A CAYLEY-BACHARACH THEOREM FOR POINTS IN $\mathbb{P}^n$

GIULIO CAVIGLIA AND ALESSANDRO DE STEFANI

Abstract. We prove a Cayley-Bacharach-type theorem for points in projective space $\mathbb{P}^n$ that lie on a complete intersection of n hypersurfaces. This is made possible by new bounds on the growth of the Hilbert function of almost complete intersections.

1. Introduction

Let $\mathbb{k}$ be an algebraically closed field. If C_1 and C_2 are two cubic curves in $\mathbb{P}^2_{\mathbb{k}}$ which meet in 9 points, and X is a cubic passing through 8 points of $C_1 \cap C_2$, then X contains the ninth point of $C_1 \cap C_2$ as well. This well-known statement extends Pappus's and Pascal's theorems, and it is one version of a series of results which are referred to as Cayley-Bacharach theorems. We refer the interested reader to the seminal work of Eisenbud, Green and Harris [EGH96], and to recent work of Kreuzer, Long and Robbiano [KLR19] for a detailed and fascinating history on the subject.

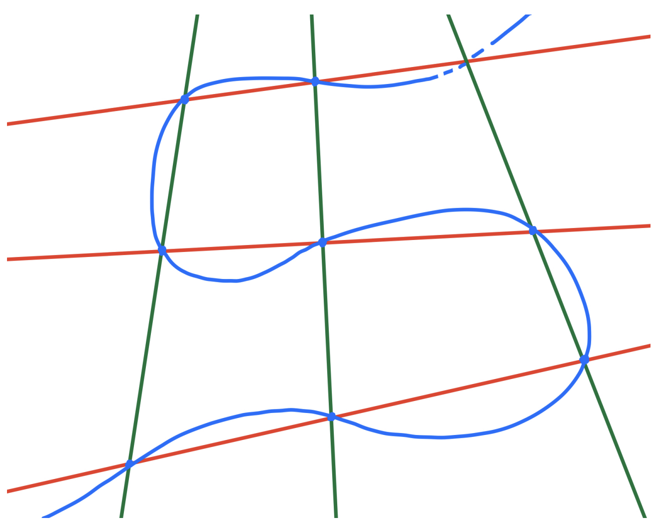

Figure 1. A sketch of the case in which C_1 and C_2 are a union of three lines.

More generally, if $C_1, C_2 \subseteq \mathbb{P}^2$ are two curves of degrees $d_1 \leqslant d_2$ meeting transversely in $d_1 d_2$ points, the Cayley-Bacharach theorem states that, if a curve X of degree $D = d_1 + d_2 - 3$ passes through all but possibly one point of $C_1 \cap C_2$, then it must contain all $d_1 d_2$ points of $C_1 \cap C_2$. In the literature, there have been several efforts to extend this theorem to a more

2010 *Mathematics Subject Classification.* Primary 14N05, 14H15, 13H15; Secondary 13A15, 13C40.

Key words and phrases. Cayley-Bacharach theorems, EGH conjecture, almost complete intersection, Hilbert function, multiplicity, lex-plus-powers ideal.

The work of the first named author was partially supported by a grant from the Simons Foundation (41000748, G.C.).

general setup [GH78, Tan00, Li19, EL19, KLR19, HLU20]. However, in most cases, the obtained results still require the hypersurface to pass through at least all but one point. In [EGH96], Eisenbud, Green and Harris suggest a different direction in which this theorem can be pushed. Namely, one can require X to contain all but a given number of points of $C_1 \cap C_2$, balancing off this additional freedom by putting more restrictions on the degree of X. This leads to a new series of conjectured inequalities on multiplicities of almost complete intersections (see [EGH96, Conjecture CB12]). More specifically, [EGH96, Conjecture CB12] can be restated by saying that the multiplicity of an almost complete intersections is bounded above by the multiplicity of a special monomial almost complete intersection of the same degrees, which in Section 2 we denote $\mathbb{L}(\mathbf{d}; D)$. By its nature, this upper bound is sharp, if true. In the literature, this improved version has been called the General Cayley-Bacharach conjecture (see [GK13]). However, in this article we will refer to the above as Cayley-Bacharach, since it is the only one we will focus on.

In $\mathbb{P}^2$, [EGH96, Conjecture CB12] and hence the Cayley-Bacharach theorem follow from a stronger conjecture of Eisenbud, Green and Harris (henceforth EGH), see Conjecture 2.1, which is known to be true in this case by [Ric04, Coo08]. In $\mathbb{P}^3$, [EGH96, Conjecture CB12] and the Cayley-Bacharach theorem have been proved by Geramita and Kreuzer [GK13, Corollary 4.4].

In Section 2, we refine the Cayley-Bacharach inequality on multiplicities of almost complete intersections of height three, and we obtain the following upper bound on their Hilbert functions.

Theorem A (see Theorem 2.8)**.** Let $S = \mathbb{k}[x_1, \ldots, x_n]$, and $\mathfrak{f} = (f_1, f_2, f_3)$ be a complete intersection of degrees $\mathbf{d} = (d_1, d_2, d_3)$. Let G be an element of degree $D \leqslant d_1 + d_2 + d_3 - 3$ such that $G \notin \mathfrak{f}$, and $\mathfrak{a} = \mathfrak{f} + (G)$ has height three. Then $\mathrm{HF}(S/\mathfrak{a}) \leqslant \mathrm{HF}(S/\mathbb{L}(\mathbf{d}; D))$, where $\mathbb{L}(\mathbf{d}; D) = (x_1^{d_1}, x_2^{d_2}, x_3^{d_3}, U_D)$ and U_D is the largest monomial with respect to the lexicographic order which has degree D and does not belong to $(x_1^{d_1}, x_2^{d_2}, x_3^{d_3})$.

Theorem A in particular gives that the multiplicity (denoted $\mathrm{e}(-)$) of an almost complete intersection of degrees $(d_1, d_2, d_3; D)$ is at most $\mathrm{e}(S/\mathbb{L}(\mathbf{d}; D))$, as conjectured in [EGH96, Conjecture CB12]. We would like to point out that the statement on Hilbert functions is stronger than the corresponding one on multiplicities. In fact, the standard techniques which usually allow one to reduce to the Artinian case might fail for this purpose (see Example 2.6). We also note that the stronger statement on Hilbert functions rather than just on multiplicities is needed in Section 3 to improve the Cayley-Bacharach theorem in $\mathbb{P}^n$, Theorem B, in a special case (see Theorem 3.10).

Using Theorem A, we immediately recover the Cayley-Bacharach theorem for points in $\mathbb{P}^3$.

Corollary A (Cayley-Bacharach in $\mathbb{P}^3$)**.** Let $\Gamma \subseteq \mathbb{P}^3$ be a complete intersection of three surfaces of degrees $\mathbf{d} = (d_1, d_2, d_3)$. If X is a surface of degree $D \leqslant \sigma = d_1 + d_2 + d_3 - 3$ which contains at least $d_1 d_2 d_3 - \mathrm{e}(S/\mathbb{L}(\mathbf{d}; D)) + 1$ points of Γ, then X contains Γ.

We refer to Section 2 for an explicit way to compute $\mathrm{e}(S/\mathbb{L}(\mathbf{d}; D))$ in terms of a new sequence $\mathsf{c} = (c_1, c_2, c_3)$, constructed from $\mathbf{d}$ and D. To give an example, if a surface of degree D in $\mathbb{P}^3$ contains at least $D^3 - D^2 + D + 1$ points of a complete intersection of three surfaces of degree D, then it must contains all D^3 of them.

An analogue of Theorem A is not known, in general, for almost complete intersections of codimension higher than three. However, a result of Francisco [Fra04] gives an upper bound on the Hilbert function of any almost complete intersection in one specific degree. In Section 3, we exhibit upper bounds for the multiplicity of almost complete intersections of any height combining a repeated use of Francisco's theorem with several other techniques (see Theorems

3.3 and 3.6). While our estimates are not in general as sharp as the ones predicted by [EGH96, Conjecture CB12], they significantly improve the best known upper bounds, due to Engheta [Eng09] and later extended by Huneke, Mantero, McCullough and Seceleanu [HMMS15], in all those circumstances in which the latter are not already sharp (see Remark 3.5).

Using these estimates, we obtain a Cayley-Bacharach-type theorem in $\mathbb{P}^n$. We refer the reader to Section 3 and Theorem 3.8 for the definition of the integer $\delta(\mathbf{d}; D)$ which appears in the statement of the theorem.

Theorem B (Cayley-Bacharach in $\mathbb{P}^n$)**.** Let $\Gamma \subseteq \mathbb{P}^n$ be a complete intersection of degrees $\mathbf{d} = (d_1, \ldots, d_n)$. If X is a hypersurface of degree $D < \sigma = \sum_{i=1}^n (d_i - 1)$ which contains at least $\delta(\mathbf{d}; D) = \prod_{i=1}^n d_i - \sum_{m=D+1}^{\tau_-} \varphi_m - \sum_{m=D+1}^{\tau^+} \varphi_m - 1$ points of Γ, then X contains Γ.

As an explicit consequence of Theorem B, if a cubic hypersurface in $\mathbb{P}^{2n}$ contains at least $3^{2n} - (6n^2 - 8n + 3)$ points of a complete intersection of $2n$ cubics, then it contains all of them.

As another application, if a hypersurface of degree $D \leqslant n$ in $\mathbb{P}^n$ contains at least $2^n - \lfloor \frac{3(n-D)^2+1}{4} \rfloor$ points of a complete intersection of n quadrics, then it contains all of them.

Finally, combining Theorem A and our new bounds on the multiplicity of almost complete intersection of any height, we improve Theorem B in case the degree D of the hypersurface X is less than d_4, see Theorem 3.10. As already pointed out, for this result it is crucial that Theorem A gives an upper bound on the Hilbert function of an almost complete intersection of codimension three, rather than on its multiplicity alone. In this scenario, when Theorem 3.10 can be applied, it drastically improves Theorem B, and it often allows one to obtain estimates which are rather close to the optimal ones of [EGH96, Conjecture CB12] (see Example 3.12).

Acknowledgments. We thank Martin Kreuzer and Lorenzo Robbiano for pointing out some inaccuracies on a previous version of the paper. We also thank the referee for useful comments and suggestions.

2. Almost complete intersections of codimension three

The goal of this section is to prove an upper bound on the Hilbert function of almost complete intersections of codimension three.

We start by setting up some notation, which will be used throughout the article. In what follows, $S = \Bbbk[x_1, \ldots, x_n]$ denotes a polynomial ring over any field $\Bbbk$. We consider the standard grading on S, that is, $\deg(x_i) = 1$ for all i. We denote by $\mathfrak{m}$ the irrelevant maximal ideal $(x_1, \ldots, x_n)S$ of S. We adopt the convention that a sum $\sum_i^j (-)$ equals zero whenever $j < i$. Given a finitely generated graded S-module M, we write $\mathrm{HF}(M)$ for the Hilbert function of M, that is, the numerical function $j \in \mathbb{Z} \mapsto \mathrm{HF}(M; j) = \dim_{\Bbbk}(M_j)$, and $\mathrm{e}(M)$ for its multiplicity. Given two graded S-modules M and N, we write $\mathrm{HF}(M) \leqslant \mathrm{HF}(N)$ to mean $\mathrm{HF}(M; j) \leqslant \mathrm{HF}(N; j)$ for all $j \in \mathbb{Z}$.

Let $\mathbf{d} = (d_1, \ldots, d_h) \in \mathbb{N}^h$, with $1 \leqslant d_1 \leqslant \cdots \leqslant d_h$. We denote by $(\mathsf{x}^{\mathbf{d}})$ the ideal $(x_1^{d_1}, \ldots, x_h^{d_h})$ of S. An ideal $\mathbb{L} \subseteq S$ is a called a $\mathbf{d}$-LPP ideal if we can write $\mathbb{L} = (\mathsf{x}^{\mathbf{d}}) + L$, where L is a lexicographic ideal (see [CM08, Definitions 4 and 5]). We now state the current version of the Eisenbud-Green-Harris conjecture [EGH93] (see [CM08]).

Conjecture 2.1 ($\mathrm{EGH}_{\mathbf{d},n}$)**.** Let I be a homogeneous ideal of $S = \Bbbk[x_1, \ldots, x_n]$, containing a regular sequence of degrees $\mathbf{d} = (d_1, \ldots, d_h)$. There exists a $\mathbf{d}$-LPP ideal containing $(\mathsf{x}^{\mathbf{d}})$ which has the same Hilbert function as I.

For $j \in \mathbb{Z}$, let $\mathbb{L}(j) = (\mathsf{x}^{\mathbf{d}}) + L$ denote the $\mathbf{d}$-LPP ideal which satisfies $\mathrm{HF}(I;j) = \mathrm{HF}(\mathbb{L}(j);j)$, and such that L is a lexsegment ideal generated in degree j. It is easy to show that I satisfies $\mathrm{EGH}_{\mathbf{d},n}$ if and only if $\mathrm{HF}(S/I;j+1) \leqslant \mathrm{HF}(S/\mathbb{L}(j);j+1)$ for all $j \in \mathbb{Z}$. Because of standard properties of lexsegment ideals, this is also equivalent to $\mathbb{L}(j)_{\geqslant j+1} \subseteq \mathbb{L}(j+1)$ for all $j \in \mathbb{Z}$, where $\mathbb{L}(j)_{\geqslant j+1}$ denotes the ideal generated by the elements of $\mathbb{L}(j)$ of degree at least $j+1$.

Conjecture $\mathrm{EGH}_{\mathbf{d},n}$ is known, among other cases, if $\mathbf{d} = (d_1, d_2)$ [Ric04, Coo08], if I contains a monomial regular sequence of degrees $\mathbf{d}$ [CL69, MP06, CK13], if the degrees of the forms in the regular sequence grow sufficiently fast [CM08], if $I = Q_1 + Q_2$, where Q_1 is generated by a regular sequence of quadrics and Q_2 is generated by general quadratic forms [HP98, Gas99], if the regular sequence factors as a product of linear forms [Abe15], and if $d_1 = \ldots = d_h = 2$ and $h \leqslant 5$ [GH19]. In general, however, the conjecture is wide-open.

One more case in which the conjecture is known is for the class of minimally licci ideals, defined by Chong [Cho16]. Chong proves that, if $\mathfrak{g} \subseteq S$ is an ideal of height three which contains a regular sequence of degrees $\mathbf{d}$ among its minimal generators, and such that $S/\mathfrak{g}$ is Gorenstein, then $\mathfrak{g}$ satisfies $\mathrm{EGH}_{\mathbf{d},n}$. The condition that the regular sequence is part of a minimal generating set for $\mathfrak{g}$ can actually be removed, as the following lemma shows.

Lemma 2.2. Let $\mathfrak{g} \subseteq \Bbbk[x_1, \ldots, x_n]$ be an ideal of height three, containing a regular sequence of degrees $\mathbf{d} = (d_1, d_2, d_3)$. If $S/\mathfrak{g}$ is Gorenstein, then $\mathfrak{g}$ satisfies $\mathrm{EGH}_{\mathbf{d},n}$.

Proof. We may harmlessly assume that $\Bbbk$ is infinite. Since $\mathfrak{g}$ contains a regular sequence of degrees $\mathbf{d}$, we can find a regular sequence f_1', f_2', f_3' of degrees $\mathbf{d}'$, with $d_i' \leqslant d_i$ for $i = 1, 2, 3$, among the minimal generators of $\mathfrak{g}$. By [Cho16, Corollary 11], there exists a $\mathbf{d}'$-LPP-ideal $\mathbb{L}$ which has the same Hilbert function as $\mathfrak{g}$. Since $\mathbb{L}$ contains $(\mathsf{x}^{\mathbf{d}'})$ which, in turn, contains $(\mathsf{x}^{\mathbf{d}})$, by [MP06, Theorem 1.2] we can find a $\mathbf{d}$-LPP ideal with the same Hilbert function as $\mathbb{L}$, and this concludes the proof. □

We now turn our attention to almost complete intersections.

Definition 2.3. Let $\mathfrak{a}$ be a homogeneous ideal of S. We say that $\mathfrak{a}$ is an almost complete intersection of degrees $(\mathbf{d}; D) = (d_1, \ldots, d_h; D)$ if $\mathrm{ht}(\mathfrak{a}) = h$, and we can write $\mathfrak{a} = \mathfrak{f} + (G)$, where the ideal $\mathfrak{f} = (f_1, \ldots, f_h)$ is generated by a regular sequence of degrees $d_1 \leqslant \cdots \leqslant d_h$, and G is an element of degree D which does not belong to $\mathfrak{f}$.

Observe that we do not require that $\mathfrak{a}$ is minimally generated by $h+1$ elements. For example, according to our definition, the ideal $\mathfrak{a} = (x_1^2, x_2^3) + (x_2^2)$ is an almost complete intersection of degrees $(2, 3; 2)$, but also a complete intersection of degrees $(2, 2)$. What is important to observe, though, is that an almost complete intersection of degrees $(\mathbf{d}; D)$ cannot be generated by a regular sequence of degrees $\mathbf{d}$.

Notation 2.4. Given integers $(\mathbf{d}; D) = (d_1, \ldots, d_h; D)$, with $D \leqslant \sum_{i=1}^h (d_i - 1)$, we let $\mathbb{L}(\mathbf{d}; D) = (\mathsf{x}^{\mathbf{d}}) + (U_D)$ be the $\mathbf{d}$-LPP ideal of $S = \Bbbk[x_1, \ldots, x_n]$ which is an almost complete intersection of degrees $(\mathbf{d}; D)$. In other words, U_D is the largest monomial with respect to the lexicographic order which has degree D, and does not belong to $(\mathsf{x}^{\mathbf{d}})$.

In order to apply Lemma 2.2 to obtain an upper bound on the multiplicity of almost complete intersections, we will use partial initial ideals with respect to the weight order $\omega = (1, 1, \ldots, 1, 0)$. For unexplained notation and terminology, we refer to [CK14] and [CDS21], where such weight order is denoted by rev_1. For convenience of the reader, we recall the main features of such an object. Let I be a homogeneous ideal in $S = \Bbbk[x_1, \ldots, x_n]$, and assume that

$\Bbbk$ is infinite. After performing a sufficiently general change of coordinates, there is a vector space decomposition $\mathrm{in}_\omega(I) = \bigoplus_{j \geqslant 0} I_{[j]} x_n^j$, where each $I_{[j]}$ is an ideal in $\overline{S} = \Bbbk[x_1, \ldots, x_{n-1}]$. This decomposition is analogous to the one in [Gre10, Section 6], where Green constructs partial elimination ideals for the lexicographic order. Observe that $I_{[0]}$ is the ideal defining the hyperplane section $S/(I + (x_n))$ viewed inside $\overline{S} \cong S/(x_n)$. In characteristic zero, the ideals $I_{[j]}$ automatically satisfy $\overline{\mathfrak{m}} I_{[j+1]} \subseteq I_{[j]}$ for all $j \geqslant 0$, where $\overline{\mathfrak{m}} = (x_1, \ldots, x_{n-1})\overline{S}$ (see [CS13, Theorem 3.2]). We will refer to this phenomenon as stability of partial general initial ideals. We may achieve this also in characteristic $p > 0$, without altering the relevant features of $\mathrm{in}_\omega(I)$, by recursively applying general distractions and partial initial ideals with respect to ω. For a description of this process, see the proof of [CK14, Theorem 4.1], or [CS16, Proposition 1.4]. We point out that, while this process may change the ideals $I_{[j]}$, it can only enlarge $I_{[0]}$.

We record these facts in a lemma, for future use.

Lemma 2.5. Let I be a homogeneous ideal in $S = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is an infinite field. With the notation introduced above, after performing a sufficiently general change of coordinates, there exist ideals $I_{[j]} \subseteq \overline{S}$ and an ideal $\widetilde{I} = \bigoplus_{j \geqslant 0} I_{[j]} x_n^j$ of S such that

- $\overline{\mathfrak{m}} I_{[j+1]} \subseteq I_{[j]}$ for all $j \geqslant 0$.
- $I + (x_n) \subseteq \widetilde{I} + (x_n)$, with equality if $\mathrm{char}(\Bbbk) = 0$.
- $\mathrm{HF}(I) = \mathrm{HF}(\widetilde{I})$.

Let $\mathfrak{a} = \mathfrak{f} + (G)$ be an almost complete intersection of degrees $(\mathbf{d}; D)$, with $D \leqslant \sum_{i=1}^h (d_i - 1)$. In order to estimate the Hilbert function and the multiplicity of $S/\mathfrak{a}$, it would be desirable to reduce to the Artinian case, without losing the relevant features of $\mathfrak{a}$. In particular, if $y \in S$ is a linear form which is regular modulo $\mathfrak{f}$, then it would be good to have that the image of G is non-zero in $S/(\mathfrak{f} + (y))$, at least for a general choice of y. While this is true if $\Bbbk$ has characteristic zero as a consequence of the proof of the forthcoming Theorem 2.7, it may be false in prime characteristic.

Example 2.6. Let $S = \Bbbk[x_1, x_2, x_3]$, with $\mathrm{char}(\Bbbk) = p > 0$, and let $\mathfrak{a} = (x_1^{p^2}, x_2^{p^2}) + (x_1 x_3^{p^2})$, which is an almost complete intersection of degrees $(p^2, p^2; p^2 + 1)$. A linear regular element for $S/(x_1^{p^2}, x_2^{p^2})$ is necessarily of the form $y = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, with $\lambda_i \in \Bbbk$, and $\lambda_3 \neq 0$. It follows that $G = x_1 x_3^{p^2}$ is zero in $S/(x_1^{p^2}, x_2^{p^2}, y)$ for any choice of y as above.

The next theorem allows us to tackle the issue illustrated by the previous example. Even if the image of G can be zero in $S/(\mathfrak{f} + (y))$ for any general linear form y, using the techniques described above we can still reduce to the Artinian case in order to estimate the Hilbert function of an almost complete intersection, even in characteristic $p > 0$.

Theorem 2.7. Let $S = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is an infinite field, and $\mathfrak{a} = \mathfrak{f} + (G)$ be an almost complete intersection of degrees $(\mathbf{d}; D) = (d_1, \ldots, d_h; D)$. If $D \leqslant \sigma = \sum_{i=1}^h (d_i - 1)$, then there exists an Artinian almost complete intersection $\overline{\mathfrak{a}} \subseteq \overline{S} = \Bbbk[x_1, \ldots, x_h]$ of degrees $(\mathbf{d}; D)$ such that $\mathrm{HF}(S/\mathfrak{a}) \leqslant \mathrm{HF}(S/\overline{\mathfrak{a}}S)$. In particular, $\mathrm{e}(S/\mathfrak{a}) \leqslant \mathrm{e}(\overline{S}/\overline{\mathfrak{a}})$.

Proof. It suffices to show the inequality on Hilbert functions and, to prove it, we proceed by induction on $\dim(S/\mathfrak{a}) \geqslant 0$. If $\dim(S/\mathfrak{a}) = 0$, then the claim is trivially true, so assume that $\dim(S/\mathfrak{a}) > 0$. After a general change of coordinates we may find a decomposition $\widetilde{\mathfrak{a}} = \bigoplus_{j \geqslant 0} \mathfrak{a}_{[j]} x_n^j$ for $\mathfrak{a}$ as in Lemma 2.5, and we may further assume that x_n is regular for $\mathfrak{f}$.

Let $S' = \Bbbk[x_1,\ldots,x_{n-1}]$, and $\mathfrak{m}' = (x_1,\ldots,x_{n-1})S'$. Since x_n is regular for $\mathfrak{f}$, the elements $\operatorname{in}_\omega(f_1),\ldots,\operatorname{in}_\omega(f_h)$ form a regular sequence of degree $\mathbf{d}$, sitting necessarily inside $\mathfrak{a}_{[0]}$. Let $\mathfrak{f}'$ be the ideal they generate inside S'.

Define $j = \inf\{t \geqslant 0 \mid \operatorname{HF}(\mathfrak{a}_{[t]}x_n^t;D) \neq \operatorname{HF}(\mathfrak{f}'x_n^t;D)\}$. Observe that j is finite, since otherwise the condition $\operatorname{HF}(\mathfrak{a}_{[t]}x_n^t;D) = \operatorname{HF}(\mathfrak{f}'x_n^t;D)$ for all $t \geqslant 0$ would imply that $\operatorname{HF}(\mathfrak{a};D) = \operatorname{HF}(\widetilde{\mathfrak{a}};D) = \operatorname{HF}(\bigoplus_{t\geqslant 0}\mathfrak{f}'x_n^t;D) = \operatorname{HF}(\mathfrak{f};D)$, contradicting our assumption that $G \in \mathfrak{a} \smallsetminus \mathfrak{f}$.

We claim that $j = 0$. If not, let $Hx_n^j \in \mathfrak{a}_{[j]}x_n^j$ be an element of degree D, so that $H \in \mathfrak{a}_{[j]} \smallsetminus \mathfrak{f}'$ is an element of degree $D-j < D$. Observe that $H \in \mathfrak{f}' : \mathfrak{m}'$ by stability, and because $\mathfrak{a}_{[j-1]}$ coincides with $\mathfrak{f}'$ up to degree $D-j+1$. It follows that H represents a non-zero element of $\operatorname{soc}(S'/\mathfrak{f}')$. If $\dim(S/\mathfrak{a}) = 1$, then we reach a contradiction since $\deg(H) < \sum_{i=0}^h (d_i-1)$, and the latter is the degree in which the socle of $S'/\mathfrak{f}'$ is concentrated. If $\dim(S/\mathfrak{a}) > 1$, then $\dim(S'/\mathfrak{f}') = \operatorname{depth}(S'/\mathfrak{f}') > 0$, therefore $\operatorname{soc}(S'/\mathfrak{f}') = 0$. A contradiction again.

Therefore $j = 0$, and $\mathfrak{a}' = \mathfrak{f}' + (H)S'$ is an almost complete intersection of degrees $(\mathbf{d};D)$, with $\dim(S'/\mathfrak{a}') = \dim(S/\mathfrak{a}) - 1$. By induction, there exists an Artinian almost complete intersection $\overline{\mathfrak{a}} \subseteq \overline{S}$ such that $\operatorname{HF}(S'/\mathfrak{a}') \leqslant \operatorname{HF}(S'/\overline{\mathfrak{a}}S')$. Because $\operatorname{HF}(S/\mathfrak{a}) \leqslant \operatorname{HF}(S/\mathfrak{a}'S)$, it follows that $\operatorname{HF}(S/\mathfrak{a}) \leqslant \operatorname{HF}(S/\overline{\mathfrak{a}}S)$, and the proof is complete. ☐

Building from Chong's work in [Cho16], and using Lemma 2.2 and Theorem 2.7, we can now prove the main result of this section.

Theorem 2.8. Let $\mathfrak{a} \subseteq S = \Bbbk[x_1,\ldots,x_n]$ be an almost complete intersections of degrees $(\mathbf{d};D) = (d_1,d_2,d_3;D)$, with $D \leqslant \sigma = d_1+d_2+d_3-3$. Then $\operatorname{HF}(S/\mathfrak{a}) \leqslant \operatorname{HF}(S/\mathbb{L}(\mathbf{d};D))$.

Proof. We may assume that $\Bbbk$ is infinite and, by Theorem 2.7, that $S/\mathfrak{a}$ is Artinian. Write $\mathfrak{a} = \mathfrak{f} + (G)$, where $\mathfrak{f}$ is generated by a regular sequence of degrees $\mathbf{d}$, and G is a form of degree D. By a standard argument of linkage (for instance, see [Mig98, Corollary 5.2.19]), for all $j \in \mathbb{Z}$ we get $\operatorname{HF}(S/\mathfrak{a};j) = \operatorname{HF}(S/\mathfrak{f};j) - \operatorname{HF}(S/\mathfrak{g};\sigma-j)$, where $\mathfrak{g} = \mathfrak{f} : \mathfrak{a}$. Since $\mathfrak{g}$ contains $\mathfrak{f}$, and it defines a Gorenstein ring, by Lemma 2.2 there is a $\mathbf{d}$-LPP ideal $\mathbb{L}$ with the same Hilbert function as $\mathfrak{g}$. If we set $\mathfrak{b} = (\mathsf{x}^{\mathbf{d}}) : \mathbb{L}$, then using linkage again we obtain that

$$\begin{aligned}\operatorname{HF}(S/\mathfrak{b};j) &= \operatorname{HF}(S/(\mathsf{x}^{\mathbf{d}});j) - \operatorname{HF}(S/\mathbb{L};\sigma-j)\\ &= \operatorname{HF}(S/\mathfrak{f};j) - \operatorname{HF}(S/\mathfrak{g};\sigma-j) = \operatorname{HF}(S/\mathfrak{a};j)\end{aligned}$$

for all $j \in \mathbb{Z}$. By [MP06, Theorem 1.2], the monomial ideal $\mathfrak{b}$ satisfies $\operatorname{EGH}_{\mathbf{d},n}$. Therefore, there exists a $\mathbf{d}$-LPP ideal $\mathbb{L}'$ with the same Hilbert function as $\mathfrak{b}$. In particular, since $\mathbb{L}'$ must contain $\mathbb{L}(\mathbf{d};D)$, we have that $\operatorname{HF}(S/\mathfrak{a}) = \operatorname{HF}(S/\mathfrak{b}) = \operatorname{HF}(S/\mathbb{L}') \leqslant \operatorname{HF}(S/\mathbb{L}(\mathbf{d};D))$. ☐

Remark 2.9. If $\mathfrak{a} \subseteq S = \Bbbk[x_1,\ldots,x_n]$ is an almost complete intersection of degrees $(\mathbf{d};D) = (d_1,\ldots,d_h;D)$, with $D > \sigma = \sum_{i=1}^h(d_i-1)$, then the conclusion of Theorem 2.8 still holds, even without assuming that $h=3$. In fact, in this scenario we have that $\mathbb{L}(\mathbf{d};D) = (\mathsf{x}^{\mathbf{d}}) + (U_D)$, where $U_D = x_1^{d_1-1}\cdots x_h^{d_h-1}x_{h+1}^{D-\sigma}$. Iterating the argument used in the proof of Theorem 2.7, we can find an almost complete intersection $\mathfrak{a}' = \mathfrak{f}' + (G') \subseteq S' = \Bbbk[x_1,\ldots,x_{h+1}]$ of degrees $(\mathbf{d};D)$ such that $\operatorname{HF}(S/\mathfrak{a}) \leqslant \operatorname{HF}(S/\mathfrak{a}'S)$. Moreover, we may assume that x_{h+1} is regular modulo $\mathfrak{f}'$. Since $\operatorname{HF}(\mathbb{L}(\mathbf{d};D)/(\mathsf{x}^{\mathbf{d}});m) \leqslant 1$ for all $m \in \mathbb{Z}$, with equality if and only if $m \geqslant D$, it follows that $\operatorname{HF}(\mathfrak{a}'/\mathfrak{f}') \geqslant \operatorname{HF}(\mathbb{L}(\mathbf{d};D)/(\mathsf{x}^{\mathbf{d}}))$, because otherwise we would have $\mathfrak{a}' \subseteq (\mathfrak{f}')^{\mathrm{sat}} = \mathfrak{f}'$. As a consequence, $\operatorname{HF}(S/\mathfrak{a}) \leqslant \operatorname{HF}(S/\mathfrak{a}'S) \leqslant \operatorname{HF}(S/\mathbb{L}(\mathbf{d};D))$.

We now show how Theorem 2.8 allows one to recover the Cayley-Bacharach theorem for points in $\mathbb{P}^3$, which has been proved by Geramita and Kreuzer [GK13, Corollary 4.4]. To do so, we introduce some notation. Let $\mathbf{d} = (d_1,d_2,d_3) \in \mathbb{N}^3$, with $1 \leqslant d_1 \leqslant d_2 \leqslant d_3$, and let

$1 \leqslant D \leqslant d_1 + d_2 + d_3 - 3$. Let $a \in \{1,2,3\}$ be such that $\sum_{i=1}^{a-1}(d_i - 1) < D \leqslant \sum_{i=1}^{a}(d_i - 1)$. We define a new sequence $\mathsf{c} = (c_1, c_2, c_3)$ as

$$c_i = \begin{cases} 1 & \text{if } 1 \leqslant i < a \\ d_a - \left(D - \sum_{i=1}^{a-1}(d_i - 1)\right) & \text{if } i = a \\ d_i & \text{if } a < i \leqslant 3. \end{cases}$$

For example, if $(\mathbf{d}; D) = (4,4,4;4)$, then $\mathsf{c} = (1,3,4)$.

Corollary 2.10. Let $\mathfrak{a} \subseteq S = \Bbbk[x_1, \ldots, x_n]$ be an almost complete intersection of degrees $(\mathbf{d}; D) = (d_1, d_2, d_3; D)$, with $D \leqslant \sigma = d_1 + d_2 + d_3 - 3$. Then $\mathrm{e}(S/\mathfrak{a}) \leqslant d_1 d_2 d_3 - c_1 c_2 c_3$.

Proof. By Theorem 2.8 we have that $\mathrm{HF}(S/\mathfrak{a}) \leqslant \mathrm{HF}(S/\mathbb{L}(\mathbf{d}; D))$. Therefore, in order to obtain an upper bound for the multiplicity of $S/\mathfrak{a}$, we may replace $\mathfrak{a}$ by $\mathbb{L}(\mathbf{d}; D) = (\mathsf{x}^{\mathbf{d}}) + (U_D)$. Since $D \leqslant \sigma$, the variable x_i does not divide U_D for any $i \geqslant 4$. Thus, after going modulo the regular sequence $x_4, \ldots, x_n$, we may assume that $S/\mathbb{L}(\mathbf{d}; D)$ is Artinian. With the notation introduced above, one can easily check that $(\mathsf{x}^{\mathbf{d}}) : U_D = (\mathsf{x}^{\mathsf{c}})$. It then immediately follows that $\mathrm{e}(S/\mathbb{L}(\mathbf{d}; D)) = \mathrm{e}(S/(\mathsf{x}^{\mathbf{d}})) - \mathrm{e}(S/((\mathsf{x}^{\mathbf{d}}) : U_D)) = d_1 d_2 d_3 - c_1 c_2 c_3$. □

Now that we have obtained Corollary 2.10, the proof of the Cayley-Bacharach theorem in $\mathbb{P}^3$, Corollary A, is immediate. In fact, in the notation of the Corollary, let $\mathfrak{f} \subseteq S = \Bbbk[\mathbb{P}^3]$ be a complete intersection defining Γ, and G be a form of degree D defining X. If $G \notin \mathfrak{f}$, by Corollary 2.10 the multiplicity of $S/(\mathfrak{f} + (G))$ is at most $d_1 d_2 d_3 - c_1 c_2 c_3$. So X contains at most this number of points of Γ, which contradicts the assumptions of Corollary A.

Remark 2.11. Corollary 2.10 is based on Lemma 2.2 and [Cho16, Corollary 11]. As these results heavily use the fact that Gorenstein ideals of codimension three are minimally licci, the same strategy cannot be used for ideals of higher codimension.

3. Almost complete intersections and Cayley-Bacharach theorems in $\mathbb{P}^n$

In order to obtain a Cayley-Bacharach type theorem for points in $\mathbb{P}^n$, we need to exhibit upper bounds on the multiplicity of almost complete intersections of height n. The strategy is to use Theorem 2.7 to first reduce to the Artinian case, and then to repeatedly apply a result on the EGH conjecture due to Francisco [Fra04], together with some symmetry considerations on certain Hilbert functions. This combination of techniques allows us to significantly improve the known upper bounds due to Engheta [Eng09, Theorem 1], and later extended by Huneke, Mantero, McCullough and Seceleanu to a more general setting [HMMS15, Theorem 2.2].

We start with an easy observation on multiplicities of unmixed ideals. Given a homogeneous ideal I, we let $\mathrm{Assh}(S/I) = \{P \in \mathrm{Ass}(S/I) \mid \dim(S/I) = \dim(S/P)\}$. An ideal is called unmixed if $\mathrm{Assh}(S/I) = \mathrm{Ass}(S/I)$.

Remark 3.1. If J is an unmixed homogeneous ideal of height h, and I is a homogeneous ideal of height h which strictly contains J, then $\mathrm{e}(S/J) > \mathrm{e}(S/I)$. In fact, there must exist $P \in \mathrm{Assh}(S/J) = \mathrm{Ass}(S/J)$ such that $J_P \subsetneq I_P$, otherwise the two ideals would coincide. As J is unmixed, and $\mathrm{Assh}(S/I) \subseteq \mathrm{Assh}(S/J) = \mathrm{Ass}(S/J)$, the associativity formula for multiplicities (for instance, see [HS06, Theorem 11.2.4]) gives

$$\begin{aligned}\mathrm{e}(S/J) &= \sum_{P \in \mathrm{Assh}(S/J)} \mathrm{e}(S/P)\ell((S/J)_P) \\ &> \sum_{P \in \mathrm{Assh}(S/J)} \mathrm{e}(S/P)\ell((S/I)_P) \geqslant \sum_{Q \in \mathrm{Assh}(S/I)} \mathrm{e}(S/Q)\ell((S/I)_Q) = \mathrm{e}(S/I).\end{aligned}$$

Notation 3.2. Let $\mathbf{d} = (d_1, \ldots, d_h)$, with $1 \leqslant d_1 \leqslant \cdots \leqslant d_h$. For $m \geqslant 2$, we consider the $\mathbf{d}$-LPP ideal $\mathbb{L}(\mathbf{d}; m-1)$ inside $\overline{S} = \Bbbk[x_1, \ldots, x_h]$. Let $\sigma = \sum_{i=1}^h (d_i - 1)$, and define

$$\varphi_m = \begin{cases} \mathrm{HF}(\overline{S}/(\mathsf{x}^{\mathbf{d}}); m) - \mathrm{HF}(\overline{S}/\mathbb{L}(\mathbf{d}; m-1); m) & \text{if } 2 \leqslant m \leqslant \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, φ_m only depends on m and on the given sequence $(\mathbf{d}; D)$. Moreover, observe that for $2 \leqslant m \leqslant \sigma$ we have $\varphi_m = \mathrm{HF}(\mathbb{L}(\mathbf{d}; m-1)/(\mathsf{x}^{\mathbf{d}}); m) = h - \dim_{\Bbbk}(((\mathsf{x}^{\mathbf{d}}) \cap (U_{m-1}))_m)$. In particular, $\varphi_m > 0$ for $2 \leqslant m \leqslant \sigma$.

Theorem 3.3. Let $\mathfrak{a} \subseteq S = \Bbbk[x_1, \ldots, x_n]$ be an Artinian almost complete intersection of degrees $(\mathbf{d}; D) = (d_1, \ldots, d_n; D)$, with $D \leqslant \sigma = \sum_{i=1}^n (d_i - 1)$. Then $\mathrm{HF}(S/\mathfrak{a}; m) \leqslant \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); m) - \varphi_m$ for all $D < m \leqslant \sigma$. In particular, $\mathrm{e}(S/\mathfrak{a}) \leqslant \prod_{i=1}^n d_i - \sum_{m=D+1}^{\sigma} \varphi_m - 1$.

Proof. Without loss of generality we may assume that $\Bbbk$ is infinite. We first prove the inequality on Hilbert functions.

We start by treating the case $D < d_1$. Under this assumption, we can find a regular sequence $f_1', \ldots, f_n'$ of degrees $\mathbf{d}' = (D, d_2, \ldots, d_n)$ inside $\mathfrak{a}$. To see this, pick G as the first element f_1'. Since $\mathfrak{a}_{\leqslant d_2}$ has height at least two, we may find an element f_2' of degree d_2 which is regular modulo f_1'. Proceding this way, we construct an ideal $\mathfrak{f}' \subseteq \mathfrak{a}$ generated by a regular sequence of degrees $\mathbf{d}'$. Observe that $\mathrm{HF}(S/\mathfrak{a}) \leqslant \mathrm{HF}(S/\mathfrak{f}') = \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}'}))$. Moreover, since $(\mathsf{x}^{\mathbf{d}'}) = (\mathsf{x}^{\mathbf{d}}) + (x_1^D)$ is a $\mathbf{d}$-LPP almost complete intersection, we have that $\mathbb{L}(\mathbf{d}; m-1) \subseteq (\mathsf{x}^{\mathbf{d}'})$ for all $D < m \leqslant \sigma$. As a consequence, $\mathrm{HF}(S/\mathfrak{a}; m) \leqslant \mathrm{HF}(S/\mathbb{L}(\mathbf{d}; m-1); m) = \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); m) - \varphi_m$ for all $D < m \leqslant \sigma$.

Now assume that $D \geqslant d_1$. If $D = \sigma$, there is nothing to show, so we may assume that $D < \sigma$. We proceed by induction on $\sigma' = \sigma - D \geqslant 1$. Assume that $\sigma' = 1$, and observe that $\varphi_\sigma = 1$. Since $S/\mathfrak{f}$ is an Artinian complete intersection, and $\mathfrak{f} \subsetneq \mathfrak{a}$, the socle of $S/\mathfrak{f}$ must be contained in $\mathfrak{a}/\mathfrak{f}$. Thus $\mathrm{HF}(S/\mathfrak{a}; \sigma) = 0 = \mathrm{HF}(S/\mathfrak{f}; \sigma) - 1 = \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); \sigma) - \varphi_\sigma$, and the desired inequality holds in this case.

Assume $\sigma' > 1$. By [Fra04, Corollary 5.2], we get $\mathrm{HF}(S/\mathfrak{a}; D+1) \leqslant \mathrm{HF}(S/\mathbb{L}(\mathbf{d}; D); D+1) = \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); D+1) - \varphi_{D+1}$. We have already observed that $\varphi_{D+1} > 0$, since $D+1 \leqslant \sigma$. In particular, the above inequality implies that $\mathrm{HF}(\mathfrak{a}; D+1) > \mathrm{HF}(\mathfrak{f}; D+1)$. Therefore, there exists an element $G' \in \mathfrak{a}$ of degree $D+1$ which does not belong to $\mathfrak{f}$. Let $\mathfrak{a}' = \mathfrak{f} + (G')$, which is an almost complete intersection of degrees $(\mathbf{d}; D+1)$. By induction, and because $\mathfrak{a}' \subseteq \mathfrak{a}$, we have that $\mathrm{HF}(S/\mathfrak{a}; m) \leqslant \mathrm{HF}(S/\mathfrak{a}'; m) \leqslant \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); m) - \varphi_m$ for all $D+1 < m \leqslant \sigma$, and this concludes the proof of the claimed inequalities on Hilbert function.

Finally, to obtain the inequality for the multiplicity, it is sufficient to observe that

$$\begin{aligned}\mathrm{e}(S/\mathfrak{a}) &= \sum_{m=0}^{\sigma} \mathrm{HF}(S/\mathfrak{a}; m) = \sum_{m=0}^{D} \mathrm{HF}(S/\mathfrak{f}; m) - 1 + \sum_{m=D+1}^{\sigma} \mathrm{HF}(S/\mathfrak{a}) \\ &\leqslant \sum_{m=0}^{\sigma} \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); m) - \sum_{m=D+1}^{\sigma} \varphi_m - 1 = \prod_{i=1}^{n} d_i - \sum_{m=D+1}^{\sigma} \varphi_m - 1.\end{aligned}$$

□

Remark 3.4. The upper bound of Theorem 3.3 for the multiplicity of an almost complete intersection holds even if $S/\mathfrak{a}$ is not Artinian, thanks to Theorem 2.7. The upper bound on Hilbert functions holds as well provided one modifies the function φ_m to take into account the number of variables of S, in addition to the degrees $(\mathbf{d}; D)$.

Remark 3.5. The bound obtained in Theorem 3.3, together with Remark 3.1, recovers and improves the one given in [HMMS15] and [Eng09]. In fact, by Theorem 2.7 we can first of all reduce to the Artinian case. If $D < \sigma$ then $\sum_{m=D+1}^{\sigma} \varphi_m \geqslant \sigma - D$, and thus $\prod_{i=1}^h d_i - \sum_{m=D+1}^{\sigma} \varphi_m - 1 \leqslant \prod_{i=1}^h d_i - \sigma + D - 1$, which is the bound given in [Eng09, HMMS15]. When $D \geqslant \sigma$, the results in [Eng09, HMMS15] just give that $\mathrm{e}(S/\mathfrak{a}) \leqslant \left(\prod_{i=1}^h d_i\right) - 1$, which is the bound given by Remark 3.1. Observe that, in the case $D \geqslant \sigma$, the bound $\mathrm{e}(S/\mathfrak{a}) \leqslant \left(\prod_{i=1}^h d_i\right) - 1$ is also the one predicted by the EGH conjecture.

We now further improve the bound of Theorem 3.3 by using that, if $\mathfrak{a} = \mathfrak{f} + (G)$ is an almost complete intersection, then the ideal $\mathfrak{g} = \mathfrak{f} : \mathfrak{a}$ defines a Gorenstein ring, hence it has symmetric Hilbert function.

Theorem 3.6. Let $\mathfrak{a} = \mathfrak{f} + (G) \subseteq S = \Bbbk[x_1, \dots, x_n]$ be an almost complete intersection of degrees $(\mathbf{d}; D) = (d_1, \dots, d_h; D)$, with $D < \sigma = \sum_{i=1}^h (d_i - 1)$. Let $\tau_- = \lfloor \frac{\sigma + D - 1}{2} \rfloor$ and $\tau^+ = \lceil \frac{\sigma + D - 1}{2} \rceil$. Then $\mathrm{e}(S/\mathfrak{a}) \leqslant \prod_{i=1}^h d_i - \sum_{m=D+1}^{\tau_-} \varphi_m - \sum_{m=D+1}^{\tau^+} \varphi_m - 2$.

Proof. We may assume that $\Bbbk$ is infinite and, by Theorem 2.7, that $S/\mathfrak{a}$ is Artinian. Let $\mathfrak{g} = \mathfrak{f} : \mathfrak{a}$. Since $\mathfrak{a}/\mathfrak{f} \cong S/\mathfrak{g}(-D)$, and $S/\mathfrak{g}$ is Gorenstein, we have that $\mathrm{HF}(\mathfrak{a}/\mathfrak{f}; D + m) = \mathrm{HF}(\mathfrak{a}/\mathfrak{f}; \sigma - m)$ for all $m \in \mathbb{Z}$. By Theorem 3.3 we have that $\mathrm{HF}(\mathfrak{a}/\mathfrak{f}; m) = \mathrm{HF}(S/\mathfrak{f}; m) - \mathrm{HF}(S/\mathfrak{a}; m) \geqslant \varphi_m$ for all $m \geqslant D + 1$. Therefore

$$\begin{aligned} \mathrm{e}(S/\mathfrak{g}) &= \sum_{m=D}^{\tau_-} \mathrm{HF}(\mathfrak{a}/\mathfrak{f}; m) + \sum_{m=D}^{\tau^+} \mathrm{HF}(\mathfrak{a}/\mathfrak{f}; m) \\ &\geqslant \sum_{m=D+1}^{\tau_-} \varphi_m + \sum_{m=D+1}^{\tau^+} \varphi_m + 2. \end{aligned}$$

Since $\mathrm{e}(S/\mathfrak{a}) = \mathrm{e}(S/\mathfrak{f}) - \mathrm{e}(S/\mathfrak{g}) = \prod_{i=1}^h d_i - \mathrm{e}(S/\mathfrak{g})$, the proof is complete. □

Remark 3.7. Since the function $m \mapsto \varphi_m$ is non-increasing for $m \geqslant 2$, Theorem 3.6 always provides a bound at least as effective at the one of Theorem 3.3.

We can finally state the main result of this section.

Theorem 3.8. Let Γ be a complete intersection of degrees $\mathbf{d} = (d_1, \dots, d_n)$ in $\mathbb{P}^n$, and X be a hypersurface of degree D, with $1 \leqslant D \leqslant \sigma = \sum_{i=1}^n (d_i - 1)$. Set $\delta(\mathbf{d}; D) = \prod_{i=1}^n d_i - \sum_{m=D+1}^{\tau_-} \varphi_m - \sum_{m=D+1}^{\tau^+} \varphi_m - 1$. If X contains at least $\delta(\mathbf{d}; D)$ points of Γ, then X contains Γ.

We omit the proof since the strategy is the same as in the case of $\mathbb{P}^3$, outlined at the end of Section 2.

Example 3.9. Let $\Gamma \subseteq \mathbb{P}^{2n}$ be a complete intersection of $2n$ cubics. If $D = 3$, with the notation of Theorem 3.6 we have that $\sigma = 4n$, $\tau_- = \tau^+ = 2n+1$, and $\sum_{m=4}^{2n+1} \varphi_m = 3n^2 - 4n + 1$. Therefore, if X is a cubic containing at least $3^{2n} - (6n^2 - 8n + 3)$ points of Γ, then X contains

Γ. For instance, if Γ is a complete intersection of four cubics in $\mathbb{P}^4$, and X is a cubic containing at least $\delta(3,3,3,3;3) = 70$ points of Γ, then it contains all 81 points of Γ. Observe that the optimal value given by the EGH conjecture in this case would be 64.

We conclude the section showing that, if either $h = 3$, or $h \geqslant 4$ and $D < d_4$, then we can improve Theorem 3.8 using the results from Section 2. In fact, with the notation of Section 2, if $h = 3$ one can take $\delta(\mathbf{d}; D) = d_1 d_2 d_3 - c_1 c_2 c_3 + 1$, by Corollary 2.10. This is a more convenient choice than the value of $\delta(\mathbf{d}; D)$ coming from Theorem 3.8, since it comes from the sharper estimates of Section 2 on Hilbert functions, which only work for almost complete intersections of height three. If $h \geqslant 4$ and $D < d_4$, we have the following theorem.

Theorem 3.10. Let $\mathfrak{a} = \mathfrak{f} + (G) \subseteq S = \Bbbk[x_1, \ldots, x_n]$ be an almost complete intersection of degrees $(\mathbf{d}; D) = (d_1, \ldots, d_h; D)$. Assume that $h \geqslant 4$ and $D < d_4$. Let $\sigma = \sum_{i=1}^{h}(d_i - 1)$, $\tau_- = \lfloor \frac{\sigma + D - 1}{2} \rfloor$ and $\tau^+ = \lceil \frac{\sigma+D-1}{2} \rceil$. Consider the ideal $\mathbb{L}(\mathbf{d}; D)$ inside $\overline{S} = \Bbbk[x_1, \ldots, x_h]$, and let

$$\delta_m = \begin{cases} \mathrm{HF}(\overline{S}/(\mathsf{x}^{\mathbf{d}}); m) - \mathrm{HF}(\overline{S}/\mathbb{L}(\mathbf{d}; D); m) & \text{for } 0 \leqslant m \leqslant d_4 \\ \varphi_m & \text{otherwise.} \end{cases}$$

Then $\mathrm{e}(S/\mathfrak{a}) \leqslant \prod_{i=1}^{h} d_i - \sum_{m=D+1}^{\tau_-} \delta_m - \sum_{m=D+1}^{\tau^+} \delta_m - 2$.

Proof. We can assume that $\Bbbk$ is infinite and, by Theorem 2.7, that $S = \overline{S}$ and $h = n$.

We start by showing that $\mathrm{HF}(S/\mathfrak{a}; m) \leqslant \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); m) - \delta_m$ for all $m \in \mathbb{Z}$. As in the proof of Theorem 3.6, this will yield the desired upper bound for $\mathrm{e}(S/\mathfrak{a})$ since the Hilbert function of $\mathfrak{a}/\mathfrak{f}$ is symmetric and $\mathrm{e}(S/(\mathsf{x}^{\mathbf{d}})) = \prod_{i=1}^{n} d_i$.

Observe that $\mathrm{HF}(S/\mathfrak{a}; m) \leqslant \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); m) - \delta_m$ is true for $m > d_4$ by Theorem 3.3. Therefore, it suffices to focus on the inequality in degrees $0 \leqslant m \leqslant d_4$.

Let $s = \max\{j \geqslant 4 \mid d_j = d_4\}$. First, assume that the elements f_1, f_2, f_3, G form a regular sequence. Then $\mathfrak{a}$ contains a regular sequence $f_1', \ldots, f_n'$ of degrees $\mathbf{d}' = (d_1, d_2, d_3, D, d_5, \ldots, d_n)$. We have $\mathrm{HF}(S/\mathfrak{a}) \leqslant \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}'})) \leqslant \mathrm{HF}(S/\mathbb{L}(\mathbf{d}; D))$, because $\mathbb{L}(\mathbf{d}; D) \subseteq \mathbb{L}$, where $\mathbb{L}$ is the $\mathbf{d}$-LPP ideal with the same Hilbert function as $(\mathsf{x}^{\mathbf{d}'})$, which exists by [MP06, Theorem 1.2]. By definition, $\mathrm{HF}(S/\mathbb{L}(\mathbf{d}; D); m) = \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); m) - \delta_m$ for all $0 \leqslant m \leqslant d_4$.

If the elements f_1, f_2, f_3, G do not form a regular sequence, then $\mathfrak{b} = (f_1, f_2, f_3, G)$ is an almost complete intersection of degrees $(\mathbf{d}''; D) = (d_1, d_2, d_3; D)$. Using Theorem 2.8 and Remark 2.9 we have that $\mathrm{HF}(S/\mathfrak{b}) \leqslant \mathrm{HF}(S/\mathbb{L}(\mathbf{d}''; D))$. For all $m < d_4$ we conclude that $\mathrm{HF}(S/\mathfrak{a}; m) = \mathrm{HF}(S/\mathfrak{b}; m) \leqslant \mathrm{HF}(S/\mathbb{L}(\mathbf{d}''; D); m) = \mathrm{HF}(S/\mathbb{L}(\mathbf{d}; D); m) = \mathrm{HF}(S/(\mathsf{x}^{\mathbf{d}}); m) - \delta_m$. For $m = d_4$, observe that the elements $f_4, \ldots, f_s$ are all minimal generators of $\mathfrak{a}_{\leqslant d_4}$, so that $\mathrm{HF}(\mathfrak{a}/\mathfrak{b}; d_4) = s - 3$. As a consequence, we get that $\mathrm{HF}(S/\mathfrak{a}; d_4) = \mathrm{HF}(S/\mathfrak{b}; d_4) - \mathrm{HF}(\mathfrak{a}/\mathfrak{b}; d_4) \leqslant \mathrm{HF}(S/\mathbb{L}(\mathbf{d}''; D); d_4) - (s-3) = \mathrm{HF}(S/\mathbb{L}(\mathbf{d}; D); d_4)$. $\square$

Remark 3.11. Theorem 3.10 shows that, in the case $h \geqslant 4$ and $D < d_4$, we may replace the value $\delta(\mathbf{d}; D)$ from Theorem 3.8 with $\prod_{i=1}^{n} d_i - \sum_{m=D+1}^{\tau_-} \delta_m - \sum_{m=D+1}^{\tau^+} \delta_m - 1$. As in the case $h = 3$, the latter choice is always more convenient to make, whenever possible. This becomes significantly more evident when $d_4 \gg D$, as the following example shows.

Example 3.12. Let $\Gamma \subseteq \mathbb{P}^4$ be a complete intersection of degrees $\mathbf{d} = (4,4,4,10)$. By Remark 3.11, if X is a quartic passing through at least 532 points of Γ, then X contains all 640 points of Γ. Notice that Theorem 3.8 would give a value of $\delta(\mathbf{d}; 4) = 612$, while the one predicted by the EGH conjecture would be 521 points.

References

[Abe15] Abed Abedelfatah. On the Eisenbud-Green-Harris conjecture. *Proc. Amer. Math. Soc.*, 143(1):105–115, 2015. 4

[CDS21] Giulio Caviglia and Alessandro De Stefani. Decomposition of local cohomology tables of modules with large E-depth. *J. Pure Appl. Algebra*, 225(6):106635, 23, 2021. 4

[Cho16] Kai Fong Ernest Chong. An application of liaison theory to the Eisenbud-Green-Harris conjecture. *J. Algebra*, 445:221–231, 2016. 4, 6, 7

[CK13] Giulio Caviglia and Manoj Kummini. Poset embeddings of Hilbert functions. *Math. Z.*, 274(3-4):805–819, 2013. 4

[CK14] Giulio Caviglia and Manoj Kummini. Poset embeddings of Hilbert functions and Betti numbers. *J. Algebra*, 410:244–257, 2014. 4, 5

[CL69] G. F. Clements and B. Lindström. A generalization of a combinatorial theorem of Macaulay. *J. Combinatorial Theory*, 7:230–238, 1969. 4

[CM08] Giulio Caviglia and Diane Maclagan. Some cases of the Eisenbud-Green-Harris conjecture. *Math. Res. Lett.*, 15(3):427–433, 2008. 3, 4

[Coo08] Susan Marie Cooper. Growth conditions for a family of ideals containing regular sequences. *J. Pure Appl. Algebra*, 212(1):122–131, 2008. 2, 4

[CS13] Giulio Caviglia and Enrico Sbarra. Generic circuits sets and general initial ideals with respect to weights. *J. Commut. Algebra*, 5(1):49–62, 2013. 5

[CS16] Giulio Caviglia and Enrico Sbarra. The lex-plus-powers inequality for local cohomology modules. *Math. Ann.*, 364(1-2):225–241, 2016. 5

[EGH93] David Eisenbud, Mark Green, and Joe Harris. Higher Castelnuovo theory. Number 218, pages 187–202. 1993. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). 3

[EGH96] David Eisenbud, Mark Green, and Joe Harris. Cayley-Bacharach theorems and conjectures. *Bull. Amer. Math. Soc. (N.S.)*, 33(3):295–324, 1996. 1, 2, 3

[EL19] Lawrence Ein and Robert Lazarsfeld. Cayley-Bacharach theorems with excess vanishing. *arXiv: 1909.08493*, 2019. 2

[Eng09] Bahman Engheta. Bound on the multiplicity of almost complete intersections. *Comm. Algebra*, 37(3):948–953, 2009. 3, 7, 9

[Fra04] Christopher A. Francisco. Almost complete intersections and the lex-plus-powers conjecture. *J. Algebra*, 276(2):737–760, 2004. 2, 7, 8

[Gas99] Vesselin Gasharov. Hilbert functions and homogeneous generic forms. II. *Compositio Math.*, 116(2):167–172, 1999. 4

[GH78] Phillip Griffiths and Joseph Harris. Residues and zero-cycles on algebraic varieties. *Ann. of Math. (2)*, 108(3):461–505, 1978. 2

[GH19] Sema Güntürkün and Melvin Hochster. The Eisenbud-Green-Harris Conjecture for defect two quadratic ideals. *To appear in Math. Res. Lett.*, 2019. 4

[GK13] Anthony V. Geramita and Martin Kreuzer. On the uniformity of zero-dimensional complete intersections. *J. Algebra*, 391:82–92, 2013. 2, 6

[Gre10] Mark L. Green. Generic initial ideals [mr1648665]. In *Six lectures on commutative algebra*, Mod. Birkhäuser Class., pages 119–186. Birkhäuser Verlag, Basel, 2010. 5

[HLU20] James Hotchkiss, Chung Ching Lau, and Brooke Ullery. The gonality of complete intersection curves. *J. Algebra*, 560:579–608, 2020. 2

[HMMS15] Craig Huneke, Paolo Mantero, Jason McCullough, and Alexandra Seceleanu. A multiplicity bound for graded rings and a criterion for the Cohen-Macaulay property. *Proc. Amer. Math. Soc.*, 143(6):2365–2377, 2015. 3, 7, 9

[HP98] Jürgen Herzog and Dorin Popescu. Hilbert functions and generic forms. *Compositio Math.*, 113(1):1–22, 1998. 4

[HS06] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006. 8

[KLR19] Martin Kreuzer, Le Ngoc Long, and Lorenzo Robbiano. On the Cayley-Bacharach property. *Comm. Algebra*, 47(1):328–354, 2019. 1, 2

[Li19] Mu-Lin Li. Virtual residue and a generalized Cayley-Bacharach theorem. *Proc. Amer. Math. Soc.*, 147(9):3765–3771, 2019. 2

[Mig98] Juan C. Migliore. *Introduction to liaison theory and deficiency modules*, volume 165 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1998. 6

[MP06] Jeffrey Mermin and Irena Peeva. Lexifying ideals. *Math. Res. Lett.*, 13(2-3):409–422, 2006. 4, 6, 10

[Ric04] Benjamin P. Richert. A study of the lex plus powers conjecture. *J. Pure Appl. Algebra*, 186(2):169–183, 2004. 2, 4

[Tan00] Sheng-Li Tan. Cayley-Bacharach property of an algebraic variety and Fujita's conjecture. *J. Algebraic Geom.*, 9(2):201–222, 2000. 2

Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA

E-mail address: gcavigli@purdue.edu

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy

E-mail address: destefani@dima.unige.it