SYZYGIES IN HILBERT SCHEMES OF COMPLETE INTERSECTIONS

GIULIO CAVIGLIA AND ALESSIO SAMMARTANO

ABSTRACT. Let $d_1, \ldots, d_c$ be positive integers and let $Y \subseteq \mathbb{P}^n$ be the monomial complete intersection defined by the vanishing of $x_1^{d_1}, \ldots, x_c^{d_c}$. For each Hilbert polynomial $p(\zeta)$ we construct a distinguished point in the Hilbert scheme $\mathrm{Hilb}^{p(\zeta)}(Y)$, called the *expansive* point. We develop a theory of expansive ideals, and show that they play for Hilbert polynomials the same role lexicographic ideals play for Hilbert functions. For instance, expansive ideals maximize number of generators and syzygies, they form descending chains of inclusions, and exhibit an extremal behavior with respect to hyperplane sections. Conjecturally, expansive subschemes provide uniform sharp upper bounds for the syzygies of subschemes $Z \in \mathrm{Hilb}^{p(\zeta)}(X)$ for all complete intersections $X = X(d_1, \ldots, d_c) \subseteq \mathbb{P}^n$. In some cases, the expansive point achieves extremal Betti numbers for the infinite free resolutions associated to subschemes in $\mathrm{Hilb}^{p(\zeta)}(Y)$. Our approach is new even in the special case $Y = \mathbb{P}^n$, where it provides several novel results and a simpler proof of a theorem of Murai and the first author.

CONTENTS

2010 *Mathematics Subject Classification.* Primary: 13D02. Secondary: 13C40; 13F20; 14C05; 14M10.
Key words and phrases. Betti numbers; free resolution; Hilbert polynomial; Clements–Lindström ring; expansive ideal.

Introduction

In this paper we investigate the extremal behavior of free resolutions of subschemes of complete intersections $X \subseteq \mathbb{P}^n$. Our motivating question is the following. Let $\mathbf{d} = (d_1, \ldots, d_c)$ be a degree sequence and $p(\zeta)$ a Hilbert polynomial: are there uniform bounds on the syzygies of $Z \subseteq X$, where $X \subseteq \mathbb{P}^n$ is a complete intersection of degrees $\mathbf{d}$ and $Z \subseteq X$ a closed subscheme with Hilbert polynomial $p(\zeta)$?

In order to address this problem, we study Hilbert schemes of Clements-Lindström schemes, i.e. complete intersections $Y \subseteq \mathbb{P}^n$ defined by the vanishing of $x_1^{d_1}, \ldots, x_c^{d_c}$. They include $Y = \mathbb{P}^n$ as special case, which is in fact interesting and non-trivial for most of our considerations. Our main contributions revolve around a new distinguished point on the Hilbert scheme $\mathrm{Hilb}^{p(\zeta)}(Y)$, called the *expansive point* (or subscheme, or ideal) and denoted by $\mathrm{Exp}(p(\zeta))$. We adopt an abstract recursive approach in defining $\mathrm{Exp}(p(\zeta))$, based on seven axioms related to hyperplane sections, cf. Theorem 3.1. In a sense, this gives rise to a theory of expansive ideals and Hilbert polynomials, which parallels the theory of lexicographic ideals and Hilbert functions.

Our main result, Theorem 4.3, states that $\mathrm{Exp}(p(\zeta))$ attains the largest possible number of i-th syzygies for a subscheme in $\mathrm{Hilb}^{p(\zeta)}(Y)$, for every homological degree i. No such theorem exists for *graded* syzygies, since each Hilbert scheme has several maximal graded Betti tables. We remark that considering expansive subschemes of Clements-Lindström schemes Y for various degree sequences $\mathbf{d}$, as opposed to just for $\mathbb{P}^n$, carries advantages. First, by taking the degree sequence into account, and restricting thus to a smaller Hilbert scheme, one obtains sharper numerical bounds on Betti numbers. A similar point of view is adopted e.g. in [13], where bounds on the number of points in intersections of quadric hypersurfaces are improved using the data of the degree sequence. More importantly, our main result extends conjecturally to arbitrary complete intersections of $\mathbb{P}^n$. In fact we show that, under the validity of the Lex Plus Powers Conjecture, $\mathrm{Exp}(p(\zeta))$ yields uniform bounds for the syzygies of subschemes $Z \in \mathrm{Hilb}^{p(\zeta)}(X)$ for *all* complete intersections $X \subseteq \mathbb{P}^n$ of degrees $\mathbf{d}$, thus giving a complete answer to our motivating problem. See Proposition 4.5 and Theorem 4.6.

We apply the theory of expansive ideals also to infinite free resolutions over complete intersections, motivated by the recent progress in this area. Our second main result, Theorem 5.3, shows that, over a quadratic Clements-Lindström ring of characteristic 0, expansive ideals achieve extremal Betti numbers for the infinite free resolution. We conjecture that this pattern holds for arbitrary degree sequences and base field.

In the case $Y = \mathbb{P}^n$, Theorem 4.3 gives a new proof of [9, Theorem 1.1], which asserts the existence of a subscheme in $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ with extremal Betti numbers. The authors remark in [9, Introduction] that the proof, of combinatorial nature, is very long and complicated, and it would be desirable to have a better understanding of the structure and construction of such extremal subschemes. We believe that, with the method developed in this work, we have found a satisfactory answer. In fact, besides providing a short and more conceptual proof, the axioms of Theorem 3.1 can be used to further illuminate the structure of expansive ideals. In particular, we prove in Theorem 6.2 that expansive ideals form descending chains of inclusions, starting with a saturated lex ideal, and each step of the chain is described explicitly. This fact serves as the basis for an efficient algorithm to compute $\mathrm{Exp}(p(\zeta))$. The problem of finding an algorithm to determine a subscheme in $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ with maximal syzygies had been suggested also in [27, Section 5].

Borel-fixed points have proved helpful in understanding the geometry of the Hilbert scheme, e.g. in questions of connectedness, smoothness, rationality, enumeration of components, and defining equations, see for instance [4, 25, 31, 33, 34, 35]. Several problems in this area remain open. Our work identifies a new distinguished Borel-fixed point, that is very different from the well-known lex point in many respects. We hope that the notion of expansive point may lead to new perspectives or applications in the geometry of Hilbert schemes.

1. Clements-Lindström rings

This section serves the purpose of fixing the basic terminology for the paper. We introduce the rings that are central to this work, and some special classes of ideals.

Let $\mathbb{N}$ denote the set of nonnegative integers. The symbol $\Bbbk$ denotes an arbitrary field. All rings considered in this work are Noetherian $\mathbb{Z}$-graded $\Bbbk$-algebras generated in degree 1, and all ideals and modules are graded; these attributes are often assumed implicitly and omitted.

If V is a $\mathbb{Z}$-graded $\Bbbk$-vector space, denote the j-th graded component by $[V]_j$. The numerical function $\mathrm{HF}(V) : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ defined by $\mathrm{HF}(V, j) = \dim_\Bbbk [V]_j$ for all $j \in \mathbb{Z}$ is called the **Hilbert function** of V. If there is a numerical polynomial $\mathrm{HP}(V) \in \mathbb{Q}[\zeta]$ such that $\mathrm{HP}(V, j) = \mathrm{HF}(V, j)$ for all $j \gg 0$, then $\mathrm{HP}(V)$ is called the **Hilbert polynomial** of V.

The maximal ideal of a ring A is denoted by $\mathfrak{m}_A$. An ideal $I \subseteq A$ is **saturated** if $I : \mathfrak{m}_A = I$, equivalently, if $\mathrm{depth}(A/I) > 0$; notice that the unit ideal $I = A$ is saturated. The saturation of $I \subseteq A$ is defined as $I : \mathfrak{m}_A^\infty = \cup_{t \geq 0} I : \mathfrak{m}_A^t$, and it is a saturated ideal with $\mathrm{HP}(I : \mathfrak{m}_A^\infty) = \mathrm{HP}(I)$.

Given a projective scheme $X = \mathrm{Proj} A$ and a polynomial $p(\zeta) \in \mathbb{Q}[\zeta]$, the **Hilbert scheme**, denoted by $\mathrm{Hilb}^{p(\zeta)}(X)$, is the scheme parametrizing the closed subschemes $Z \subseteq X$ with $\mathrm{HP}(Z) = p(\zeta)$. As it is common in the literature, we often identify a closed subscheme $Z \subseteq X$ with its saturated ideal $I_Z \subseteq A$ and with the point on the Hilbert scheme parametrizing it, and sometimes we extend attributes of one object to the other two. For instance we may talk about strongly stable subschemes or lex points on the Hilbert scheme, and we adopt the following:

Convention 1.1. If $I \subseteq A$ is an ideal, the expression "$I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj} A)$" means that I is saturated and $\mathrm{HP}(A/I) = p(\zeta)$.

Let A be a ring and M a finite A-module. The integers $\beta^A_{i,j}(M) = \dim_{\Bbbk}[\mathrm{Tor}^i_A(M,\Bbbk)]_j$ and $\beta^A_i(M) = \dim_{\Bbbk} \mathrm{Tor}^i_A(M,\Bbbk)$ are the **graded Betti numbers** and the **(total) Betti numbers** of M, respectively.

Convention 1.2. We will often use $\mathbb{N} \cup \{\infty\}$ as index set and as range for exponents. We adopt standard conventions on ∞, namely that $m < \infty$ and $\infty - m = \infty$ for all $m \in \mathbb{N}$. If r is an element in a ring then we set $r^\infty := 0$. If $d = \infty$, the expression "$\ell < d$" means "$\ell \in \mathbb{N}$".

Definition 1.3. A **Clements-Lindström ring** is a ring of the form

$$A = \frac{\Bbbk[x_1,\dots,x_m]}{\left(x_1^{d_1},\dots,x_m^{d_m}\right)}$$

for some sequence of integers $d_1 \le d_2 \le \cdots \le d_m$ with $d_i \in \mathbb{N}\cup\{\infty\}$ We emphasize that $x_i^\infty := 0$. Thus, when $d_1 = \infty$ the ring A is simply a polynomial ring. On the other hand, when $d_m < \infty$ the ring A is Artinian, and its only saturated ideal is the unit ideal.

For the remainder of this section, let A denote an arbitrary Clements-Lindström ring.

An ideal $I \subseteq A$ is monomial if it is the image of a monomial ideal of $\Bbbk[x_1,\dots,x_m]$. We denote by $<_{\mathrm{lex}}$ the lexicographic monomial order in A induced by $x_1 > x_2 > \cdots > x_m$. A monomial ideal $I \subseteq A$ is **lex** if $[I]_j$ is a vector space generated by an initial segment of monomials with respect to $<_{\mathrm{lex}}$ for every j, equivalently, if I is the image of a lex ideal of $\Bbbk[x_1,\dots,x_m]$. The saturation of a lex ideal is again lex. A theorem of Clements and Lindström, which includes classical results of Macaulay and Kruskal-Katona as special cases, states that lex ideals classify Hilbert functions in A:

Proposition 1.4 ([12])**.** *Let A be a Clements-Linström ring and $I \subseteq A$ an ideal. There exists a unique lex ideal $L \subseteq A$ such that* $\mathrm{HF}(L) = \mathrm{HF}(I)$.

If $\mathcal{H} : \mathbb{Z} \to \mathbb{N}$ is the Hilbert function of some ideal of A, we denote by $\mathrm{Lex}(\mathcal{H}, A)$ the unique lex ideal $L \subseteq A$ with $\mathrm{HF}(L) = \mathcal{H}$. If $I \subseteq A$ we define $\mathrm{Lex}(I) := \mathrm{Lex}(\mathrm{HF}(I), A)$.

A monomial ideal $I \subseteq A$ is **almost lex** if the last variable x_m is a non-zerodivisor on A/I and $\frac{I+(x_m)}{(x_m)}$ is a lex ideal of the Clements-Lindström ring $\frac{A}{(x_m)}$. In particular, almost lex ideals are saturated. Observe that a lex ideal is not, in general, almost lex.

A monomial ideal $I \subseteq A$ is **strongly stable** if for every nonzero monomial $\mathbf{u} \in I$ and x_h dividing $\mathbf{u}$, then $\frac{x_k \mathbf{u}}{x_h} \in I$ for all $k < h$. It suffices to check this condition for the monomial minimal generators $\mathbf{u}$ of I. When A is a polynomial ring, strongly stable ideals are fixed under the action of the Borel group. A strongly stable ideal $I \subseteq A$ is saturated if and only if the last variable x_m is a non-zerodivisor on A/I; when $\dim A > 0$, this is equivalent to the fact that x_m does not divide any monomial minimal generator of I. The saturation of a strongly stable ideal is again strongly stable. Both lex ideals and almost lex ideals are strongly stable.

Examples 1.5. Let $d_1 = 2, d_2 = 3, d_3 = d_4 = \infty$. The associated Clements-Lindström ring is $A = \frac{\Bbbk[x_1,x_2,x_3,x_4]}{(x_1^2,x_2^3)}$. Consider the following ideals of A:

- $\big(x_1x_2^2,\, x_1x_2x_3\big)$ is both lex and almost lex;
- $\big(x_1x_2,\, x_1x_3,\, x_1x_4^2,\, x_2^2x_3\big)$ is lex but not almost lex, since it is not saturated;
- $\big(x_1x_2,\, x_1x_3,\, x_2^2\big)$ is almost lex but not lex, since $x_1x_4 >_{\text{lex}} x_2^2$;
- $\big(x_1x_2,\, x_2^2\big)$ is strongly stable and saturated, but neither lex nor almost lex.

Remark 1.6 (The lex point)**.** For every $p(\zeta) \in \mathbb{Q}[\zeta]$ such that $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A) \neq \emptyset$, there is exactly one lex ideal in $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A)$. The show existence, take $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A)$ and let $L = \mathrm{Lex}(I) : \mathfrak{m}_A^\infty$. It follows that L is a saturated lex ideal of A with $\mathrm{HP}(I) = \mathrm{HP}(L)$. If L, L' are two saturated lex ideals with $\mathrm{HP}(L) = \mathrm{HP}(L')$, then $\mathrm{HF}(L, d) = \mathrm{HF}(L', d)$ for $d \gg 0$, thus $[L]_d = [L']_d$. Let $K = \big([L]_d\big) \subseteq A$, it follows that $L = L' = K : \mathfrak{m}_A^\infty$, proving uniqueness. In the case of $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ it is known that the lex point is smooth [35], however this is unknown for Clements-Lindström schemes [30].

If $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A) \neq \emptyset$, we denote by $\mathrm{Lex}(p(\zeta), A)$ the unique lex ideal in $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A)$. We emphasize that the lex ideal of a Hilbert function and the lex ideal of a Hilbert polynomial are different concepts, and both are relevant for this work. The notation $\mathrm{Lex}(I)$ is reserved for the lex ideal with the same Hilbert function as the ideal I.

By Remark 1.6 the set of monomial subschemes in $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A)$ is non-empty whenever $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A) \neq \emptyset$. On the other hand, this set is always finite, as the next discussion shows.

Remark 1.7. There are finitely many monomial subschemes in each $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A)$. To see this, since the preimage in the polynomial ring of a saturated monomial ideal of A is again saturated and monomial, it suffices to treat the case when A is a polynomial ring. There is a well-known upper bound, due to Gotzmann [21], for the Castelnuovo-Mumford regularity of a saturated ideal $J \subseteq A$ in terms of $\mathrm{HP}(A/J)$. This implies the desired conclusion, since there are finitely many monomial ideals generated in bounded degrees.

We remark that there are algorithms to produce all the strongly stable points or almost lex points of $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$, see for instance [1, 11, 27]. These algorithms can be extended with minor modifications to the case of Clements-Lindström rings A.

Example 1.8. Let $d_1 = 2, d_2 = 3, d_3 = d_4 = \infty$ and consider the associated Clements-Lindström ring $A = \frac{\Bbbk[x,y,z,w]}{(x^2,y^3)}$. For $p(\zeta) = 3\zeta + 5$ the strongly stable ideals in $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}A)$ are

$$\begin{array}{lll} \big(xy, xz^5\big) & \big(xy^2, xyz^2, xz^3\big) & \big(xy^2, xyz, xz^4\big) \\ \big(xy^2, xyz, y^2z^3\big) & \big(xy^2, xyz^2, y^2z^2\big) & \big(xy, y^2z^4\big)\,. \end{array}$$

The ideals in the first row are almost lex, and $\mathrm{Lex}(3\zeta + 5, A) = \big(xy, xz^5\big)$.

2. Decomposition of monomial ideals

We introduce an inductive decomposition of monomial ideals in Clements-Lindström rings. This decomposition is particularly effective for strongly stable and (almost) lex ideals; it will play a fundamental role in the construction of expansive ideals in Section 3.

For the rest of the paper, we fix the following notation:

$$\begin{aligned} S &= \Bbbk[x_1, x_2, \dots, x_n, x_{n+1}] & R &= S/\big(x_1^{d_1}, \dots, x_n^{d_n}\big) \\ \overline{S} &= \Bbbk[x_1, x_2, \dots, x_{n-1}, x_{n+1}] & \overline{R} &= \overline{S}/\big(x_1^{d_1}, \dots, x_{n-1}^{d_{n-1}}\big) \\ \widetilde{S} &= \Bbbk[x_1, x_2, \dots, x_{n-1}, x_n] & \widetilde{R} &= \widetilde{S}/\big(x_1^{d_1}, \dots, x_n^{d_n}\big) \end{aligned} \tag{2.1}$$

where $n \in \mathbb{N}$ and $d_1 \leq d_2 \leq \cdots \leq d_n$ is any sequence with $d_i \in \mathbb{N} \cup \{\infty\}$. In other words, we will always set $d_{n+1} = \infty$, so that $x_{n+1}^{d_{n+1}} = 0$ and it will be omitted. In this way, the Clements-Lindström rings R and $\overline{R}$ always have positive Krull dimension, whereas the Clements-Lindström ring $\widetilde{R}$ may be Artinian. The rings $\overline{R}$ and $\overline{S}$ are defined only if $n > 0$. When $n = 0$ we have $R = S = \Bbbk[x_1]$, and the only saturated ideals are the zero ideal and the unit ideal. Observe that $\overline{S}$ and $\overline{R}$ are algebra retracts of S and R, respectively, and they may be regarded either as subrings or as factor rings; similarly for $\widetilde{S}$ and $\widetilde{R}$. By abuse of notation, the symbols x_i will be used to denote elements in different rings.

Since $R = \widetilde{R}[x_{n+1}]$ there is a tight relation between invariants of ideals of R and $\widetilde{R}$; we summarize the main formulas in the following remark.

Remark 2.1. Let $I \subseteq R$ be a saturated ideal such that $I : x_{n+1} = I$. Denote by $\widetilde{I} = \frac{I+(x_{n+1})}{(x_{n+1})} \subseteq \widetilde{R}$ the image of I in $\widetilde{R}$. Then $\mathrm{HF}(\widetilde{I}, d) = \mathrm{HF}(I, d) - \mathrm{HF}(I, d-1)$ for $d \in \mathbb{Z}$ and $\mathrm{HP}(\widetilde{I}, \zeta) = \mathrm{HF}(I, \zeta) - \mathrm{HF}(I, \zeta - 1)$. Furthermore, for all i, j we have

$$\begin{aligned} \beta_{i,j}^{\widetilde{S}}(\widetilde{R}/\widetilde{I}) &= \beta_{i,j}^{S}(R/I) & \beta_{i,j}^{S}(\widetilde{R}/\widetilde{I}) &= \beta_{i,j}^{S}(R/I) + \beta_{i-1,j-1}^{S}(R/I) \\ \beta_{i,j}^{\widetilde{R}}(\widetilde{R}/\widetilde{I}) &= \beta_{i,j}^{R}(R/I) & \beta_{i,j}^{R}(\widetilde{R}/\widetilde{I}) &= \beta_{i,j}^{R}(R/I) + \beta_{i-1,j-1}^{R}(R/I). \end{aligned}$$

If I is monomial or strongly stable, then so is $\widetilde{I}$. Conversely, given any strongly stable $K \subseteq \widetilde{R}$, the extension $KR \subseteq R$ of K to R is a saturated strongly stable ideal whose image in $\widetilde{R}$ is K.

For a monomial ideal $I \subseteq R$ there exist uniquely determined monomial ideals $I_\ell \subseteq \overline{R}$ such that the following decomposition of $\overline{R}$-modules holds

$$I = \bigoplus_{\ell=0}^{d_n - 1} I_\ell x_n^\ell. \tag{2.2}$$

The set of components $\{I_\ell\}$ is finite if $d_n < \infty$, infinite otherwise. Throughout the paper, the notation I_ℓ will always refer to this decomposition; it should not be confused with graded components, denoted instead by $[I]_j$.

Example 2.2. Let $R = \Bbbk[x_1, x_2, x_3, x_4]/\big(x_1^3, x_2^3, x_3^4\big)$, so that $\overline{R} = \Bbbk[x_1, x_2, x_4]/\big(x_1^3, x_2^3\big)$. The components of the ideal $I = \big(x_1^2,\, x_1x_2^2x_3,\, x_1x_2x_3^2,\, x_1x_3^3,\, x_2^2x_3^2\big) \subseteq R$ are the $\overline{R}$-ideals

$$I_0 = \big(x_1^2\big), \quad I_1 = \big(x_1^2, x_1x_2^2\big), \quad I_2 = \big(x_1^2, x_1x_2, x_2^2\big), \quad I_3 = \big(x_1, x_2^2\big).$$

We are going to record some elementary properties of the decomposition (2.2). First, we define a partial order $\preceq$ among univariate polynomials with rational coefficients.

Definition 2.3. Let $p(\zeta), q(\zeta) \in \mathbb{Q}[\zeta]$. We set $p(\zeta) \preceq q(\zeta)$ if $q(\zeta) - p(\zeta)$ is a non-negative constant polynomial, i.e., if the coefficients of positive degree coincide in $p(\zeta)$ and $q(\zeta)$ and the constant terms satisfy $p(0) \leq q(0)$.

Recall that the constant term of a Hilbert polynomial is always an integer, carrying the same information as the arithmetic genus.

Proposition 2.4. *Let $I \subseteq J \subseteq R$ be saturated strongly stable ideals. The quotient J/I is a free module over $\Bbbk[x_{n+1}]$ via restriction of scalars. The following conditions are equivalent*

(i) $\mathrm{HP}(I) \preceq \mathrm{HP}(J)$
(ii) $\mathrm{rank}_{\Bbbk[x_{n+1}]}\,(J/I) < \infty$
(iii) $\dim_{\Bbbk}\big(\widetilde{J}/\widetilde{I}\big) < \infty$

and if these conditions are satisfied then $\mathrm{HP}(J) - \mathrm{HP}(I) = \mathrm{rank}_{\Bbbk[x_{n+1}]}\,(J/I) = \dim_{\Bbbk}\big(\widetilde{J}/\widetilde{I}\big)$.

Proof. Let $M = J/I$ and $\widetilde{M} = \widetilde{J}/\widetilde{I}$, with notation as in Remark 2.1. Then $M \cong \widetilde{M} \otimes_{\widetilde{R}} \widetilde{R}[x_{n+1}] \cong \widetilde{M} \otimes_{\Bbbk} \Bbbk[x_{n+1}]$, implying the first statement and the equality $\mathrm{rank}_{\Bbbk[x_{n+1}]}\,(J/I) = \dim_{\Bbbk}\big(\widetilde{J}/\widetilde{I}\big)$. Since $I \subseteq J$, we have $\mathrm{HP}(I) \preceq \mathrm{HP}(J)$ if and only if $\mathrm{HP}(M) = \mathrm{HP}(J) - \mathrm{HP}(I)$ is a constant, equivalently M has Krull dimension at most 1, equivalently M is a finite $\Bbbk[x_{n+1}]$-module. Finally, the rank of a finite free $\Bbbk[x_{n+1}]$-module is equal to its Hilbert polynomial. □

In the next proposition we list basic properties of the decomposition (2.2).

Proposition 2.5. *Let R be a Clements-Lindström ring and $I \subseteq R$ a monomial ideal.*

(1) *The sequence $\{I_\ell\}$ is a non-decreasing chain of ideals of $\overline{R}$.*
(2) *If $d_n = \infty$ the sequence $\{I_\ell\}$ is eventually constant, and the limit is equal to the ideal*

$$I_\infty := \frac{I : (x_n)^\infty + (x_n)}{(x_n)} \subseteq \overline{R}.$$

(3) *I is strongly stable if and only if I_ℓ is strongly stable for all ℓ and $(x_1, \ldots, x_{n-1})I_\ell \subseteq I_{\ell-1}$ for all $\ell \geq 1$.*
(4) *If I is strongly stable, then I is saturated if and only if I_ℓ is saturated for every ℓ.*
(5) $\mathrm{HP}\big((x_1, \ldots, x_n)I\big) \preceq \mathrm{HP}\big(I\big)$.

Now assume that I is strongly stable and saturated.

(6) *$I_{\ell+1}/I_\ell$ is a finite free $\Bbbk[x_{n+1}]$-module of rank equal to the integer* $\mathrm{HP}(I_{\ell+1}) - \mathrm{HP}(I_\ell)$.
(7) *If $d_n = \infty$ then* $\mathrm{HP}(I_\infty, \zeta) = \mathrm{HP}(I, \zeta) - \mathrm{HP}(I, \zeta - 1)$.
(8) $\mathrm{HP}(I_\ell, \zeta) - \mathrm{HP}(I, \zeta) + \mathrm{HP}(I, \zeta - 1)$ *is a constant polynomial. In particular, the coefficient of ζ^h in $\mathrm{HP}(I_\ell)$, where $h > 0$, is independent of ℓ and uniquely determined by* $\mathrm{HP}(I)$.
(9) $\mathrm{HP}(I_{\ell_1}) \preceq \mathrm{HP}(I_{\ell_2})$ *for all $\ell_1 \leq \ell_2 < d_n$.*
(10) *There exists a saturated strongly stable ideal $J \subseteq I$ such that* $\mathrm{HP}(I) - \mathrm{HP}(J) = 1$.

Proof. (1) follows immediately from (2.2) since the ideal I is closed under multiplication by x_n.

Assume $d_n = \infty$. Since $\overline{R}$ is Noetherian, the non-decreasing sequence $\{I_\ell\}$ stabilizes. Choose $\ell_0 \in \mathbb{N}$ so that $I_\ell = I_{\ell_0}$ for all $\ell \geq \ell_0$, and consider the ideal $J = \bigoplus_{\ell=0}^{d_n-1} I_{\ell_0} x_n^\ell \subseteq R$. Then we have $J : x_n = J$, $I \subseteq J$, and $x_n^{\ell_0} J \subseteq I$. It follows that $J = I : (x_n)^\infty$, and thus $I_\infty = I_{\ell_0} = \frac{J+(x_n)}{(x_n)} = \frac{I:(x_n)^\infty + (x_n)}{(x_n)} \subseteq \overline{R}$, proving (2).

(3) holds by definition of strongly stable ideal.

(4) follows since a strongly stable ideal is saturated if and only if the last variable x_{n+1} does not divide any of its monomial minimal generators, under our assumption that $\dim R > 0$.

(5) holds because the R-module $I/(x_1, \dots, x_n)I$ has Krull dimension at most 1, hence its Hilbert polynomial is a non-negative constant.

(6) follows from (4) and Proposition 2.4, since by (3) $I_{\ell+1}/I_\ell$ is annihilated by $(x_1, \dots, x_{n-1})$ and thus it is a finite $\Bbbk[x_{n+1}]$-module.

With notation as in the proof of (2), let $J = I : (x_n)^\infty = \bigoplus_{\ell=0}^{d_n-1} I_{\ell_0} x_n^\ell \subseteq R$. Since x_n is a non-zerodivisor on R/J and $I_\infty = \frac{J+(x_n)}{(x_n)}$, we have $\mathrm{HP}(I_\infty, \zeta) = \mathrm{HP}(J, \zeta) - \mathrm{HP}(J, \zeta - 1)$. However, from (6) we see that J/I has Krull dimension at most 1, so that $\mathrm{HP}(J) - \mathrm{HP}(I)$ is a constant polynomial, and (7) follows.

If $d_n < \infty$ then R has Krull dimension 1, the coefficient of ζ^h in $\mathrm{HP}(I_\ell)$ and $\mathrm{HP}(I)$ is 0 for $h > 0$, thus (8) holds in this case. Now assume $d_n = \infty$. By (6) we have that $\mathrm{HP}(I_\ell) - \mathrm{HP}(I_{\ell-1}) \in \mathbb{N}$, hence the coefficient of ζ^h in $\mathrm{HP}(I_\ell)$ is independent of ℓ for $h > 0$. Therefore, the claim (8) reduces to the case of the component I_∞, and follows thus from (7).

(9) follows immediately from (1) and (6).

Let $\mathbf{u}_1, \dots, \mathbf{u}_t$ be the minimal monomial generators of I ordered decreasingly in $<_{\mathrm{lex}}$. Then the ideal $J = (\mathbf{u}_1, \dots, \mathbf{u}_{t-1}, x_1\mathbf{u}_t, \dots, x_n\mathbf{u}_t)$ satisfies (10). $\square$

3. The expansive point in the Hilbert scheme

This section represents the core of the paper: here we introduce the expansive point on the Hilbert scheme of a Clements-Lindström scheme. We develop a machine to deal with expansive ideals both from an abstract and computational perspective. The reader may choose to skip the proof of Theorem 3.1 – the rest of the paper relies on the axioms (A1) through (A7), but does not use the proof. Recall our standing notation (2.1) on Clements-Lindström rings.

Theorem 3.1. *Let R be a Clements-Lindström ring. For every polynomial $p(\zeta) \in \mathbb{Q}[\zeta]$ such that $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R) \neq \emptyset$ there exists a unique ideal $\mathrm{Exp}(p, R) \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$, called the* **expansive ideal** *with Hilbert polynomial $p(\zeta)$, such that the following axioms are satisfied:*

- (A1) $\mathrm{Exp}(p, R)$ *is strongly stable;*
- (A2) *the components $\mathrm{Exp}(p, R)_\ell \subseteq \overline{R}$ are expansive for all ℓ;*
- (A3) *given two polynomials $p(\zeta) \preceq p'(\zeta)$ we have $\mathrm{Exp}(p', R) \subseteq \mathrm{Exp}(p, R)$;*
- (A4) $(x_1, \dots, x_n)\mathrm{Exp}(p, R)$ *is expansive;*
- (A5) *if $q(\zeta) = \mathrm{HP}\big(\overline{R}/\mathrm{Exp}(p, R)_k\big) - 1$ is such that $\mathrm{Hilb}^{q(\zeta)}\big(\mathrm{Proj}\overline{R}\big) \neq \emptyset$, for some $k < d_n$, then for all $h < k$ we have $\mathrm{Exp}(p, R)_h \subseteq (x_1, \dots, x_{n-1})^{k-h}\mathrm{Exp}\big(q(\zeta), \overline{R}\big)$;*
- (A6) *if $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ is strongly stable then for every $0 \leq \rho \leq d_n - 1$ we have*
$$\sum_{\ell=0}^{\rho} \mathrm{HP}\big(\mathrm{Exp}(p, R)_\ell\big) \preceq \sum_{\ell=0}^{\rho} \mathrm{HP}(J_\ell);$$
- (A7) *if $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ is strongly stable then $\mathrm{HP}\big((x_1, \dots, x_n)\mathrm{Exp}(p, R)\big) \preceq \mathrm{HP}\big((x_1, \dots, x_n)J\big)$.*

We define $\mathrm{Exp}(I) = \mathrm{Exp}(p(\zeta), R)$ if $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$.

Proof of Theorem 3.1. We prove the theorem by induction on the number of variables $n + 1$. For the base of the induction $n = 0$ there is nothing to prove, so we assume $n > 0$. Fix a polynomial $p(\zeta)$ such that $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R) \neq \emptyset$. It follows by Proposition 2.5 (8) that for any two strongly stable $I, J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ and $0 \leq h, k < d_n$, the polynomial $\mathrm{HP}(I_h) - \mathrm{HP}(J_k)$ is a constant, so $\mathrm{HP}(I_h) \preceq \mathrm{HP}(J_k)$ or $\mathrm{HP}(I_h) \succeq \mathrm{HP}(J_k)$. Moreover, by Remarks 1.6 and 1.7 the set of strongly stable ideals in $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ is finite and non-empty. We can choose a strongly stable $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ satisfying the following condition: if $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ is also strongly stable and $\sum_{\ell=0}^{\rho} \mathrm{HP}(J_\ell) \preceq \sum_{\ell=0}^{\rho} \mathrm{HP}(I_\ell)$ for every $\rho < d_n$, then $\mathrm{HP}(I_\ell) = \mathrm{HP}(J_\ell)$ for every ℓ. In other words, the sequence of polynomials $\big\{ \sum_{\ell=0}^{\rho} \mathrm{HP}(I_\ell)\big\}_{\rho=0}^{d_n-1}$ is componentwise minimal, among all strongly stable ideals in $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$. With this choice of I we define

$$E = \bigoplus_{\ell < d_n} \mathrm{Exp}(I_\ell) x_n^\ell \subseteq R. \tag{3.1}$$

Note that $\mathrm{HP}(E_\ell) = \mathrm{HP}(I_\ell)$ for all ℓ, and $\mathrm{HP}(E) = \mathrm{HP}(I)$. Thus E is an ideal of R, as Proposition 2.5 (9) and (A3) imply that the sequence of components $\{\mathrm{Exp}(I_\ell)\}$ is non-decreasing. The variable x_{n+1} does not divide any monomial minimal generator of E, since E_ℓ is saturated for all ℓ by induction; it follows that E is saturated, so that $E \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$. We claim that E satisfies the axioms (A1) through (A7). The axiom (A2) holds by construction.

By (A1) the components of E are strongly stable ideals of $\overline{R}$. By Proposition 2.5 (3) we have $(x_1, \dots, x_{n-1})I_\ell \subseteq I_{\ell-1}$. By Proposition 2.5 (5) and (8) we have $\mathrm{HP}\big((x_1, \dots, x_{n-1})I_\ell\big) \preceq \mathrm{HP}(I_{\ell-1})$, equivalently, $\mathrm{HP}\big(\overline{R}/(x_1, \dots, x_{n-1})I_\ell\big) \succeq \mathrm{HP}\big(\overline{R}/I_{\ell-1}\big)$, and by (A3) it follows that

$\mathrm{Exp}\big((x_1,\dots,x_{n-1})I_\ell\big) \subseteq E_{\ell-1}$. By (A7) $\mathrm{HP}\big((x_1,\dots,x_{n-1})E_\ell\big) \preceq \mathrm{HP}\big((x_1,\dots,x_{n-1})I_\ell\big)$, i.e. $\mathrm{HP}\big(\overline{R}/(x_1,\dots,x_{n-1})E_\ell\big) \succeq \mathrm{HP}\big(\overline{R}/(x_1,\dots,x_{n-1})I_\ell\big)$. Since $(x_1,\dots,x_{n-1})E_\ell$ is expansive by (A4), axiom (A3) yields $(x_1,\dots,x_{n-1})E_\ell \subseteq \mathrm{Exp}\big((x_1,\dots,x_{n-1})I_\ell\big)$. Combining the two inclusions we derive $(x_1,\dots,x_{n-1})E_\ell \subseteq E_{\ell-1}$, and by Proposition 2.5 (3) E satisfies (A1).

Next, we prove that E verifies (A5). Assume by contradiction there exist $h < k < d_n$ such that $\mathrm{Hilb}^{q(\zeta)}(\mathrm{Proj}\overline{R}) \neq \emptyset$ and $E_h \not\subseteq F := (x_1,\dots,x_{n-1})^{k-h}\mathrm{Exp}\big(q(\zeta),\overline{R}\big) \subseteq \overline{R}$, where $q(\zeta) = \mathrm{HP}\big(\overline{R}/E_k\big) - 1$. Pick h,k so that $k-h$ is the least possible. By (A4) F is expansive. We have $\mathrm{HP}(F) \preceq \mathrm{HP}\big(\mathrm{Exp}\big(q(\zeta),\overline{R}\big)\big)$ by Proposition 2.5 (5). Since $\mathrm{HP}(E_h) \preceq \mathrm{HP}(E_k)$ and $\mathrm{HP}\big(\mathrm{Exp}\big(q(\zeta),\overline{R}\big)\big) = \mathrm{HP}(E_k)+1$ we conclude that $\mathrm{HP}(F)$ and $\mathrm{HP}(E_h)$ differ by an integer. As $E_h \not\subseteq F$ and both ideals are expansive, (A3) implies that $F \subsetneq E_h$. Let $\eta \le h$ be minimal such that $E_\eta = E_h$ and $\chi > k$ be maximal such that $E_k = E_\chi$.

We exhibit a strongly stable $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ generating a contradiction. Set

$$J = \bigoplus_{\ell\neq\eta,\chi} E_\ell x_n^\ell \oplus \mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E_\eta\big)+1,\overline{R}\big)x_n^\eta \oplus \mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E_\chi\big)-1,\overline{R}\big)x_n^\chi \subseteq R. \tag{3.2}$$

Note that both $\mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E_\eta\big)+1,\overline{R}\big)$ and $\mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E_\chi\big)-1,\overline{R}\big)$ exist by induction, since the corresponding Hilbert schemes are nonempty. For the former, by Proposition 2.5 (10) there exists some ideal $\mathcal{J} \subseteq E_h$ with $\mathrm{HP}\big(\overline{R}/\mathcal{J}\big) = \mathrm{HP}\big(\overline{R}/E_h\big)+1$. For the latter, the Hilbert scheme of $q(\zeta)$ is nonempty by assumption. We observe that J is an ideal of R, as its components form a non-decreasing sequence: by (A3), since we already know that E is an ideal, it only remains to check the two inclusions $E_{\eta-1} \subseteq \mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E_\eta\big)+1,\overline{R}\big)$ and $\mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E_\chi\big)-1,\overline{R}\big) \subseteq E_{\chi+1}$, but they follow by the choice of η,χ. Clearly J is monomial, and it is saturated since x_{n+1} does not divide its minimal generators.

We prove that J is strongly stable. By Proposition 2.5 (3) it suffices to show that $(x_1,\dots,x_{n-1})J_\ell \subseteq J_{\ell-1}$ for each ℓ, since all J_ℓ are strongly stable. Using Proposition 2.5 (5) and (8), we see that the $\mathrm{HP}\big(\overline{R}/(x_1,\dots,x_{n-1})J_\ell\big) - \mathrm{HP}\big(\overline{R}/J_{\ell-1}\big)$ is an integer. Both $(x_1,\dots,x_{n-1})J_\ell$ and $J_{\ell-1}$ are expansive, hence it suffices to show $\mathrm{HP}\big(\overline{R}/(x_1,\dots,x_{n-1})J_\ell\big) - \mathrm{HP}\big(\overline{R}/J_{\ell-1}\big) \ge 0$. There are three cases that do not follow from E being strongly stable. If $\ell = \chi = \eta+1$, necessarily $\chi = k$ and $\eta = h$, hence $F = (x_1,\dots,x_{n-1})^{k-h}\mathrm{Exp}\big(q(\zeta),\overline{R}\big) \subsetneq E_h$ becomes $(x_1,\dots,x_{n-1})J_\chi \subsetneq E_\eta$, so $\mathrm{HP}(E_\eta) - \mathrm{HP}\big((x_1,\dots,x_{n-1})J_\chi\big) > 0$ and $\mathrm{HP}\big(\overline{R}/(x_1,\dots,x_{n-1})J_\ell\big) - \mathrm{HP}\big(\overline{R}/J_{\ell-1}\big) \ge 0$. If $\ell = \eta+1 < \chi$, we must prove $\mathrm{HP}\big(\overline{R}/(x_1,\dots,x_{n-1})J_{\eta+1}\big) - \mathrm{HP}\big(\overline{R}/J_\eta\big) \ge 0$, i.e. $\mathrm{HP}\big(\overline{R}/(x_1,\dots,x_{n-1})E_{\eta+1}\big) - \mathrm{HP}\big(\overline{R}/E_\eta\big) - 1 \ge 0$, equivalently, $(x_1,\dots,x_{n-1})E_{\eta+1} \subsetneq E_\eta$. But if this were false, then $(x_1,\dots,x_{n-1})E_{\eta+1} = E_\eta$, since E is strongly stable. In particular $E_{\eta+1} \neq E_\eta$, forcing $\eta = h$, and the pair $h+1,k$ would contradict the choice of h,k. Finally, if $\ell = \chi > \eta+1$, we must show $\mathrm{HP}\big(\overline{R}/(x_1,\dots,x_{n-1})J_\chi\big) - \mathrm{HP}\big(\overline{R}/J_{\chi-1}\big) \ge 0$. If this were false, then $E_{\chi-1} \subsetneq (x_1,\dots,x_{n-1})E_\chi$, forcing $\chi = k$, and $\mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E_{k-1}\big)+1,\overline{R}\big) \subseteq (x_1,\dots,x_{n-1})\mathrm{Exp}\big(q(\zeta)+1,\overline{R}\big)$. We obtain $(x_1,\dots,x_{n-1})^{k-h-1}\mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E_{k-1}\big)+1,\overline{R}\big) \subseteq (x_1,\dots,x_{n-1})^{k-h}\mathrm{Exp}\big(q(\zeta)+1,\overline{R}\big) = F \subsetneq E_h$, and the pair $h,k-1$ contradicts the choice of h,k. Thus J is strongly stable.

From (3.2) it follows immediately that $\sum_{\ell=0}^{\rho} \mathrm{HP}(J_\ell) \preceq \sum_{\ell=0}^{\rho} \mathrm{HP}(E_\ell)$ for every $\rho < d_n$ and $\sum_{\ell=0}^{\eta} \mathrm{HP}(J_\ell) \prec \sum_{\ell=0}^{\eta} \mathrm{HP}(E_\ell)$. This yields a contradiction to the choice of I, proving (A5).

In order to verify (A3), we prove a stronger statement:

(†) if $E, E' \subseteq R$ are saturated monomial ideals satisfying (A1), (A2), (A5), and such that $\mathrm{HP}(E') - \mathrm{HP}(E)$ is an integer, then $E \subseteq E'$ or $E' \subseteq E$.

Since $\mathrm{HP}(E') - \mathrm{HP}(E) \in \mathbb{Z}$, by Proposition 2.5 (8) $\mathrm{HP}(E'_h) - \mathrm{HP}(E_k) \in \mathbb{Z}$ for every h, k. Suppose $E \neq E'$, and let h be the least index such that $E_h \neq E'_h$. By axiom (A3) either $E_h \subsetneq E'_h$ or $E'_h \subsetneq E_h$; assume, for instance, that $E_h \subsetneq E'_h$. Assume by contradiction that there exists $h < k < d_n$ with $E_k \not\subseteq E'_k$. Using (A3) again we find that $E'_k \subsetneq E_k$ and $\mathrm{HP}(E'_k) \prec \mathrm{HP}(E_k)$, so $\mathrm{HP}\big(\overline{R}/E_k\big) \preceq \mathrm{HP}\big(\overline{R}/E'_k\big) - 1$ and $\mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E'_k\big) - 1, \overline{R}\big) \subseteq E_k$. Since E is strongly stable, we have $(x_1, \ldots, x_{n-1})^{k-h} E_k \subseteq E_h$. By (A5) we get $E'_h \subseteq (x_1, \ldots, x_{n-1})^{k-h}\mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E'_k\big) - 1, \overline{R}\big)$. We derive the contradiction

$$E_h \subsetneq E'_h \subseteq (x_1, \ldots, x_{n-1})^{k-h}\mathrm{Exp}\big(\mathrm{HP}\big(\overline{R}/E'_k\big) - 1, \overline{R}\big) \subseteq (x_1, \ldots, x_{n-1})^{k-h} E_k \subseteq E_h.$$

This proves the claim (†), which implies the axiom (A3) and also the uniqueness of the expansive ideal for each Hilbert polynomial.

Next, we show (A6). We must prove that $\big\{ \sum_{\ell=0}^{\rho} \mathrm{HP}(E_\ell)\big\}_{\rho=0}^{d_n-1}$ is the unique minimal sequence, with respect to componentwise comparison by $\preceq$, among all strongly stable ideals $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj} R)$. Let $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj} R)$ be strongly stable such that $\big\{ \sum_{\ell=0}^{\rho} \mathrm{HP}(J_\ell)\big\}_{\rho=0}^{d_n-1}$ is also minimal with respect to componentwise comparison. As in (3.1) we define

$$E' = \bigoplus_{\ell < d_n} \mathrm{Exp}(J_\ell) x_n^\ell \subseteq R$$

and, by the same proof as for E, it follows that E' is a saturated strongly stable ideal of R satisfying (A1), (A2), (A5). By (†) we deduce that $E = E'$, proving thus (A6).

Next, we show that E verifies (A7). Let $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj} R)$ be strongly stable. Applying (A6) with $\rho = 0$ we have $\mathrm{HP}\big(\overline{R}/J_0\big) \preceq \mathrm{HP}\big(\overline{R}/E_0\big)$, thus $E_0 \subseteq \mathrm{Exp}(J_0)$ by (A3), and therefore $\mathrm{HP}\big((x_1, \ldots, x_{n-1})E_0\big) \preceq \mathrm{HP}\big((x_1, \ldots, x_{n-1})\mathrm{Exp}(J_0)\big)$. On the other hand, by (A7) we have $\mathrm{HP}\big((x_1, \ldots, x_{n-1})\mathrm{Exp}(J_0)\big) \preceq \mathrm{HP}\big((x_1, \ldots, x_{n-1})J_0\big)$. Combining the inequalities we obtain $\mathrm{HP}\big((x_1, \ldots, x_{n-1})E_0\big) \preceq \mathrm{HP}\big((x_1, \ldots, x_{n-1})J_0\big)$. Now consider decompositions

$$(x_1, \ldots, x_n)E = (x_1, \ldots, x_{n-1})E_0 \oplus \bigoplus_{\ell=0}^{d_n-2} E_\ell x_n^{\ell+1}, \tag{3.3}$$

$$(x_1, \ldots, x_n)J = (x_1, \ldots, x_{n-1})J_0 \oplus \bigoplus_{\ell=0}^{d_n-2} J_\ell x_n^{\ell+1}. \tag{3.4}$$

The desired inequality $\mathrm{HP}\big((x_1, \ldots, x_n)E\big) \preceq \mathrm{HP}\big((x_1, \ldots, x_n)J\big)$ follows from additivity of $\mathrm{HP}(-)$ on direct sums and axiom (A6) applied with $\rho = d_n - 2$.

Finally, we show that E verifies (A4). It follows from (3.3) that $(x_1,\ldots,x_n)E$ is a saturated ideal satisfying (A1), (A2), (A5). Repeating the construction (3.1) for the polynomial $p'(\zeta) = \mathrm{HP}\big(\overline{R}/(x_1,\ldots,x_n)E\big)$ yields another ideal $E' \in \mathrm{Hilb}^{p'(\zeta)}(\mathrm{Proj}R)$ satisfying (A1), (A2), (A5). Applying (†) to E' and $(x_1,\ldots,x_n)E$ we conclude that $(x_1,\ldots,x_n)E = E'$, which means that $(x_1,\ldots,x_n)E$ is the expansive ideal with Hilbert polynomial $p'(\zeta)$. □

We remark that expansive ideals satisfy also an extremal property with respect to higher hyperplane sections, comparable to the inequalities for lex ideals proved in [18, 19, 24].

Corollary 3.2. *Let $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ be strongly stable, then $\mathrm{HP}\big(\mathrm{Exp}(p)+(x_n^h)\big) \preceq \mathrm{HP}\big(J+(x_n^h)\big)$ for every h.*

Proof. It follows from (A6) and additivity of $\mathrm{HP}(-)$ on the decompositions

$$J + (x_n^h) = \bigoplus_{\ell=0}^{h-1} J_\ell\, x_n^\ell \oplus \bigoplus_{\ell=h}^{d_n-1} \overline{R}x_n^\ell \qquad \mathrm{Exp}(p) + (x_n^h) = \bigoplus_{\ell=0}^{h-1} \mathrm{Exp}(p)_\ell\, x_n^\ell \oplus \bigoplus_{\ell=h}^{d_n-1} \overline{R}x_n^\ell.$$

□

We conclude the section with a comment. The approach undertaken in Theorem 3.1 has the advantage of identifying extremal properties that play a crucial role in estimating syzygies. On the other hand, the structure of $\mathrm{Exp}(p)$ remains somewhat obscure, and the axioms are impractical for the purpose of computing examples. We are going to fill this gap in Section 6.

4. Maximal syzygies

We present the main application of expansive ideals: the existence of sharp upper bounds for the syzygies of subschemes of a Clements-Linström scheme. Our treatment relies entirely on the axioms of Theorem 3.1. The extension of the result to arbitrary complete intersections in $\mathbb{P}^n$ is also discussed. We keep the notation of the previous sections, and in particular (2.1).

First, the main result of [26] allows to perform an important reduction to almost lex ideals.

Lemma 4.1. *For any $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ there exists an almost lex $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ with $\mathrm{HF}(I) = \mathrm{HF}(J)$ and $\beta^S_{i,j}(R/I) \leq \beta^S_{i,j}(R/J)$ for all i,j.*

Proof. Since $I \subseteq R$ is saturated, there exists a linear form $\ell \in [S]_1$ that is a non-zerodivisor on R/I. Up to a change of coordinates in S, we may assume that $\ell = x_{n+1}$. With the notation of Remark 2.1, consider $\widetilde{J} = \mathrm{Lex}(\widetilde{I}) \subseteq \widetilde{R}$ and let $J = \widetilde{J}R \subseteq R$. By [26, Theorem 8.1] we obtain $\beta^{\widetilde{S}}_{i,j}(\widetilde{R}/\widetilde{I}) \leq \beta^{\widetilde{S}}_{i,j}(\widetilde{R}/\widetilde{J})$ for all i,j, and the conclusion follows from Remark 2.1. □

In the next lemma we consider the natural $\mathbb{Z}^{n+1}$-grading on R.

Lemma 4.2. *Let M be a finite $\mathbb{Z}^{n+1}$-graded R-module that is a free $\Bbbk[x_{n+1}]$-module of finite rank $c \in \mathbb{N}$ via restriction of scalars. For every $i \in \mathbb{N}$ we have*

(i) $\beta_i^S(M) \leq c \cdot \beta_i^S(\Bbbk[x_{n+1}])$ *and* $\beta_i^R(M) \leq c \cdot \beta_i^R(\Bbbk[x_{n+1}])$;
(ii) *if* $\operatorname{ann}_R(M) = (x_1, \dots, x_n)$, *then* $\beta_i^S(M) = c \cdot \beta_i^S(\Bbbk[x_{n+1}])$ *and* $\beta_i^R(M) = c \cdot \beta_i^R(\Bbbk[x_{n+1}])$.

Proof. We prove (ii) first. Let $m_1, \dots, m_s$ be minimal $\mathbb{Z}^{n+1}$-graded R-module generators of M. The assumptions imply the isomorphisms of R-modules $M \cong Rm_1 \oplus \cdots \oplus Rm_s$ and $Rm_h \cong \Bbbk[x_{n+1}]$ for every h, so that $s = c$ and the formulas for the Betti numbers follow. To prove (i), we may assume $c > 1$. Let $M' = (x_1, \dots, x_n)M$ and $M'' = M/M'$. Both M' and M'' are finite $\mathbb{Z}^{n+1}$-graded R-modules. As $\Bbbk[x_{n+1}]$-modules via restriction of scalars, M' is free of rank less than c, whereas M'' is also free, by multidegree reasons, and it satisfies (ii). The conclusions follow by induction on c from the exact sequence $0 \to M' \to M \to M'' \to 0$. □

Theorem 4.3. *Let* $S = \Bbbk[x_1, \dots, x_{n+1}]$ *be a polynomial ring and* $R = S/(x_1^{d_1}, \dots, x_n^{d_n})$ *a Clements-Lindström ring, where* $2 \leq d_1 \leq \cdots \leq d_n \leq \infty$. *For each polynomial* $p(\zeta)$ *we have*

$$\beta_i^S\big(R/I\big) \leq \beta_i^S\big(R/\operatorname{Exp}(p(\zeta))\big)$$

for all $I \in \operatorname{Hilb}^{p(\zeta)}(\operatorname{Proj} R)$ *and all* $i \geq 0$.

Proof. We proceed by induction on n and the case $n = 0$ is trivial, so let $n > 0$. By Lemma 4.1, we may assume without loss of generality that I is a strongly stable monomial ideal. Let $\mathcal{I}, \mathcal{E}$ denote the preimages of $I, \operatorname{Exp}(p) \subseteq R$ in the polynomial ring S. We have decompositions

$$\mathcal{I} = \bigoplus_{\ell=0}^{\infty} \mathcal{I}_\ell x_n^\ell, \qquad \mathcal{E} = \bigoplus_{\ell=0}^{\infty} \mathcal{E}_\ell x_n^\ell \tag{4.1}$$

where $\mathcal{I}_\ell, \mathcal{E}_\ell$ are ideals of $\overline{S}$. Specifically, $\mathcal{I}_\ell \subseteq \overline{S}$ is the preimage of $I_\ell \subseteq \overline{R}$ if $\ell < d_n$, and $\mathcal{I}_\ell = \overline{S}$ if $d_n \leq \ell < \infty$; likewise for $\mathcal{E}_\ell$. Since $R/I \cong S/\mathcal{I}$ and $R/\operatorname{Exp}(p) \cong S/\mathcal{E}$, we must prove that $\beta_i^S(\mathcal{I}) \leq \beta_i^S(\mathcal{E})$ for all i. The variable x_n is a non-zerodivisor on $S, \mathcal{I}, \mathcal{E}$, so it suffices to prove $\beta_i^{\overline{S}}(\mathcal{I}/x_n\mathcal{I}) \leq \beta_i^{\overline{S}}(\mathcal{E}/x_n\mathcal{E})$ for all i.

Let $\mathcal{J} \subseteq \overline{S}$ denote the preimage of $\operatorname{Exp}(I_0) \subseteq \overline{R}$. Since $\overline{S}/\mathcal{I}_0 \cong R/I_0$ and $\overline{S}/\mathcal{J} \cong \overline{R}/\operatorname{Exp}(I_0)$, by induction we have $\beta_i^{\overline{S}}(\mathcal{I}_0) \leq \beta^{\overline{S}}(\mathcal{J})$ for every $i \geq 0$. By Corollary 3.2 we have $\operatorname{HP}\big(\operatorname{Exp}(p) + (x_n)\big) \preceq \operatorname{HP}\big(I + (x_n)\big)$. Note that $I_0 \cong \frac{I+(x_n)}{(x_n)}$ and $\operatorname{Exp}(p)_0 \cong \frac{\operatorname{Exp}(p)+(x_n)}{(x_n)}$, so $\operatorname{HP}(\operatorname{Exp}(p)_0) \preceq \operatorname{HP}(I_0)$. By (A3) we conclude that $\operatorname{Exp}(p)_0 \subseteq \operatorname{Exp}(I_0)$, and hence $\mathcal{E}_0 \subseteq \mathcal{J}$. By Proposition 2.4 the quotient $\mathcal{J}/\mathcal{E}_0 \cong \operatorname{Exp}(I_0)/\operatorname{Exp}(p)_0$ is a free $\Bbbk[x_{n+1}]$-module over of rank $c_0 = \operatorname{HP}\big(\operatorname{Exp}(I_0)/\operatorname{Exp}(p)_0\big) = \operatorname{HP}\big(I+(x_n)\big) - \operatorname{HP}\big(\operatorname{Exp}(p)+(x_n)\big)$. Using the short exact sequence $0 \to \mathcal{E}_0 \to \mathcal{J} \to \mathcal{J}/\mathcal{E}_0 \to 0$ and Lemma 4.2 (i) we obtain

$$\beta_i^{\overline{S}}(\mathcal{I}_0) \leq \beta_i^{\overline{S}}(\mathcal{J}) \leq \beta_i^{\overline{S}}(\mathcal{E}_0) + \beta_i^{\overline{S}}(\mathcal{J}/\mathcal{E}_0) \leq \beta_i^{\overline{S}}(\mathcal{E}_0) + c_0\beta_i^{\overline{S}}\big(\Bbbk[x_{n+1}]\big). \tag{4.2}$$

Suppose first that $d_n = \infty$. From (4.1) we deduce decompositions of $\overline{S}$-modules

$$\frac{\mathcal{I}}{x_n\mathcal{I}} \cong \mathcal{I}_0 \oplus \bigoplus_{\ell=0}^{\infty} \frac{\mathcal{I}_\ell}{\mathcal{I}_{\ell-1}} \cong \mathcal{I}_0 \oplus \bigoplus_{\ell=0}^{\infty} \frac{I_\ell}{I_{\ell-1}}, \qquad \frac{\mathcal{E}}{x_n\mathcal{E}} \cong \mathcal{E}_0 \oplus \bigoplus_{\ell=0}^{\infty} \frac{\mathcal{E}_\ell}{\mathcal{E}_{\ell-1}} \cong \mathcal{E}_0 \oplus \bigoplus_{\ell=0}^{\infty} \frac{\operatorname{Exp}(p)_\ell}{\operatorname{Exp}(p)_{\ell-1}}. \tag{4.3}$$

By Proposition 2.5 (6) the $\overline{S}$-modules $\bigoplus_{\ell=0}^{\infty} \frac{I_\ell}{I_{\ell-1}}$ and $\bigoplus_{\ell=0}^{\infty} \frac{\mathrm{Exp}(p)_\ell}{\mathrm{Exp}(p)_{\ell-1}}$ are free $\Bbbk[x_{n+1}]$-modules of rank $c_1 = \mathrm{HP}(I_\infty) - \mathrm{HP}(I_0) \in \mathbb{N}$ and $c_2 = \mathrm{HP}(\mathrm{Exp}(p)_\infty) - \mathrm{HP}(\mathrm{Exp}(p)_0) \in \mathbb{N}$, respectively. Moreover, by 2.5 (3) these modules are annihilated by $(x_1, \ldots, x_{n-1}) \subseteq \overline{S}$. Using Lemma 4.2 (ii) and combining with (4.2) we obtain

$$\beta_i^{\overline{S}}(\mathcal{I}/x_n\mathcal{I}) = \beta_i^{\overline{S}}(\mathcal{I}_0) + \beta_i^{\overline{S}}\left(\bigoplus_{\ell=0}^{\infty} \frac{\mathcal{I}_\ell}{\mathcal{I}_{\ell-1}}\right) = \beta_i^{\overline{S}}(\mathcal{I}_0) + c_1\beta_i^{\overline{S}}\big(\Bbbk[x_{n+1}]\big)$$
$$\leq \beta_i^{\overline{S}}(\mathcal{E}_0) + (c_0 + c_1)\beta_i^{\overline{S}}\big(\Bbbk[x_{n+1}]\big).$$

We claim that $\beta_i^{\overline{S}}(\mathcal{E}/x_n\mathcal{E}) = \beta_i^{\overline{S}}(\mathcal{E}_0) + (c_0 + c_1)\beta_i^{\overline{S}}\big(\Bbbk[x_{n+1}]\big)$, concluding the proof in this case. This follows from (4.3) and Lemma 4.2 (ii) once we show that $\bigoplus_{\ell=0}^{\infty} \frac{\mathrm{Exp}(p)_\ell}{\mathrm{Exp}(p)_{\ell-1}}$ has rank $c_0 + c_1$ as $\Bbbk[x_{n+1}]$-module, that is, $c_2 = c_0 + c_1$. But this is true by definition of c_0, c_1, c_2, additivity of Hilbert polynomials, and the fact that $\mathrm{HP}(I_\infty) = \mathrm{HP}(\mathrm{Exp}(p)_\infty)$ by Proposition 2.5 (7).

Now suppose that $d_n < \infty$. The decompositions of $\overline{S}$-modules obtained from (4.1) become

$$\text{(4.4)} \qquad \frac{\mathcal{I}}{x_n\mathcal{I}} \cong \mathcal{I}_0 \oplus \bigoplus_{\ell=0}^{d_n-1} \frac{I_\ell}{I_{\ell-1}} \oplus \frac{\overline{R}}{I_{d_n-1}}, \qquad \frac{\mathcal{E}}{x_n\mathcal{E}} \cong \mathcal{E}_0 \oplus \bigoplus_{\ell=0}^{d_n-1} \frac{\mathrm{Exp}(p)_\ell}{\mathrm{Exp}(p)_{\ell-1}} \oplus \frac{\overline{R}}{\mathrm{Exp}(p)_{d_n-1}}.$$

Our goal is to estimate $\beta_i^{\overline{S}}(\overline{R}/I_{d_n-1})$. By induction $\beta_i^{\overline{S}}(\overline{R}/I_{d_n-1}) \leq \beta_i^{\overline{S}}\big(\overline{R}/\mathrm{Exp}(I_{d_n-1})\big)$ for all $i \geq 0$. Note that both $R, \overline{R}$ have Krull dimension 1, since $d_n < \infty$, hence all Hilbert polynomials of ideals are constant. By additivity of $\mathrm{HP}(-)$ we have the formulas

$$\mathrm{HP}(I) = \sum_{\ell=0}^{d_n-1} \mathrm{HP}(I_\ell) \qquad \mathrm{HP}\big(I + (x_n^{d_n-1})\big) = \sum_{\ell=0}^{d_n-2} \mathrm{HP}(I_\ell) + \mathrm{HP}(\overline{R})$$
$$\mathrm{HP}(\mathrm{Exp}(p)) = \sum_{\ell=0}^{d_n-1} \mathrm{HP}(\mathrm{Exp}(p)_\ell) \qquad \mathrm{HP}\big(\mathrm{Exp}(p) + (x_n^{d_n-1})\big) = \sum_{\ell=0}^{d_n-2} \mathrm{HP}(\mathrm{Exp}(p)_\ell) + \mathrm{HP}(\overline{R}).$$

By Corollary 3.2 we have $\mathrm{HP}\big(\mathrm{Exp}(p) + (x_n^{d_n-1})\big) \preceq \mathrm{HP}\big(I + (x_n^{d_n-1})\big)$. Since $\mathrm{HP}(I) = \mathrm{HP}(\mathrm{Exp}(p))$, the formulas above imply that $\mathrm{HP}(I_{d_n-1}) \preceq \mathrm{HP}(\mathrm{Exp}(p)_{d_n-1})$, and using (A3) we deduce $\mathrm{Exp}(I_{d_n-1}) \subseteq \mathrm{Exp}(p)_{d_n-1}$. From the short exact sequence

$$0 \to \frac{\mathrm{Exp}(p)_{d_n-1}}{\mathrm{Exp}(I_{d_n-1})} \to \frac{\overline{R}}{\mathrm{Exp}(I_{d_n-1})} \to \frac{\overline{R}}{\mathrm{Exp}(p)_{d_n-1}} \to 0$$

we obtain

$$\text{(4.5)} \qquad \beta_i^{\overline{S}}(\overline{R}/I_{d_n-1}) \leq \beta_i^{\overline{S}}\big(\overline{R}/\mathrm{Exp}(I_{d_n-1})\big) \leq \beta_i^{\overline{S}}\left(\frac{\mathrm{Exp}(p)_{d_n-1}}{\mathrm{Exp}(I_{d_n-1})}\right) + \beta_i^{\overline{S}}\left(\frac{\overline{R}}{\mathrm{Exp}(p)_{d_n-1}}\right).$$

Finally, we are going to use (4.4) to give an upper bound for $\beta_i^{\overline{S}}(\mathcal{I}/x_n\mathcal{I})$. As before, the $\overline{S}$-modules $\bigoplus_{\ell=0}^{d_n-1} \frac{I_\ell}{I_{\ell-1}}$ and $\bigoplus_{\ell=0}^{d_n-1} \frac{\mathrm{Exp}(p)_\ell}{\mathrm{Exp}(p)_{\ell-1}}$ are annihilated by $(x_1, \ldots, x_{n-1}) \subseteq \overline{S}$, and by Proposition 2.5 (6) they are free $\Bbbk[x_{n+1}]$-modules of ranks $c_1' = \mathrm{HP}(I_{d_n-1}) - \mathrm{HP}(I_0)$ and $c_2' = \mathrm{HP}\big(\mathrm{Exp}(p)_{d_n-1}\big) - \mathrm{HP}\big(\mathrm{Exp}(p)_0\big)$, respectively. By Proposition 2.4, the module $\frac{\mathrm{Exp}(p)_{d_n-1}}{\mathrm{Exp}(I_{d_n-1})}$

is also free over $\Bbbk[x_{n+1}]$, of rank $c_3 = \mathrm{HP}\big(\mathrm{Exp}(p)_{d_n-1}\big) - \mathrm{HP}\big(\mathrm{Exp}(I_{d_n-1}\big)$. Combining the decomposition (4.4) and the bounds (4.2), (4.5), and using Lemma 4.2 (i) we find

$$\begin{aligned}
\beta_i^{\overline{S}}(\mathcal{I}/x_n\mathcal{I}) &= \beta_i^{\overline{S}}(\mathcal{I}_0) + \beta_i^{\overline{S}}\left(\bigoplus_{\ell=0}^{d_n-1}\frac{I_\ell}{I_{\ell-1}}\right) + \beta_i^{\overline{S}}\left(\frac{\overline{R}}{I_{d_n-1}}\right)\\
&\leq \left[\beta_i^{\overline{S}}(\mathcal{E}_0) + c_0\beta_i^{\overline{S}}\big(\Bbbk[x_{n+1}]\big)\right] + c_1'\beta_i^{\overline{S}}\big(\Bbbk[x_{n+1}]\big) + \left[c_3\beta_i^{\overline{S}}\big(\Bbbk[x_{n+1}]\big) + \beta_i^{\overline{S}}\left(\frac{\overline{R}}{\mathrm{Exp}(p)_{d_n-1}}\right)\right]\\
&= \beta_i^{\overline{S}}(\mathcal{E}_0) + (c_0 + c_1' + c_3)\beta_i^{\overline{S}}\big(\Bbbk[x_{n+1}]\big) + \beta_i^{\overline{S}}\left(\frac{\overline{R}}{\mathrm{Exp}(p)_{d_n-1}}\right).
\end{aligned}$$

The expression in the last line is equal to $\beta_i^{\overline{S}}(\mathcal{E}/x_n\mathcal{E})$ because of (4.4), Lemma 4.2 (ii), and the fact that $c_2' = c_0 + c_1' + c_3$. This concludes the proof. □

We remark that, in the case of $\mathbb{P}^n$, the existence of a point in $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ satisfying the conclusion of Theorem 4.3 was proved in [9, Theorem 1.1], cf. the Introduction.

Remark 4.4. The numerical bounds on the Betti numbers provided by Theorem 4.3 do not depend on $\Bbbk$, as it follows from the combinatorial formula of [28, Proposition 2.1].

Given integers $d_1 \leq d_2 \leq \cdots \leq d_n \leq \infty$, we say that an ideal is a complete intersection of degree sequence $d_1, \ldots, d_n$ if it is generated by a regular sequence $f_1, \ldots, f_c$ with $d_i = \deg(f_i)$ for every $i \leq c := \max\{j : d_j < \infty\}$. We emphasize that a complete intersection of degree sequence $d_1, \ldots, d_n$ may have codimension $c \leq n$.

A remarkable consequence of Theorem 4.3 is the fact that, conjecturally, the expansive subscheme $\mathrm{Exp}(p(\zeta), R) \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ has the largest number of syzygies among all subschemes $Z \in \mathrm{Hilb}^{p(\zeta)}(X)$ of *any* complete intersection $X \subseteq \mathbb{P}^n$ of degree sequence $d_1, \ldots, d_n$. To justify this claim, we recall the statements of two famous conjectures on complete intersections. For our purposes, it is convenient to state them in terms of ideals of $\widetilde{S} = \Bbbk[x_1, \ldots, x_n]$.

- *Eisenbud-Green-Harris Conjecture*: If $I \subseteq \widetilde{S}$ contains a regular sequence of degree sequence $d_1, \ldots, d_n$, then there exists a lex ideal $L \subseteq \widetilde{S}$ with $\mathrm{HF}(I) = \mathrm{HF}\big(L + (x_1^{d_1}, \ldots, x_n^{d_n})\big)$.
- *Lex-plus-powers Conjecture*: If $I \subseteq \widetilde{S}$ contains a regular sequence of degree sequence $d_1, \ldots, d_n$ and if there exists a lex ideal $L \subseteq \widetilde{S}$ with $\mathrm{HF}(I) = \mathrm{HF}\big(L + (x_1^{d_1}, \ldots, x_n^{d_n})\big)$, then $\beta_{i,j}^{\widetilde{S}}(I) \leq \beta_{i,j}^{\widetilde{S}}\big(L + (x_1^{d_1}, \ldots, x_n^{d_n})\big)$ for all i, j.

The first conjecture was proposed in [13], whereas the second one is attributed to Charalambous and Evans in [17]. Despite the apparently independent statements, it is known that the Lex-plus-powers Conjecture implies the Eisenbud-Green-Harris Conjecture.

Proposition 4.5. *Let $X \subseteq \mathbb{P}^n$ be a complete intersection with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n \leq \infty$. If the Lex-plus-powers conjecture is true, then $\beta_i^S\big(S/I_Z\big) \leq \beta_i^S\big(R/\mathrm{Exp}(p)\big)$ for every closed subscheme $Z \in \mathrm{Hilb}^{p(\zeta)}(X)$ and all $i \geq 0$.*

Proof. As in Lemma 4.1, we may assume that x_{n+1} is a non-zerodivisor on S/I_Z, and we consider $\widetilde{I} = \frac{I_Z+(x_{n+1})}{(x_{n+1})} \subseteq \widetilde{S}$. By assumption the Lex-plus-powers Conjecture and the Eisenbud-Green-Harris Conjecture hold, therefore there exists a lex ideal $\widetilde{L} \subseteq \widetilde{R}$ such that $\mathrm{HF}(\widetilde{S}/\widetilde{I}) = \mathrm{HF}(\widetilde{R}/\widetilde{L})$ and $\beta^{\widetilde{S}}_{i,j}(\widetilde{S}/\widetilde{I}) \leq \beta^{\widetilde{S}}_{i,j}(\widetilde{R}/\widetilde{L})$ for all $i,j \geq 0$. The extension $L = \widetilde{L}R \subseteq R$ is an almost lex ideal, and using Remark 2.1 we deduce $\mathrm{HF}(L) = \mathrm{HF}(I_Z)$ and $\beta^S_{i,j}(S/I_Z) \leq \beta^S_{i,j}(R/L)$ for all $i,j \geq 0$. By Theorem 4.3 we have $\beta^S_i(R/L) \leq \beta^S_i(R/\mathrm{Exp}(p))$ for all $i \geq 0$, and this concludes the proof. □

In particular, by [10, Main Theorem] we obtain the following result.

Theorem 4.6. *Assume* $\mathrm{char}(\Bbbk) = 0$. *Let* $X \subseteq \mathbb{P}^n_{\Bbbk}$ *be a complete intersection with degree sequence such that* $d_j > \sum_{h=1}^{j-1}(d_h - 1)$ *for all* $j \geq 3$. *Then* $\beta^S_i\big(S/I_Z\big) \leq \beta^S_i\big(R/\mathrm{Exp}(p)\big)$ *for every closed subscheme* $Z \in \mathrm{Hilb}^{p(\zeta)}(X)$ *and all* $i \geq 0$.

5. Infinite free resolutions

In this section we investigate bounds for the Betti numbers of infinite free resolutions over a Clements-Lindström ring. Minimal free resolutions over complete intersections have attracted much attention in the past few years, and significant progress has been achieved in the description of their asymptotic behavior, see e.g. [6, 14, 15]. We conjecture that expansive subschemes exhibit extremal infinite free resolutions, and prove this conjecture for quadratic Clements-Linström rings in characteristic zero. We also deduce the extremality of the deviations of expansive subschemes of $\mathbb{P}^n$, and in particular the extremality of the Poincaré series.

We begin by proposing the following problem.

Conjecture 5.1. *Let* R *be a Clements-Lindström ring. We have* $\beta^R_i(I) \leq \beta^R_i(\mathrm{Exp}(p))$ *for every* $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ *and every* $i \geq 0$.

When the ground field has characteristic zero, [29, Theorem 1.4] reduces the problem to almost lex ideals, proceeding as in Lemma 4.1.

Lemma 5.2. *Assume that* $\mathrm{char}(\Bbbk) = 0$. *For every* $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ *there exists an almost lex* $J \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ *such that* $\mathrm{HF}(I) = \mathrm{HF}(J)$ *and* $\beta^R_{i,j}(I) \leq \beta^R_{i,j}(J)$ *for all* i,j.

The following theorem is the main result of this section. The proof employs a construction from [2, 16, 20].

Theorem 5.3. *Assume that* $\mathrm{char}(\Bbbk) = 0$. *Let* R *be a Clements-Lindström ring with* $d_j \in \{2, \infty\}$ *for every* j. *We have* $\beta^R_i\big(I\big) \leq \beta^R_i\big(\mathrm{Exp}(p)\big)$ *for every* $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ *and every* $i \geq 0$.

Proof. We proceed by induction on n, and the case $n = 0$ is trivial, so assume $n > 0$. By Lemma 5.2 we may assume that I is strongly stable. In addition to the notation established in (2.1), in this proof we consider the "intermediate" ring

$$T = \frac{S}{(x_1^{d_1}, \dots, x_{n-1}^{d_{n-1}})}$$

so that $R = T/(x_n^{d_n})$. By assumption, we either have $d_n = \infty$, in which case $T = R$, or $d_n = 2$.

Consider the ideal $\mathcal{I} \subseteq T$ generated by the monomials of T corresponding to the minimal generators of $I \subseteq R$. Notice that $\mathcal{I}$ is smaller than the preimage of I in T if $d_n = 2$, whereas $\mathcal{I} = I$ if $d_n = \infty$. Since x_n is a non-zerodivisor on T and $\mathcal{I}$, and $T/(x_n) \cong \overline{R}$, we have $\beta^T_{i,j}(\mathcal{I}) = \beta^{\overline{R}}_{i,j}(\mathcal{I}/x_n\mathcal{I})$. We have a decomposition of $\overline{R}$-modules

$$\frac{\mathcal{I}}{x_n\mathcal{I}} = I_0 \oplus \bigoplus_{\ell=1}^{d_n-1} \frac{I_\ell}{I_{\ell-1}}.$$

By induction $\beta^{\overline{R}}_i(I_0) \leq \beta^{\overline{R}}_i(\mathrm{Exp}(I_0))$. In the proof of Theorem 4.3 we established that $\mathrm{Exp}(p)_0 \subseteq \mathrm{Exp}(I_0)$, and that $\frac{\mathrm{Exp}(I_0)}{\mathrm{Exp}(p)_0}$ is a free $\Bbbk[x_{n+1}]$-module via restriction of scalars of rank $c_0 = \mathrm{HP}(I_0) - \mathrm{HP}(\mathrm{Exp}(p)_0)$. By Lemma 4.2 (i) we obtain

$$\beta^{\overline{R}}_i(I_0) \leq \beta^{\overline{R}}_i(\mathrm{Exp}(I_0)) \leq \beta^{\overline{R}}_i(\mathrm{Exp}(p)_0) + c_0\beta^{\overline{R}}_i(\Bbbk[x_{n+1}]). \tag{5.1}$$

We also saw, using Propositions 2.4 and 2.5, that the $\overline{R}$-module $\oplus_{\ell=1}^{d_n-1} \frac{I_\ell}{I_{\ell-1}}$ is annihilated by $(x_1, \dots, x_{n-1})$, and is a free $\Bbbk[x_{n+1}]$-module of rank $c_1 = \mathrm{HP}(I_{d_n-1}) - \mathrm{HP}(I_0)$. By Lemma 4.2 (ii)

$$\beta^T_i(\mathcal{I}) = \beta^{\overline{R}}_i(I_0) + c_1\beta^{\overline{R}}_i(\Bbbk[x_{n+1}]). \tag{5.2}$$

If $d_n = \infty$ then the formula (5.2) becomes $\beta^R_i(I) = \beta^{\overline{R}}_i(I_0) + c_1\beta^{\overline{R}}_i(\Bbbk[x_{n+1}])$, and likewise we obtain $\beta^R_i(\mathrm{Exp}(p)) = \beta^{\overline{R}}_i(\mathrm{Exp}(p)_0) + c_2\beta^{\overline{R}}_i(\Bbbk[x_{n+1}])$ where $c_2 = \mathrm{HP}(\mathrm{Exp}(p)_\infty) - \mathrm{HP}(\mathrm{Exp}(p)_0)$. By Proposition 2.5 (7) we have $\mathrm{HP}(\mathrm{Exp}(p)_\infty) = \mathrm{HP}(I_\infty)$ and therefore $c_2 = c_0 + c_1$, and combining with (5.1) we conclude that $\beta^R_i(I) \leq \beta^R_i(\mathrm{Exp}(p))$ as desired.

Now assume that $d_n = 2$. We regard $R, \overline{R}$, and T as $\mathbb{Z}^{n+1}$-graded, but we also consider the $\mathbb{Z}$-grading induced by the variable x_n. If M is $\mathbb{Z}^{n+1}$-graded T-module we define $\sigma(M)$ to be the vector space consisting of the graded components of M with x_n-degrees 0 or 1. Clearly, σ defines an exact functor from the category of $\mathbb{Z}^{n+1}$-graded T-modules to the category of $\mathbb{Z}^{n+1}$-graded $\Bbbk$-vector spaces.

Let $\mathbb{F}$ be the minimal $\mathbb{Z}^{n+1}$-graded free resolution of $\mathcal{I}$ over T. The x_n-twists in this resolution are all equal to 0 or 1: this follows from the fact that $\mathbb{F} \otimes_T \frac{T}{(x_n)}$ is a minimal $\mathbb{Z}^n$-graded free resolution of $\mathcal{I}/x_n\mathcal{I}$ over $\overline{R}$, and that $\mathcal{I}/x_n\mathcal{I}$ is generated in x_n-degrees $0, 1$. The complex $\mathbb{E} = \sigma(\mathbb{F})$ is acyclic and minimal, in the sense that the image of its differential lies in $(x_1, \dots, x_{n+1})\mathbb{E}$. Each direct summand in $\mathbb{F}$ has the form $T(-\delta_1, \dots, -\delta_n, -\delta_{n+1})$ with $\delta_n \in \{0, 1\}$; the corresponding summand in $\mathbb{E}$ is a factor ring of $R = T/(x_n^2)$, namely

$$\sigma\big(T(-\delta_1, \dots, -\delta_n, -\delta_{n+1})\big) \cong \frac{R}{(x_n^{2-\delta_n})}(-\delta_1, \dots, -\delta_n, -\delta_{n+1}).$$

The cyclic R-module on the right hand side is free if and only if $\delta_n = 0$. In fact, $\mathbb{E}$ is an acyclic minimal $\mathbb{Z}^{n+1}$-graded complex of (not necessarily free) finitely generated R-modules. Since all the x_n-twists in $\mathbb{F}$ are in $\{0,1\}$, every free summand of $\mathbb{F}$ contributes with a non-zero summand in $\mathbb{E}$. In other words, the numbers of generators in every homological degree i is the same for $\mathbb{F}$ and $\mathbb{E}$, and this number is $\beta_i^T(\mathcal{I})$. Among the direct summands of $\mathbb{E}$, the free modules are precisely those coming from copies of T in $\mathbb{F}$ with x_n-twist equal to 0. These modules form themselves another complex $\mathbb{E}'$, which is again minimal and acyclic, but it is even free. In fact, $\mathbb{E}'$ is the minimal free resolution of I_0 over $\overline{R}$, since I_0 is the truncation of $\mathcal{I}$ in x_n-degree 0, and $\overline{R}$ is the truncation of T in x_n-degree 0. We conclude that in homological degree i in $\mathbb{E}$ we have exactly $\beta_i^{\overline{R}}(I_0)$ free summands, i.e. copies of R.

To summarize, $\mathbb{E}$ is an acyclic minimal complex of $\mathbb{Z}^{n+1}$-graded R-modules, it has $\beta_i^T(\mathcal{I})$ generators in homological degree i, of which $\beta_i^{\overline{R}}(I_0)$ generate a free module R, whereas the remaining ones generate a non-free module isomorphic to $R/(x)$. The number of non-free summands of $\mathbb{E}$ in homological degree i is therefore $\beta_i^T(\mathcal{I}) - \beta_i^{\overline{R}}(I_0) = c_1\beta_i^{\overline{R}}(\Bbbk[x_{n+1}])$ by (5.2). Note also that the 0-homology of $\mathbb{E}$ is $\sigma(T/\mathcal{I}) = R/I$.

Let E_j denote the module in homological degree j in $\mathbb{E}$. The differentials of $\mathbb{E}$ can be lifted to a complex of complexes, namely a double complex $\mathbb{D}_I$ of R-modules where the j-th vertical complex is the minimal free resolution of F_j. By construction, the double complex $\mathbb{D}_I$ is free. Furthermore, it is minimal, and the total complex $\mathrm{Tot}(\mathbb{D}_I)$ is a minimal $\mathbb{Z}^{n+1}$-graded free resolution of R/I over R, cf. [16, Proposition 5.6], [2, Theorem 1.3], or [20, Theorem 2.10]. Recall that the R-module $R/(x)$ has an infinite minimal free resolution over R with $\beta_i^R(R/(x)) = 1$ for every $i \in \mathbb{N}$. It follows that in $\mathbb{D}_I$, for each $i \geq 0$, we have

- $(*)$ $\beta_i^{\overline{R}}(I_0)$ summands in homological bidegree $(i,0)$ arising from the free summands of $\mathbb{E}$;
- $(**)$ $c_1\beta_i^{\overline{R}}(\Bbbk[x_{n+1}])$ summands in homological bidegree (i,j) for all $j \geq 0$, arising from the non-free summands of $\mathbb{E}$;

where the first coordinate is horizontal and the second coordinate vertical. We conclude that the Betti numbers of a saturated strongly stable $I \subseteq R$ depend only on those of $I_0 \subseteq \overline{R}$ and on the number $c_1 = \mathrm{HP}(I_1) - \mathrm{HP}(I_0)$.

The same construction for $\mathrm{Exp}(p)$ yields a double complex $\mathbb{D}_{\mathrm{Exp}(p)}$. Let $c_2' = \mathrm{HP}(\mathrm{Exp}(p)_1) - \mathrm{HP}(\mathrm{Exp}(p)_0)$. We observed in the proof of Theorem 4.3 that $\mathrm{HP}(I_{d_n-1}) \preceq \mathrm{HP}(\mathrm{Exp}(p)_{d_n-1})$, that is, $\mathrm{HP}(I_1) \preceq \mathrm{HP}(\mathrm{Exp}(p)_1)$. We deduce that $c_2' \geq c_0 + c_1$. Finally, we compare the contribution of the two types of summands $(*)$ and $(**)$ to the double complexes $\mathbb{D}_I$ and $\mathbb{D}_{\mathrm{Exp}(p)}$:

- $(*)$ For every $i \geq 0$, by (5.1), $\mathbb{D}_I$ has at most $c_0\beta_i^{\overline{R}}(\Bbbk[x_{n+1}])$ more summands in position $(i,0)$ than $\mathbb{D}_{\mathrm{Exp}(p)}$, among those arising from the free summands of $\mathbb{E}$.
- $(**)$ For every $i,j \geq 0$, $\mathbb{D}_{\mathrm{Exp}(p)}$ has at least $(c_2' - c_1)\beta_i^{\overline{R}}(\Bbbk[x_{n+1}])$ more summands in position (i,j) than $\mathbb{D}_I$, among those arising from the non-free summands of $\mathbb{E}$.

Thus $\mathbb{D}_{\mathrm{Exp}(p)}$ has at least as many copies of R as $\mathbb{D}_I$ in every position (i, j). This concludes the proof, since $\beta_i^R(I), \beta_i^R(\mathrm{Exp}(p))$ are the Betti numbers of $\mathrm{Tot}(\mathbb{D}_I), \mathrm{Tot}(\mathbb{D}_{\mathrm{Exp}(p)})$ respectively. $\square$

In the remainder of this section, we explore deviations and Poincaré series of expansive subschemes. The deviations of a ring A are a sequence of integers $\{\varepsilon_i(A)\}_{i\geq 1}$ measuring several homological or cohomological data of A. Examples include: the generators of a Tate resolution of A over a polynomial ring, as well as a Tate resolution of $\Bbbk$ over A; the ranks of the modules in a cotangent complex of A; the dimensions of the components of the homotopy Lie algebra $\pi(A)$ of A. We refer to [3, Sections 7 and 10] for definitions and background.

Lemma 5.4. *Let $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ be strongly stable. We have the inclusion of vector spaces of linear forms $[\mathrm{Exp}(p)]_1 \subseteq [I]_1$.*

Proof. We may assume $I \neq R$. Since $\mathrm{Exp}(p)$ is saturated and strongly stable, we have $[\mathrm{Exp}(p)]_1 = \langle x_1, \ldots, x_m\rangle_{\Bbbk}$ for some $0 \leq m \leq n$. If $m = n$ then $\mathrm{Exp}(p) = (x_1, \ldots, x_n) \subseteq R$ is the only strongly stable ideal in $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$, so $I = \mathrm{Exp}(p)$. If $m < n$ then $[\mathrm{Exp}(p)]_1 = [\mathrm{Exp}(p)_0]_1$. We proceed by induction on n, and for $n = 0$ there is nothing to show. By axiom (A6) we have $\mathrm{HP}(\mathrm{Exp}(p)_0) \preceq \mathrm{HP}(I_0)$, thus $\mathrm{Exp}(p)_0 \subseteq \mathrm{Exp}(I_0)$ by (A3). By induction we have $[\mathrm{Exp}(I_0)]_1 \subseteq [I_0]_1$, hence $[\mathrm{Exp}(p)]_1 = [\mathrm{Exp}(p)_0]_1 \subseteq [\mathrm{Exp}(I_0)]_1 \subseteq [I_0]_1 \subseteq [I]_1$. $\square$

A consequence of Theorem 4.3 and the results of [5] is the fact that an expansive subscheme of $\mathbb{P}^n$ has maximal deviations in its Hilbert scheme.

Corollary 5.5. *Let $S = \Bbbk[x_1, \ldots, x_{n+1}]$. We have $\varepsilon_i(S/I) \leq \varepsilon_i(S/\mathrm{Exp}(p))$ for every $I \in \mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ and all $i \geq 1$.*

Proof. We may assume, as in Lemma 4.1, that $I : x_{n+1} = I$. Let $\widetilde{I} = \frac{I+(x_{n+1})}{(x_{n+1})} \subseteq \widetilde{S}$ and $\widetilde{L} = \mathrm{Lex}(\widetilde{I}) \subseteq \widetilde{S}$. By [5, Theorem 3.4] we have $\varepsilon_i(\widetilde{S}/\widetilde{I}) \leq \varepsilon_i(\widetilde{S}/\widetilde{L})$ for all $i \geq 2$. It follows from [3, Proposition 7.1.6] that $\varepsilon_i(S/I) \leq \varepsilon_i(S/L)$ for all $i \geq 2$, where $L = \widetilde{L}S \subseteq S$. The ideals L and $\mathrm{Exp}(p)$ are strongly stable, and this implies that S/L and $S/\mathrm{Exp}(p)$ are Golod rings by [22, Theorem 4]. Now by [5, Proposition 3.2] we derive that $\varepsilon_i(S/L) \leq \varepsilon_i(S/\mathrm{Exp}(p))$ for all $i \geq 2$. Finally, for $i = 1$ the deviation $\varepsilon_1(A)$ is equal to the embedding dimension of A, cf. [3, Corollary 7.1.5], therefore $\varepsilon_1(S/L) \leq \varepsilon_1(S/\mathrm{Exp}(p))$ by Lemma 5.4. $\square$

In particular, an expansive subscheme of $\mathbb{P}^n$ has maximal Poincaré series, that is, the generating function of the dimensions of $\mathrm{Tor}^A_\bullet(\Bbbk, \Bbbk)$ or $\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$.

Corollary 5.6. *Let $S = \Bbbk[x_1, \ldots, x_{n+1}]$. We have $\beta_i^{S/I}(\Bbbk) \leq \beta_i^{S/\mathrm{Exp}(p))}(\Bbbk)$ for every $I \in \mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ and all $i \geq 0$.*

Proof. Apply Corollary 5.5 and [3, Remark 7.1.1]. $\square$

6. Computation of expansive ideals

In this section we explore further properties of expansive subschemes. The main goal is to construct the generators of the expansive ideal from the Hilbert polynomial, in an explicit manner that avoids the recursive decomposition into expansive components. This will be done in Theorem 6.2. As a result, we provide a simple and efficient algorithm to construct the expansive point.

We begin by proving that expansive ideals are almost lex. Recall (2.1) and Remark 2.1.

Proposition 6.1. *The ideal* $\mathrm{Exp}\big(p(\zeta)\big) \subseteq R$ *is almost lex.*

Proof. For simplicity we denote $E = \mathrm{Exp}\big(p(\zeta)\big)$ and $\widetilde{E} = \frac{E+(x_{n+1})}{(x_{n+1})} \subseteq \widetilde{R}$. By definition, we must show that $\widetilde{E}$ is a lex ideal of $\widetilde{R}$. The decomposition (2.2) for E yields

$$\widetilde{E} = \bigoplus_{\ell=0}^{d_n-1} \widetilde{E}_\ell x_n^\ell \qquad \text{where} \quad \widetilde{E}_\ell = \frac{E_\ell + (x_{n+1})}{(x_{n+1})} \subseteq \frac{\overline{R}}{(x_{n+1})} = \frac{R}{(x_n, x_{n+1})}$$

in other words E_ℓ is the extension of $\widetilde{E}_\ell$ to $\overline{R}$. By induction on n, each $\widetilde{E}_\ell$ is a lex ideal of $\frac{R}{(x_n,x_{n+1})}$. Moreover, $\widetilde{E}$ is strongly stable since E is strongly stable. Now let $\widetilde{I} = \mathrm{Lex}(\widetilde{E}) \subseteq \widetilde{R}$ and decompose $\widetilde{I} = \oplus_{\ell=0}^{d_n-1} \widetilde{I}_\ell x_n^\ell \subseteq \widetilde{R}$. The recursive criterion for lex ideals in Clements-Lindström rings proved in [8, Proof of Theorem 3.3, Lemma 3.7, Lemma 3.8], cf. also [10, Remark 3.1], implies that $\widetilde{I}_\ell$ is lex for every ℓ and that the following inequalities hold for every $\rho, \tau \geq 0$

$$\sum_{\ell=0}^{\rho} \mathrm{HF}\big(\widetilde{I}_\ell, \tau - \ell\big) \leq \sum_{\ell=0}^{\rho} \mathrm{HF}\big(\widetilde{E}_\ell, \tau - \ell\big). \tag{6.1}$$

Consider the extension $I = \widetilde{I}R \subseteq R$. Then $I = \oplus_{\ell=0}^{d_n-1} I_\ell x_n^\ell$ where $I_\ell \subseteq \overline{R}$ is the extension of $\widetilde{I}_\ell$ to $\overline{R}$. For every $\delta \in \mathbb{N}$ we have $\mathrm{HF}(I_\ell, \delta) = \sum_{\tau=0}^{\delta} (\widetilde{I}_\ell, \tau)$, $\mathrm{HF}(E_\ell, \delta) = \sum_{\tau=0}^{\delta} (\widetilde{E}_\ell, \tau)$, therefore, adding the inequalities (6.1), we obtain for every $\rho, \delta \geq 0$

$$\sum_{\ell=0}^{\rho} \mathrm{HF}\left(I_\ell, \delta - \ell\right) \leq \sum_{\ell=0}^{\rho} \mathrm{HF}\left(E_\ell, \delta - \ell\right). \tag{6.2}$$

By definition of $\widetilde{I}$ we have $\mathrm{HF}(\widetilde{I}) = \mathrm{HF}(\widetilde{E})$, which implies $\mathrm{HF}(I) = \mathrm{HF}(E)$ and in particular $\mathrm{HP}(I) = \mathrm{HP}(E)$. By Proposition 2.5 (8) it follows that $\mathrm{HP}(I_h) - \mathrm{HP}(E_k) \in \mathbb{Z}$ for every h, k. Combining this fact with (6.2) we deduce that $\sum_{\ell=0}^{\rho} \mathrm{HP}\left(I_\ell\right) \preceq \sum_{\ell=0}^{\rho} \mathrm{HP}\left(E_\ell\right)$ for every $\rho \geq 0$. However, by axiom (A6) we have the opposite inequalities $\sum_{\ell=0}^{\rho} \mathrm{HP}\left(I_\ell\right) \succeq \sum_{\ell=0}^{\rho} \mathrm{HP}\left(E_\ell\right)$, for every $\rho \geq 0$, so that $\sum_{\ell=0}^{\rho} \mathrm{HP}\left(I_\ell\right) = \sum_{\ell=0}^{\rho} \mathrm{HP}\left(E_\ell\right)$ and hence $\mathrm{HP}\left(I_\ell\right) = \mathrm{HP}\left(E_\ell\right)$ for every ℓ. We claim that $I_\ell = E_\ell$ for every $\ell \in \mathbb{N}$. Assume otherwise, and choose the least ρ such that $I_\rho \neq E_\rho$. It follows from (6.2) that $\mathrm{HF}(I_\rho, \delta) \leq \mathrm{HF}(E_\rho, \delta)$ for every δ, and since $I_\rho, E_\rho \subseteq \overline{R}$ are almost lex we conclude that $I_\rho \subsetneq E_\rho$. However, Proposition 2.4 yields $\mathrm{HP}(I_\rho) \neq \mathrm{HP}(E_\rho)$, contradiction. We have proved that $E = I$, that is, $\widetilde{E} = \widetilde{I}$, so $\widetilde{E}$ is lex as desired. □

In the following theorem we employ the *opposite lex order* on R, denoted by $<_{\mathrm{opp}}$, that is the lexicographic monomial order induced by the opposite order on the variables $x_{n+1} > x_n > x_{n-1} > \cdots > x_1$. The usual lex order is denoted by $<_{\mathrm{lex}}$. Furthermore, let $\mathcal{G}(-)$ denote the set of minimal monomial generators of a monomial ideal, and $[\mathcal{G}(-)]_d$ those of degree d.

Theorem 6.2. *Let R be a Clements-Lindström ring and $p(\zeta)$ such that $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R) \neq \emptyset$. Let $p'(\zeta) = p(\zeta) - p(\zeta - 1)$ and $\widetilde{L} = \mathrm{Lex}(p'(\zeta), \widetilde{R})$. There exists a chain of almost lex ideals of R*

$$E^{(0)} \supseteq E^{(1)} \supseteq \cdots \supseteq E^{(c-1)} \supseteq E^{(c)}$$

such that $E^{(c)} = \mathrm{Exp}(p, R)$, $E^{(0)} = \widetilde{L}R \subseteq R$ is the extension to R, and for each $k = 0, \ldots, c-1$ we have $\frac{E^{(k)}}{E^{(k+1)}} = \Bbbk[x_{n+1}]\mathbf{u}^{(k)}$, where $\mathbf{u}^{(k)}$ is the following monomial of $E^{(k)}$

$$\mathbf{u}^{(k)} = \min_{<_{\mathrm{opp}}} \left\{ \min_{<_{\mathrm{lex}}} \left[\mathcal{G}\big(E^{(k)}\big) \right]_d \,:\, d \in \mathbb{N} \right\}.$$

Proof. Denote $E = \mathrm{Exp}(p, R)$ and $\widetilde{E} = \frac{E + (x_{n+1})}{(x_{n+1})} \subseteq \widetilde{R}$. By Proposition 6.1 $\widetilde{E}$ is a lex ideal of $\widetilde{R}$, and by Remark 2.1 $\mathrm{HP}(\widetilde{E}, \zeta) = \mathrm{HP}(E, \zeta) - \mathrm{HP}(E, \zeta - 1)$. The saturation $\widetilde{L} = \widetilde{E} : \mathfrak{m}_{\widetilde{R}}^{\infty} \subseteq \widetilde{R}$ is a saturated lex ideal containing $\widetilde{E}$ and with $\mathrm{HP}(\widetilde{L}) = \mathrm{HP}(\widetilde{E})$, so $\widetilde{L} = \mathrm{Lex}(p'(\zeta), \widetilde{R})$. Let $E^{(0)} = \widetilde{L}R \subseteq R$, so $E \subseteq E^{(0)}$ and $\mathrm{HP}(E^{(0)}) - \mathrm{HP}(E)$ is the non-negative integer $c = \dim_{\Bbbk}(\widetilde{L}/\widetilde{E})$, cf. Proposition 2.4.

We prove the theorem by induction on c. The case $c = 0$ is trivial, so assume $c > 0$. Denoting $E' = \mathrm{Exp}\big(p(\zeta) - 1, R\big)$ and $\widetilde{E}' = \frac{E' + (x_{n+1})}{(x_{n+1})} \subseteq \widetilde{R}$, we have $E \subseteq E'$ by axiom (A3), and $\mathrm{HP}(E'/E) = 1$. Taking images in $\widetilde{R}$, the quotient $\widetilde{E}'/\widetilde{E}$ is a 1-dimensional vector space generated by a monomial $\mathbf{u}$ of $\widetilde{E}'$, necessarily $\mathbf{u} \in \mathcal{G}(E')$. Furthermore, since $\widetilde{E}$ is a lex ideal of $\widetilde{R}$, $\mathbf{u}$ must be the lowest monomial with respect to $<_{\mathrm{lex}}$ in its graded component of $\widetilde{E}'$, so $\mathbf{u} = \min_{<_{\mathrm{lex}}} \left[\mathcal{G}\big(E'\big)\right]_d$ for some d. Since $\widetilde{E} : \mathfrak{m}_{\widetilde{R}}^{\infty} = \widetilde{E}' : \mathfrak{m}_{\widetilde{R}}^{\infty}$ and the theorem holds for E' by induction, it remains to show that $\mathbf{u} = \min_{<_{\mathrm{opp}}} \left\{ \min_{<_{\mathrm{lex}}} \left[\mathcal{G}\big(E'\big)\right]_d \,:\, d \in \mathbb{N} \right\}$. In other words, given $\mathbf{v} = \min_{<_{\mathrm{lex}}} \left[\mathcal{G}\big(E'\big)\right]_{d'}$ with $d \neq d'$, we must show that $\mathbf{v} >_{\mathrm{opp}} \mathbf{u}$.

Let $I \subseteq E'$ be the almost lex ideal such that $\widetilde{E}'/\widetilde{I}$ is the 1-dimensional vector space $\langle \mathbf{v} \rangle$. Notice that $\mathrm{HP}(I) = \mathrm{HP}(E)$. Let $\mathbf{u} = x_1^{u_1} \cdots x_m^{u_m} \cdots x_n^{u_n}$ and $\mathbf{v} = x_1^{v_1} \cdots x_m^{v_m} \cdots x_n^{v_n}$ where $m = \max\{i \,:\, u_i \neq v_i\}$. In order to conclude the proof, we must show that $u_m > v_m$. If $m < n$, it follows by construction of I and E that $I_\ell = E_\ell$ for all $\ell \neq u_n$, therefore we must also have $\mathrm{HP}(I_{u_n}) = \mathrm{HP}(E_{u_n})$. Up to replacing I, E by I_{u_n}, E_{u_n} and repeating this process, we may assume that $m = n$. If $u_m < v_m$ then $I_\ell = E_\ell$ for all $\ell < u_m$ and $I_{u_m} \subsetneq E_{u_m}$, thus $\sum_{\ell=0}^{u_m} \mathrm{HP}(I_\ell) \prec \sum_{\ell=0}^{u_m} \mathrm{HP}(E_\ell)$ contradicting axiom (A6). □

Remark 6.3. The unique lex ideal in $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj}R)$ (cf. Remark 1.6) admits a corresponding construction. Specifically, there are lex ideals $L^{(0)} \supseteq L^{(1)} \supseteq \cdots \supseteq L^{(c-1)} \supseteq L^{(c)}$ of R such that $L^{(c)} = \mathrm{Lex}(p(\zeta), R)$, $L^{(0)} = \mathrm{Lex}\big(p'(\zeta), \widetilde{R}\big)R$, and $\frac{L^{(k)}}{L^{(k+1)}} = \Bbbk[x_{n+1}]\mathbf{w}^{(k)}$, but this time

$$\mathbf{w}^{(k)} = \min_{<_{\mathrm{lex}}} \left[\mathcal{G}\big(L^{(k)}\big)\right]_\delta \qquad \text{where} \quad \delta = \max\left\{ d \in \mathbb{N} \,:\, \left[\mathcal{G}\big(L^{(k)}\big)\right]_d \neq \emptyset \right\}.$$

By the same proof as Theorem 6.2, it suffices to verify the following statement: let $L' \subseteq R$ be a saturated lex ideal, $\mathbf{v} = \min_{<_{\text{lex}}} \left[\mathcal{G}\big(L'\big)\right]_d$ for some $d < \max\left\{d \in \mathbb{N} : \left[\mathcal{G}\big(L'\big)\right]_d \neq \emptyset\right\}$, and let $I \subseteq E'$ be the almost lex ideal such that $\widetilde{L}'/\widetilde{I}$ is the 1-dimensional vector space $\langle \mathbf{v} \rangle$, then I is not lex. By assumption there exists $\mathbf{w} = \min_{<_{\text{lex}}} \left[\mathcal{G}\big(L'\big)\right]_\delta$ with $\delta > d$. Since L' is lex, $\mathbf{w}$ is a minimal generator, $x_{n+1}^{\delta-d}\mathbf{v} \in L'$ is not a minimal generator, and the two monomials have the same degree, it follows that $\mathbf{w} <_{\text{lex}} x_{n+1}^{\delta-d}\mathbf{v}$. However, $\mathbf{w} \in I$ but $x_{n+1}^{\delta-d}\mathbf{v} \notin I$, so I is not lex, as desired.

Theorem 6.2 readily translates into an algorithm to compute $\text{Exp}(p(\zeta)$ from $p(\zeta)$, sketched below. For the sake of completeness, we also include an algorithm to compute $\text{Lex}(p(\zeta))$. These algorithms have been implemented by the authors in Macaulay2 [23].

Algorithm 6.4 (The expansive ideal of a Hilbert polynomial)**.** Let R be a Clements-Lindström ring and $p(\zeta) \in \mathbb{Q}[\zeta]$ with $\text{Hilb}^{p(\zeta)}(\text{Proj}R) \neq \emptyset$.

- If $p(\zeta) = 0$, return $\text{Exp}\big(p(\zeta), R\big) := R$.
- If $p(\zeta) \neq 0$, let $p'(\zeta) = p(\zeta) - p(\zeta - 1)$, $L^{(0)} := \text{Lex}\big(p'(\zeta), \widetilde{R}\big)R$, $c = p(\zeta) - \text{HP}(R/L^{(0)})$. For each $k = 1, \ldots, c$ let $\mathbf{u}_1, \ldots, \mathbf{u}_t$ be the minimal generators of $L^{(k-1)}$ so that $\mathbf{u}_t = \min_{<_{\text{opp}}} \left\{\min_{<_{\text{lex}}} \left[\mathcal{G}\big(H^{(k-1)}\big)\right]_d : d \in \mathbb{N}\right\}$. Set $L^{(k)} := (\mathbf{u}_1, \ldots, \mathbf{u}_{t-1}, x_1\mathbf{u}_t, x_2\mathbf{u}_t, \ldots, x_n\mathbf{u}_t)$.
- Return $\text{Exp}\big(p(\zeta), R\big) = L^{(c)}$.

Algorithm 6.5 (The lex ideal of a Hilbert polynomial)**.** Let R be a Clements-Lindström ring and $p(\zeta) \in \mathbb{Q}[\zeta]$ with $\text{Hilb}^{p(\zeta)}(\text{Proj}R) \neq \emptyset$.

- If $p(\zeta) = 0$, return $\text{Lex}\big(p(\zeta), R\big) := R$.
- If $p(\zeta) \neq 0$, let $p'(\zeta) = p(\zeta) - p(\zeta - 1)$, $H^{(0)} := \text{Lex}\big(p'(\zeta), \widetilde{R}\big)R$, $c = p(\zeta) - \text{HP}(R/H^{(0)})$. For each $k = 1, \ldots, c$, let $\mathbf{w}_1, \ldots, \mathbf{w}_t$ be the minimal generators of $H^{(k-1)}$ ordered so that either $\deg(\mathbf{w}_i) < \deg(\mathbf{w}_{i+1})$ or $\deg(\mathbf{w}_i) = \deg(\mathbf{w}_{i+1})$ and $\mathbf{w}_i >_{\text{lex}} \mathbf{w}_{i+1}$. Set $H^{(k)} := (\mathbf{w}_1, \ldots, \mathbf{w}_{t-1}, x_1\mathbf{w}_t, x_2\mathbf{w}_t, \ldots, x_n\mathbf{w}_t)$.
- Return $\text{Lex}\big(p(\zeta), R\big) = H^{(c)}$.

The last result of this section shows that the lex point and the expansive point on $\text{Hilb}^{p(\zeta)}(\text{Proj}R)$ are as different as they can be: they are almost never equal, and if they are, then there is only one strongly stable point on the Hilbert scheme. In the case of $\mathbb{P}^n$ it follows that if $\text{Exp}(p(\zeta), S) = \text{Lex}(p(\zeta), S)$ then $\text{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ is rational, irreducible, and smooth [25, 35].

Observe that a saturated lex $L \subseteq R$ is necessarily of the form

$$L = \big(x_1^{a_1+1}, x_1^{a_1}x_2^{a_2+1}, \ldots, x_1^{a_1}\cdots x_{r-1}^{a_{r-1}}x_r^{a_r+1}\big) \tag{6.3}$$

for some integers $a_i \geq 0$, $r \leq n$, and such that $x_1^{a_1}\cdots x_{r-1}^{a_{r-1}}x_r^{a_r+1} \neq 0$. Note that some of the other generators may be 0, if we have $a_i + 1 = d_i$ for some i.

Proposition 6.6. *Let R be a Clements-Lindström ring and $p(\zeta) \in \mathbb{Q}[\zeta]$. With notation as in Theorem 6.2 and Remark 6.3 we have* $\text{Lex}(p(\zeta), R) = \text{Exp}(p(\zeta), R)$ *if and only if one of the following occurs*

(1) $\mathrm{Lex}(p(\zeta), R) = L^{(0)}$;
(2) $L^{(0)}$ *is generated in a single degree and* $\mathrm{Lex}(p(\zeta), R) = L^{(1)}$;
(3) $L^{(0)}$ *is principal and* $\mathrm{Lex}(p(\zeta), R) = L^{(2)}$;
(4) $d_{n-1} < \infty$ *and* $\mathrm{Lex}(p(\zeta), R) = \left(x_1^{d_1-1} \cdots x_{n-1}^{d_{n-1}} x_n^{\alpha}\right)$ *for some* $\alpha \in \mathbb{N}$.

In this case $\mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj} R)$ *contains only one strongly stable point.*

Proof. It is easy to check, using Theorem 6.2 and Remark 6.3, that in each case (1), (2), (3), (4) we get $\mathrm{Exp}(p(\zeta), R) = E^{(c)} = L^{(c)} = \mathrm{Lex}(p(\zeta), R)$. Assume now that $L = \mathrm{Lex}(p(\zeta), R) = \mathrm{Exp}(p(\zeta), R)$, and its generators are as in (6.3). Then

$$E^{(0)} = L^{(0)} = \left(x_1^{a_1+1}, x_1^{a_1}x_2^{a_2+1}, \dots, x_1^{a_1} \cdots x_{r-2}^{a_{r-2}+1}, x_1^{a_1} \cdots x_{r-1}^{a_{r-1}}\right).$$

This implies that the $L^{(k)}$'s are the only almost lex I such that $L^{(0)} \subseteq I \subseteq L$, hence $E^{(k)} = L^{(k)}$ for all $k \leq c$. In particular, $\mathrm{Lex}(p(\zeta), R) = \mathrm{Exp}(p(\zeta), R)$ implies $\mathrm{Lex}(p(\zeta) - b, R) = \mathrm{Exp}(p(\zeta) - b, R)$ for every $b \in \mathbb{N}$ for which the Hilbert scheme is nonempty. Observe that the generators of a saturated lex ideal ordered as in (6.3) are non-decreasing in degree, decreasing in $<_{\mathrm{lex}}$, and increasing in $<_{\mathrm{opp}}$. It follows from Theorem 6.2 and Remark 6.3 that, whenever $\mathrm{Lex}(q(\zeta), R) = \mathrm{Exp}(q(\zeta), R)$ for some $q(\zeta)$, we have $\mathrm{Lex}(q(\zeta)+1, R) = \mathrm{Exp}(q(\zeta)+1, R)$ if and only if $\mathrm{Lex}(q(\zeta))$ is generated in a single degree. Finally notice that if $\mathrm{Lex}(q(\zeta)+1, R)$ is generated in a single degree, then $\mathrm{Lex}(q(\zeta), R)$ is necessarily principal, and $\mathrm{Lex}(q(\zeta) + 1)$ is principal only if $\mathrm{Lex}(q(\zeta), R) = \left(x_1^{d_1-1} \cdots x_{n-1}^{d_{n-1}} x_n^{\alpha}\right)$ for some $\alpha \in \mathbb{N}$. This forces one of (1), (2), (3), or (4) to occur.

To prove the last statement, let $I \in \mathrm{Hilb}^{p(\zeta)}(\mathrm{Proj} R)$ be strongly stable. Note that $L_0 = \mathrm{Exp}(p(\zeta), R)_0 = \mathrm{Lex}(p(\zeta), R)_0 \subseteq \overline{R}$ is both expansive and lex. Let $r(\zeta) = \mathrm{HP}(\overline{R}/L_0)$. By (A6) $\mathrm{HP}(L_0) \preceq \mathrm{HP}(I_0)$, i.e. $\mathrm{HP}(\overline{R}/I_0) = r(\zeta) - b$ for some $b \in \mathbb{N}$. As observed above, this implies $\mathrm{Exp}(r(\zeta) - b, \overline{R}) = \mathrm{Lex}(r(\zeta) - b, \overline{R})$, thus by induction on n we obtain $I_0 = L_0$. If case (1) holds, then $L = L_0 R$, as x_n does not divide the generators of L. On the other hand, $I_0 R \subseteq I$. We have $\mathrm{HP}(L_0) = \mathrm{HP}(I_0)$ so $\mathrm{HP}(L_0 R) = \mathrm{HP}(I_0 R)$, and $\mathrm{HP}(I) = \mathrm{HP}(L)$, implying $I = I_0 R$ by Proposition 2.4, so $I = L$ as desired. For the other cases (2), (3), and (4), it suffices to observe that if $\mathrm{Lex}(q(\zeta), R)$ is generated in a single degree for some $q(\zeta)$, then $L(q(\zeta) + 1, R)$ is the only saturated strongly stable ideal $H \subseteq \mathrm{Lex}(q(\zeta), R)$ with $\mathrm{HP}(R/H) = q(\zeta) + 1$. ☐

7. Examples

We conclude the paper by exhibiting examples of expansive points in some Hilbert schemes, constructed by the methods of Section 6, and numerical bounds on Betti numbers obtained by the results of Section 4.

We begin with the analysis of expansive subschemes of dimension 0. It follows by axiom (A4) that the 0-dimensional subschemes defined by $(x_1, \dots, x_n)^{\delta}$ are expansive for every $\delta \geq 0$. More generally, we can characterize all of them explicitly.

Example 7.1 (0-dimensional subschemes)**.** Let $c \in \mathbb{N}$, then $\mathrm{Exp}(c)$ is the unique almost lex ideal of $\mathrm{Hilb}^c(\mathrm{Proj}R)$ generated in at most two consecutive degrees. Equivalently, $\mathrm{Exp}(c) = (x_1, \ldots, x_n)^\delta + I$ where $\delta = \min\{d : \mathrm{HP}\big(R/(x_1, \ldots, x_n)^d\big) \geq c\}$ and $I \subseteq R$ is an almost lex ideal generated in degree $\delta - 1$. Note that I is necessarily generated by the first $\mathrm{HP}\big(R/(x_1, \ldots, x_n)^\delta\big) - c$ monomials of $[\widetilde{R}]_{\delta-1}$ in the lex order. This statement follows by induction on c, using the chain of ideals in Theorem 6.2. In the special case of $\mathrm{Hilb}^c(\mathbb{P}^n)$, we recover the main result of [36].

Example 7.2. Assume $\mathrm{char}(\Bbbk) \neq 2, 3$. The simplest known reducible Hilbert scheme of points is $\mathrm{Hilb}^8(\mathbb{P}^4)$, see [7]. It is the union of two irreducible components of dimension 32 and 25. The expansive subscheme is $E = (x_1, \ldots, x_4)^3 + (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2, x_1x_4)$, and it lies in the intersection of the two components. To verify this, consider the vector space $W = [\widetilde{S}/\widetilde{E}]_2 = \langle x_2x_4, x_3x_4, x_4^2\rangle_\Bbbk$ and the bilinear form $B : \big([\widetilde{S}]_1 \otimes W\big)^{\otimes 2} \to \bigwedge^3 W \cong \Bbbk$ given by $B(\ell_1 \otimes q_1, \ell_2 \otimes q_2) = \ell_1\ell_2 \wedge q_1 \wedge q_2$. Then B is degenerate, since $B(x_2 \otimes x_2x_4, \ell_2 \otimes q_2) = x_2\ell_2 \wedge x_2x_4 \wedge q_2 = 0$ for every ℓ_2, q_2, so the conclusion follows from [7, Theorem 1.3].

Example 7.3. We exhibit three situations, found in [33], where the lex point and the expansive point are the only two strongly stable points of the Hilbert scheme.

(1) Let $p(\zeta) = \binom{\zeta+n-2}{n-2} + \zeta + 1$ with $n \geq 4$. Then $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ is the union of two irreducible components $\mathcal{H}, \mathcal{H}'$, whose general points are respectively a line and an $(n-2)$-plane in general position, and a line intersecting an $(n-2)$-plane union an isolated point [33, Section 3]. The lex point $\mathrm{Lex}(p(\zeta)) = (x_1, x_2^2, x_2x_3, \ldots, x_2x_{n-2}, x_2x_{n-1}^2, x_2x_{n-1}x_n)$ lies in the interior of $\mathcal{H}'$. The expansive point $\mathrm{Exp}(p(\zeta)) = (x_1^2, x_1x_2, \ldots, x_1x_n, x_2^2, x_2x_3, \ldots, x_2x_{n-1})$ lies in the intersection $\mathcal{H} \cap \mathcal{H}'$.

(2) Let $p(\zeta) = \binom{\zeta+n-2}{n-2} + 2$ with $n \geq 4$. Then $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ is irreducible, and its general point parametrizes an $(n-2)$-plane and 2 isolated points [33, Section 5.1]. We have $\mathrm{Lex}(p(\zeta)) = (x_1, x_2^2, x_2x_3, \ldots, x_2x_{n-1}, x_2x_n^2)$ and $\mathrm{Exp}(p(\zeta)) = (x_1, x_2)(x_1, \ldots, x_n)$. The $\mathrm{GL}(n+1)$-orbit of $\mathrm{Exp}(p(\zeta))$ is the singular locus of $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$.

(3) Let $p(\zeta) = \binom{\zeta+n}{n} - \binom{\zeta+n-d}{n} + 3$. Then $\mathrm{Hilb}^{p(\zeta)}(\mathbb{P}^n)$ is smooth, and its general point parametrizes a hypersurface of degree d with 3 isolated points [33, Section 5.2]. We have $\mathrm{Lex}(p(\zeta)) = x_1^d(x_1, x_2, \ldots, x_{n-1}, x_n^3)$ and $\mathrm{Exp}(p(\zeta)) = x_1^d(x_1, x_2, \ldots, x_{n-2}, x_{n-1}^2, x_{n-1}x_n, x_n^2)$.

Example 7.4 (Twisted cubics)**.** The Hilbert scheme $\mathrm{Hilb}^{3\zeta+1}(\mathbb{P}^3)$ is described in [32]. It is the union of two rational smooth irreducible components $\mathcal{H}, \mathcal{H}'$, whose general point parametrizes respectively a twisted cubic and a plane cubic union a point in $\mathbb{P}^3$. There are three strongly stable points in $\mathrm{Hilb}^{3\zeta+1}(\mathbb{P}^3)$. The point (x^2, xy, y^2) lies in the interior of $\mathcal{H}$, and it is the generic initial ideal of the twisted cubic with respect to $<_{\mathrm{lex}}$. The point $\mathrm{Lex}(3\zeta + 1) = (x, y^4, y^3z)$ lies in the interior of $\mathcal{H}'$. Finally, $\mathrm{Exp}(3\zeta + 1) = (x^2, xy, xz, y^3)$ lies in the intersection $\mathcal{H} \cap \mathcal{H}'$ and gives the most degenerate curve in this Hilbert scheme, namely a line tripled in the plane with a spatial embedded point. The universal deformation space of $\mathrm{Exp}(3\zeta + 1)$ is studied in [32, Lemma 6] to deduce the rationality of $\mathcal{H}, \mathcal{H}'$, and $\mathcal{H} \cap \mathcal{H}'$.

Example 7.5. Let $R = \Bbbk[x_1, x_2, x_3, x_4, x_5]/(x_1^2, x_2^3, x_3^3)$. The upper bounds on the syzygies of $I \in \mathrm{Hilb}^{7\zeta}(\mathrm{Proj}R)$ are

$$\beta_1^S(R/I) \leq 7, \quad \beta_2^S(R/I) \leq 13, \quad \beta_3^S(R/I) \leq 9, \quad \beta_4^S(R/I) \leq 2.$$

If we ignore the data of the degree sequence and regard $\mathrm{Proj}R/I$ as a subscheme in $\mathrm{Hilb}^{7\zeta}(\mathbb{P}^4)$, we obtain the coarser bounds

$$\beta_1^S(R/I) \leq 19, \quad \beta_2^S(R/I) \leq 42, \quad \beta_3^S(R/I) \leq 33, \quad \beta_4^S(R/I) \leq 9.$$

Example 7.6. Let $\mathcal{S} \subseteq \mathbb{P}^4$ be a complete intersection of a quadric and a cubic hypersurface. Using Theorem 4.6 we find that a 1-dimensional subscheme $\mathcal{C} \in \mathrm{Hilb}^{5\zeta+10}(\mathcal{S})$ has syzygies bounded by $\beta_0^S(I_\mathcal{C}) \leq 17, \beta_1^S(I_\mathcal{C}) \leq 39, \beta_2^S(I_\mathcal{C}) \leq 32, \beta_3^S(I_\mathcal{C}) \leq 9$.

Example 7.7. An elliptic quartic $\mathcal{C} \subseteq \mathbb{P}^3_\mathbb{C}$ is the complete intersection of 2 quadric surfaces. For any 0-dimensional subscheme $\mathcal{Z} \subseteq \mathcal{C}$ we claim that the following bounds hold

$$\beta_0^S(I_\mathcal{Z}) \leq 6, \; \beta_1^S(I_\mathcal{Z}) \leq 9, \; \beta_2^S(I_\mathcal{Z}) \leq 4.$$

To see this, let $R = \mathbb{C}[x, y, z, w]/(x^2, y^2)$. It follows from Example 7.1 that $E = \mathrm{Exp}(\mathrm{HP}(S/I_\mathcal{Z}), R)$ is either one of $(x, y, z), (x, y, z^2), (x, yz, z^2)$ or it has the form $(xyz^\alpha, xz^{\alpha+1+\delta_1}, yz^{\alpha+1+\delta_2}, z^{\alpha+2+\delta_3})$ for some integers $\alpha \in \mathbb{N}$ and $0 \leq \delta_1 \leq \delta_2 \leq \delta_3 \leq 1$. The claim follows now from Theorem 4.6 computing a resolution of R/E. Observe that results that do not take degree sequences into account (such as those of [9] or [36]) do not yield any bounds in this example, since the Betti numbers of arbitrary 0-dimensional subschemes $\mathcal{Z} \subseteq \mathbb{P}^n$ are obviously unbounded.

Acknowledgments. The second author would like to thank Ritvik Ramkumar for some helpful conversations.

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(Giulio Caviglia) Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, USA

Email address: `gcavigli@purdue.edu`

(Alessio Sammartano) Department of Mathematics, University of Notre Dame, 255 Hurley, Notre Dame, IN 46556, USA

Email address: `asammart@nd.edu`