., Cambridge UP, 1915-16 (reprinted by Chelsea, NY, in one vol. in 1960).

[14] J. Milnor, Topology from the differentiable viewpoint, UP Virginia, 1965.

[15] G. de Rham, Differentiable manifolds, (English transl.) Springer, NY 1984.

[16] F. Sottile, Special Schubert calculus is real, Electronic Res. Announcements, AMS, 5 (1999) 35-39.

[17] R. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Univ. Press, Cambridge, 1999.

[18] X. Wang, Grassmannian, central projection, and output feedback pole assignment of linear systems. IEEE Trans. Automat. Control 41 (1996), no. 6, 786–794.

[19] D. White, Sign-balanced posets, J. Combin. Theory, 95 (2001), 1-38.

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