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Koszul property of projections of the Veronese cubic surface

Giulio Caviglia[a,*], Aldo Conca[b]

[a] *Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA*
[b] *Dipartimento di Matematica, Universitá di Genova, Via Dodecaneso 35, 16146 Genova, Italy*

Dedicated to Tito Valla, our teacher and friend

Abstract

Let $V \subset \mathbf{P}^9$ be the Veronese cubic surface. We classify the projections of V to $\mathbf{P}^8$ whose coordinate rings are Koszul. In particular we obtain a purely theoretical proof of the Koszulness of the pinched Veronese, a result obtained originally by Caviglia using filtrations, deformations and computer assisted computations. To this purpose we extend, to certain complete intersections, results of Conca, Herzog, Trung and Valla concerning homological properties of diagonal algebras.
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1. Introduction

Koszul algebras were originally introduced by Priddy [18] in his study of homological properties of graded (non-commutative) algebras arising from various constructions in algebraic topology. Given a field K, a positively graded K-algebra $A = \oplus_{i \in \mathbf{N}} A_i$ with $A_0 = K$ is Koszul if the field K, viewed as a A-module via the identification $K = A/A_+$, has a linear free resolution.

* Corresponding author.
  E-mail addresses: gcavigli@math.purdue.edu (G. Caviglia), conca@dima.unige.it (A. Conca).

In the very interesting volume [17] Polishchuk and Positselski discuss various surprising aspects of Koszulness.

In the commutative setting Koszul algebras can be characterized by means of the relative Castelnuovo–Mumford regularity. Paraphrasing Hochster [13, p. 887], one can say that life is worth living in standard graded algebras when all the finitely generated graded modules have finite (relative) Castelnuovo–Mumford regularity. Such algebras are exactly the commutative Koszul algebras; see Avramov and Eisenbud [3] and Avramov and Peeva [4].

Let K be a field and $R = K[x_0, x_1, x_2]$. The pinched Veronese is the K-subalgebra of R generated by all the monomials of degree 3 with the exception of $x_0 x_1 x_2$. It can be seen as the coordinate ring of a projection from a point of the Veronese cubic surface $V_{2,3}$ of $\mathbf{P}^9$, that is, the embedding of $\mathbf{P}^2$ in $\mathbf{P}^9$ with the forms of degree 3. It is a "generic" projection with respect to the stratification of the ambient space by the secant varieties of $V_{2,3}$. That is, the center of the projection is outside the second secant variety $\sec_2(V_{2,3})$ of $V_{2,3}$ and the third secant is the ambient space $\mathbf{P}^9$.

In the nineties Sturmfels asked, in a conversation with Peeva, whether the pinched Veronese is Koszul and the problem became quickly known as a benchmark example to test new theorems and techniques. The first author of the present paper proved in [5] that the pinched Veronese is indeed Koszul by using a combination of arguments based on filtrations, deformations and computer assisted computations.

More generally, one can ask the same question for any projection of $V_{2,3}$ to $\mathbf{P}^8$. In particular, one can ask whether the projection of $V_{2,3}$ to $\mathbf{P}^8$ from a point that does not belong to $\sec_2(V_{2,3})$ is Koszul. The goal of the paper is to show that this is indeed the case. As a special case, we obtain a entirely theoretical proof of the Koszulness of the pinched Veronese.

To achieve this result we develop in Section 2 homological arguments that generalize results of Conca, Herzog, Trung and Valla [7]. Given a standard $\mathbf{Z}^2$-graded K-algebra S and a cyclic subgroup Δ of $\mathbf{Z}^2$ one considers the "diagonal" subalgebra S_Δ of S defined as $\oplus_{v \in \Delta} S_v$. Similarly, for every $\mathbf{Z}^2$-graded S-module M one defines the S_Δ-module M_Δ as $\oplus_{v \in \Delta} M_v$. For every element $w \in \mathbf{Z}^2$, the shifted copy $S(w)$ of S is defined as the $\mathbf{Z}^2$-graded S module whose v-th component is S_{v+w}.

In the transfer of homological information from S to S_Δ it is crucial to bound the homological invariants of the shifted-diagonal modules $S(w)_\Delta$ as S_Δ-modules. When S is a bigraded polynomial ring, it is proved in [7] that the modules $S(w)_\Delta$ have a linear S_Δ-free resolution. We extend this result to the main diagonal $\Delta = (1, 1)\mathbf{Z}$ of certain bigraded complete intersections; see Section 2.

In Section 3 we prove that, given a complete intersection I of 3 quadrics in a polynomial ring R, the K-subalgebra of R generated by the cubics in I is Koszul. This is done by constructing a complex whose homology vanishes along the relevant diagonal. Finally in Section 4 we reinterpret the result of Section 3 to get the classification of the projections to $\mathbf{P}^8$ of the cubic Veronese surface with a Koszul coordinate ring.

2. Generalities and preliminary results

Let K be a field and A be a standard graded K-algebra, that is, a commutative algebra of the form $A = \oplus_{i \in \mathbf{N}} A_i$ such that $A_0 = K$, $\dim_K A_1$ is finite and A is generated as a K-algebra by A_1. In other words, a standard graded K-algebra can be written as $A = S/I$ where $S = K[x_1, \ldots, x_n]$ is a polynomial ring over K equipped with the graded structure induced by the assignment $\deg(x_i) = 1$ for every i and I is a homogeneous ideal. The algebra A is said to

be quadratic if its defining ideal I is generated by quadrics and G-quadratic if I has a Gröbner basis of quadrics (with respect to some coordinate system of S and some term order).

Given a non-zero graded A-module M one defines the (relative) Castelnuovo–Mumford regularity as

$$\text{reg}_A(M) = \sup\{j - i : \text{Tor}_i^A(M, K)_j \neq 0\} \in \mathbf{Z} \cup \{+\infty\}.$$

One says that M has a linear A-resolution if M is generated in a single degree, say d, and $\text{reg}_A(M) = d$. Also, A is Koszul if the residue field K has a linear A-resolution, i.e. $\text{reg}_A K = 0$. It turns out that a standard graded K-algebra A is Koszul if and only if $\text{reg}_A(M)$ is finite for every finitely generated graded A-module M; see [3, Theorem 1] and [4, Theorem 2].

It is known that Koszul algebras are quadratic and that G-quadratic algebras are Koszul. Both implications are, in general, strict; see Eisenbud, Reeves and Totaro [11].

One defines the graded Poincaré series of M as

$$P_M^A(t, s) = \sum_{i,j} \dim_K \text{Tor}_i^A(M, K)_j s^j t^i \in \mathbf{Z}[s][|t|].$$

Lemma 2.1. *Let A be a standard graded K-algebra and let B be a graded quotient of A. Let M be a finitely generated graded B-module. Assume $\text{reg}_A(B) \leq 1$. Then we have the following.*

(1) $\text{reg}_B(M) \leq \text{reg}_A(M)$.
(2) *If M has a A-linear resolution then M has a B-linear resolution.*
(3) *If A is Koszul then B is Koszul as well.*

Proof. Notice that (3) is a special case of (2) and (2) is a special case of (1). To prove (1) we may assume $\text{reg}_A(M)$ is finite (otherwise there is nothing to prove), say $a = \text{reg}_A(M)$. The Cartan–Eilenberg spectral sequence, in the graded setting, induces a coefficientwise inequality

$$P_M^B(t, s) \leq \frac{P_M^A(t, s)}{1 + t - t P_B^A(t, s)};$$

see Avramov [2, Proposition 3.3.2]. Set $G(t, s) = t P_B^A(t, s) - t$ so that, by assumption, we may write $G(t, s) = \sum_{i \geq 2} g_i(s) t^i$ with $g_i(s)$ polynomials of degree $\leq i$. It follows that the terms $s^j t^i$ that appear with a non-zero coefficient in $1/(1 - G) = \sum_{k \geq 0} G^k$ satisfy $j \leq i$. But then the terms $s^j t^i$ that appear with a non-zero coefficient in $P_M^A(t, s)/(1 - G)$ satisfy $j \leq i + a$ and the desired inequality follows. $\square$

Let $R = \oplus_{(i,j) \in \mathbf{Z}^2} R_{(i,j)}$ be a bigraded standard K-algebra. Here standard means that $R_{(0,0)} = K$ and that R is generated as a K-algebra by the K-vector spaces $R_{(1,0)}$ and $R_{(0,1)}$ of finite dimension. Let Δ be the diagonal $(1, 1)\mathbf{Z} \subset \mathbf{Z}^2$. We set $R_\Delta = \oplus_{i \in \mathbf{Z}} R_{(i,i)}$ and observe that R_Δ is the K-subalgebra of R generated by $R_{(1,1)}$ and hence it is a standard graded K-algebra. For every bigraded R-module $M = \oplus_{(i,j) \in \mathbf{Z}^2} M_{(i,j)}$ we set $M_\Delta = \oplus_{i \in \mathbf{Z}} M_{(i,i)}$. Notice that M_Δ is a module over R_Δ. We may think of $-_\Delta$ as a functor from the category of bigraded R-modules and maps of degree 0 to that of graded R_Δ-modules with maps of degree 0. Clearly $-_\Delta$, being a selection of homogeneous components, is an exact functor.

For $(a, b) \in \mathbf{Z}^2$ let $R(-a, -b)$ be the shifted copy of R. The diagonal module $R(-a, -b)_\Delta$ is a R_Δ-submodule of R. The homological properties of $R(-a, -b)_\Delta$ play an important role in

the transfer of homological information from R to R_Δ. Note that $R(-a, -b)_\Delta$ is generated by $R_{(0,a-b)}$ if $a \geq b$ and by $R_{(b-a,0)}$ if $b \geq a$. More precisely, one has:

$$R(-a, -b)_\Delta = \begin{cases} R(0, a - b)_\Delta(-a) & \text{if } a \geq b \\ R(b - a, 0)_\Delta(-b) & \text{if } b \geq a. \end{cases}$$

We have the following.

Proposition 2.2. *Let $S = K[x_1, \ldots, x_m, t_1, \ldots, t_n]$ be the polynomial ring bigraded by $\deg x_i = (1, 0)$ for $i = 1, \ldots, m$ and $\deg t_i = (0, 1)$ for $i = 1, \ldots, n$. Let I be an ideal of S generated by a regular sequence of elements of S all of bidegree $(2, 1)$ and $R = S/I$. Then we have the following.*

(1) *R_Δ is Koszul.*
(2) *For every $(a, b) \in \mathbf{Z}^2$ the module $R(-a, -b)_\Delta$ has a linear R_Δ-resolution, i.e., $\operatorname{reg}_{R_\Delta} R(-a, -b)_\Delta = \max(a, b)$.*

Proof. Let h be the codimension of I. We argue by induction on h. If $h = 0$ then R_Δ is the Segre product of $K[x_1, \ldots, x_m]$ and $K[t_1, \ldots, t_n]$ that is Koszul, indeed G-quadratic; see for instance [11]. In this case, statement (2) is proved in [7, proof of Theorem 6.2]. Assume $h > 0$. We may write $R = T/(f)$ where f is a T-regular element of bidegree $(2, 1)$ and where T is defined by a S-regular sequence of length $h - 1$ of elements of bidegree $(2, 1)$. We have a short exact sequence of T-modules:

$$0 \to T(-2, -1) \to T \to R \to 0 \tag{2.1}$$

and applying Δ we have an exact sequence of T_Δ-modules:

$$0 \to T(-2, -1)_\Delta \to T_\Delta \to R_\Delta \to 0.$$

By induction we know T_Δ is Koszul and that $\operatorname{reg}_{T_\Delta} T(-2, -1)_\Delta = 2$. Hence

$$\operatorname{reg}_{T_\Delta} R_\Delta \leq 1.$$

By Lemma 2.1(3) we may conclude that R_Δ is Koszul. In order to prove (2) we divide the discussion into three cases.

Case 1. $a = b$. Just observe that $R(-a, -a)_\Delta = R_\Delta(-a)$.

Case 2. $b > a$. We first shift (2.1) by $(-a, -b)$ and then apply Δ. We get a short exact sequence of T_Δ-modules:

$$0 \to T(-a - 2, -b - 1)_\Delta \to T(-a, -b)_\Delta \to R(-a, -b)_\Delta \to 0.$$

So we have:

$$\operatorname{reg}_{T_\Delta} R(-a, -b)_\Delta \leq \max\{\operatorname{reg}_{T_\Delta} T(-a, -b)_\Delta, \operatorname{reg}_{T_\Delta} T(-a - 2, -b - 1)_\Delta - 1\}.$$

By induction and since we are assuming $b > a$ we deduce that

$$\operatorname{reg}_{T_\Delta} R(-a, -b)_\Delta = b.$$

Since we have shown already that $\operatorname{reg}_{T_\Delta} R_\Delta \leq 1$, by Lemma 2.1(1) we can deduce that $\operatorname{reg}_{R_\Delta} R(-a, -b)_\Delta = b$.

Case 3. $a > b$. Set $P = (t_1, \ldots, t_n) \subset S$. As already observed we have $R(-a, -b)_\Delta = R(0, a - b)_\Delta(-a)$. So we have to prove that $\operatorname{reg}_{R_\Delta} R(0, u)_\Delta = 0$ for every $u > 0$. We consider

the minimal free (bigraded) resolution of S/P^u as an S-module (that is a Eagon–Northcott complex)

$$\mathbf{F} : 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

with $F_0 = S$ and $F_i = S^{\#}(0, -u - i + 1)$ for $i > 0$ where # denotes some integer depending on n, u and i that is irrelevant in our discussion. The homology of $\mathbf{F} \otimes R$ is $\operatorname{Tor}^S(S/P^u, R)$. We may as well compute $\operatorname{Tor}^S(S/P^u, R)$ as the homology of $S/P^u \otimes \mathbf{G}$ where $\mathbf{G}$ is a free resolution of R as an S-module. By assumption, we may take $\mathbf{G}$ to be the Koszul complex on a sequence of elements of bidegree $(2, 1)$. It follows that:

$$H_i(\mathbf{F} \otimes R) = \begin{cases} \text{a subquotient of } (S/P^u)^{\#}(-2i, -i) & \text{if } 0 \leq i \leq h \\ 0 & \text{if } i > h. \end{cases}$$

Shifting with $(0, u)$ and applying Δ we have a complex $(\mathbf{F} \otimes R(0, u))_\Delta$ that, we claim, has no homology at all. Shifting and applying Δ are compatible operations with taking homology. Therefore to prove that $(\mathbf{F} \otimes R(0, u))_\Delta$ has no homology we have only to check that

$$[(S/P^u)(-2i, -i + u)]_\Delta = 0$$

for all i and that is obvious by degree reasons. So we have an exact complex of R_Δ-modules:

$$0 \to R^{\#}(0, -n)_\Delta \to \cdots \to R^{\#}(0, -1)_\Delta \to R^{\#}_\Delta \to R(0, u)_\Delta \to 0.$$

Since we know (by Case 2) that $\operatorname{reg}_{R_\Delta} R(0, -i)_\Delta = i$ we may conclude (see [7, Lemma 6.3]) that $\operatorname{reg}_{R_\Delta} R(0, u)_\Delta = 0$ as desired. $\quad\square$

3. Diagonal algebras of cubic forms

Let I be a homogeneous complete intersection ideal of codimension r generated by elements of degree d in a polynomial ring R over a field K. Let $c, e \in \mathbf{N}$ and consider the K-subalgebra $A_{c,e}$ of R generated by the forms of degree $c + ed$ in I^e, i.e. $A_{c,e} = K[(I^e)_{ed+c}]$. If one gives the standard $\mathbf{Z}^2$-graded structure to the Rees algebra $\operatorname{Rees}(I)$ of I, then the ring $A_{c,e}$ can be seen as the (c, e)-diagonal subalgebra of $\operatorname{Rees}(I)$, that is, $A_{c,e} = \oplus_{i \in \mathbf{N}} \operatorname{Rees}(I)_{(ic, ie)}$.

Corollary 6.10 in [7] asserts that $A_{c,e}$ is quadratic if $c \geq d/2$ and Koszul if $c \geq d(r - 1)/r$. The authors of [7] ask (at p. 900) whether $A_{c,e}$ is Koszul also for $d/2 \leq c < d(r - 1)/r$. The first instance of this problem occurs for $d = 2$ and $r = 3$, i.e. 3 quadrics, where the only possible value for c is 1. The goal of this section is to treat this case that, as we will see in the following section, is also the crucial case for the classification problem discussed in the introduction.

Let $I = (g_1, g_2, g_3)$ be a complete intersection of quadrics in $R = K[x_1, \ldots, x_n]$. Consider the Rees algebra

$$\operatorname{Rees}(I) = R[g_1 t, g_2 t, g_3 t] \subset R[t]$$

of I with its standard bigraded structure induced by $\deg x_i = (1, 0)$ and $\deg g_i t = (0, 1)$. It can be realized as a quotient of the polynomial ring

$$S = K[x_1, \ldots, x_n, t_1, t_2, t_3]$$

bigraded with $\deg x_i = (1, 0)$ and $\deg t_i = (0, 1)$, by the ideal J generated by the 2-minors of

$$X = \begin{pmatrix} g_1 & g_2 & g_3 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Let f_1, f_2, f_3 be the 2-minors of X with the appropriate sign, say f_i equals to $(-1)^{i+1}$ times the minor of X obtained by deleting the i-th column. Hence

$$J = I_2(X) = (f_1, f_2, f_3).$$

The sign convention is chosen so that the rows of the matrix X are syzygies of f_1, f_2, f_3. By construction we have

$$\text{Rees}(I)_\Delta = K[I_3] = K[x_i g_j : i = 1, \ldots, n, \ j = 1, 2, 3]$$

where $\Delta = (1, 1)\mathbf{Z} \subset \mathbf{Z}^2$. Our goal is to prove that $K[I_3]$ is Koszul.

For later use we record the following.

Lemma 3.1. (1) f_1, f_2 *form a regular S-sequence.*
(2) $(f_1, f_2) : f_3 = (g_3, t_3)$.
(3) $(f_1, f_2) : t_3 = J$.
(4) $(t_3, f_1, f_2) : g_3 = (t_1, t_2, t_3)$.

Proof. (1) The ideal J is prime and hence f_1, f_2 have no common factors.
(2) The inclusion $\supseteq$ follows because the rows of X are syzygies of f_1, f_2, f_3. Clearly $(g_3, t_3) \supseteq (f_1, f_2)$. Hence the equality follows if one shows that g_3, t_3, f_3 is a regular sequence. But that is obvious because the variable t_3 does not appear in the polynomials g_3 and f_3, and f_3 is irreducible being a minimal generator of a prime ideal.
(3) The inclusion $\supseteq$ follows because the second row of X is a syzygy of f_1, f_2, f_3. The equality follows because J is prime and contains (f_1, f_2).
(4) For the inclusion $\supseteq$ simply note that $f_1 = -g_3 t_2 \mod (t_3)$ and $f_2 = g_3 t_1 \mod (t_3)$. The other inclusion follows because (t_1, t_2, t_3) is a prime ideal containing (t_3, f_1, f_2). $\square$

Set $B = S/(f_1, f_2)$. Since f_1, f_2 is a regular S-sequence of elements of bidegree $(2, 1)$ we may apply to B the results of Proposition 2.2. One has also $B/f_3 B = \text{Rees}(I)$. We will prove the following.

Theorem 3.2. *We have:*
(1) $\text{reg}_{B_\Delta}(\text{Rees}(I)_\Delta) = 1$;
(2) $\text{Rees}(I)_\Delta$ *is Koszul.*

Since, by construction, $K[I_3] = \text{Rees}(I)_\Delta$, as a corollary of Theorem 3.2 we have the following.

Corollary 3.3. *Let I be a complete intersection of 3 quadrics in $K[x_1, \ldots, x_n]$. Then $K[I_3]$ is Koszul.*

Proof of Theorem 3.2. First of all we note that (2) follows from (1) and Lemma 2.1 since, by Proposition 2.2, we know that B_Δ is Koszul.

It remains to prove (1). Since $f_3 t_3 = 0$ in B we have a complex

$$\mathbf{F} : \cdots \to B(-4, -4) \xrightarrow{t_3} B(-4, -3) \xrightarrow{f_3} B(-2, -2) \xrightarrow{t_3} B(-2, -1) \xrightarrow{f_3} B \to 0,$$

i.e. $F_i = B(-i, -i)$ if i is even, $F_i = B(-i-1, -i)$ if i is odd. The homology of $\mathbf{F}$ can be described by using Lemma 3.1:

$$H_i(\mathbf{F}) = \begin{cases} \text{Rees}(I) & \text{if } i = 0 \\ 0 & \text{if } i \text{ is even and positive} \\ S/(t_1, t_2, t_3)(-i-3, -i) & \text{if } i \text{ is odd and positive.} \end{cases}$$

The assertion for $i = 0$ holds by construction, and for i even and positive it holds because of Lemma 3.1(3). Finally for i odd and positive by Lemma 3.1(2) we have

$$H_i(\mathbf{F}) = (t_3, g_3)/(t_3, f_1, f_2)(-i - 1, -i).$$

Hence $H_i(\mathbf{F})$ is cyclic generated by the residue class of $g_3 \bmod (t_3, f_1, f_2)$ that has degree $(-i - 3, -i)$. Using Lemma 3.1(4) and keeping track of the degrees we get the desired result. Note that we have $H_i(\mathbf{F}_\Delta) = H_i(\mathbf{F})_\Delta = 0$ for every $i > 0$ and $H_0(\mathbf{F}_\Delta) = \mathrm{Rees}(I)_\Delta$. We may deduce from [7, Lemma 6.3] that

$$\mathrm{reg}_{B_\Delta} \mathrm{Rees}(I)_\Delta \leq \sup\{\mathrm{reg}_{B_\Delta}(F_i)_\Delta - i\}.$$

Since B is defined by a regular sequence of elements of bidegree $(2, 1)$ we may apply Proposition 2.2 and get

$$\mathrm{reg}_{B_\Delta}(F_i)_\Delta = \begin{cases} i & \text{if } i \text{ is even} \\ i + 1 & \text{if } i \text{ is odd.} \end{cases}$$

Summing up, we obtain $\mathrm{reg}_{B_\Delta}(\mathrm{Rees}(I)_\Delta) = 1$. $\square$

Remark 3.4. (1) In 3.2(2) one cannot replace $\mathrm{Rees}(I)$ with a ring of the form $S/I_2(Y)$ where $Y = (y_{ij})$ is a 2×3 matrix with $\deg y_{1j} = (2, 0)$ and $\deg y_{2j} = (0, 1)$ and $I_2(Y)$ has codimension 2. For instance for

$$H = I_2 \begin{pmatrix} x_1^2 & x_2^2 & 0 \\ 0 & t_2 & t_3 \end{pmatrix}$$

and $S = K[x_1, x_2, x_3, t_1, t_2, t_3]$ one has that $(S/H)_\Delta$ is not Koszul as can be checked by using Macaulay 2 [12]. In other words, in the proof of 3.2 the fact that J is a prime ideal plays a crucial role.

(2) The coordinate ring of the pinched Veronese can be realized as S_Δ/J_Δ, where

$$J = I_2 \begin{pmatrix} x_1^2 & x_2^2 & x_3^2 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Within the Segre product $S_\Delta = K[x_i t_j : 1 \leq i, j \leq 3]$ the ideal J_Δ has a Gröbner deformation to $H_\Delta = (x_1^2 t_2 t_i, x_1^2 t_3 t_i, x_2^2 t_3 t_i : i = 1, 2, 3)$. In particular, this shows that the pinched Veronese has a nice quadratic Gröbner deformation within the Segre ring S_Δ. But this is, unfortunately, not enough to prove that it is Koszul.

Remark 3.5. With the notation introduced at the beginning of the section, we have shown above that $A_{1,1}$ is Koszul for $r = 3$ and $d = 2$. The statements and the proofs of Proposition 2.2 and Theorem 3.2 generalize immediately to the case of diagonal $(1, e)\mathbf{Z}$ and one obtains that $A_{1,e}$ is Koszul as well. Moreover the case $c = 0$ is obvious because $A_{0,e}$ is the e-th Veronese ring of a polynomial ring in 3 variables. So we may conclude that, for $r = 3$ and $d = 2$, $A_{c,e}$ is Koszul for all c and e.

4. Projections to $\mathbf{P}^8$ of the Veronese cubic embedding of $\mathbf{P}^2$

Let c, n be positive integers and V be a vector space of dimension $n + 1$ over a field K of characteristic 0 or $>c$. The Veronese embedding of $\mathbf{P}(V)$ with the forms of degree c can be identified with the map

$$v_{n,c} : \mathbf{P}(V) \to \mathbf{P}(S_c(V))$$

sending $[x]$ to $[x^c]$. Here $S_c(V)$ denotes the degree c component of the symmetric algebra $S(V)$. The coordinate ring of $\mathbf{P}(V)$ is $S(V^*)$. Denote by $V_{n,c}$ the image of $v_{n,c}$. Denote by $(g, f) \to g \circ f$ the natural bilinear form

$$S_c(V^*) \times S_c(V) \to K$$

that is GL(V)-equivariant. By taking a non-zero $F \in S_c(V)$ we may consider the projection ϕ_F from $\mathbf{P}(S_c(V))$ to $\mathbf{P}(W)$ where $W = S_c(V)/\langle F \rangle$. The coordinate ring (of the closure of) the image of the composition $\phi_F \circ v_{n,d}$ gets identified with the K-subalgebra of $S(V^*)$ generated by $U_F = \{g \in S_c(V^*) : g \circ F = 0\}$, a space of forms of degree c of codimension 1. After fixing a basis $y_0, \ldots, y_n$ for V and the dual basis $x_0, \ldots, x_n$ for V^*, then $S(V) = K[y_0, \ldots, y_n]$ and $S(V^*) = K[x_0, \ldots, x_n]$. Furthermore $g(x_0, \ldots, x_n) \circ f(y_0, \ldots, y_n)$ gets identified with the action of the differential operator $g(\partial/\partial y_0, \ldots, \partial/\partial y_n)$ on $f(y_0, \ldots, y_n)$.

Returning now to the Veronese cubic embedding $V_{2,3}$ of $\mathbf{P}^2$ in $\mathbf{P}^9$, we may summarize the discussion above as follows. The projection of $V_{2,3}$ to $\mathbf{P}^8$ from a point $[F] \in \mathbf{P}^9 = \mathbf{P}(S_3(V))$, where $S_3(V) = K[y_0, y_1, y_2]_3$, has coordinate ring $K[U_F]$ where $U_F = \{g \in K[x_0, x_1, x_2]_3 : g \circ F = 0\}$. The construction is compatible with the PGL$_3(K)$ action; hence forms F and F_1 that are in the same PGL$_3(K)$-orbit will give projections with isomorphic coordinate rings.

Denote by $\sec_i(V_{2,3})$ the i-th secant variety of $V_{2,3}$. One knows that $\sec_1(V_{2,3})$ is a 5-dimensional variety defined by the 3-minors of a 3×6 matrix of linear forms, a catalecticant matrix; see Kanev [15] and Raicu [19]. It is also known that $\sec_2(V_{2,3})$ is a quartic hypersurface, defined by the Aronhold invariant, that coincides with the Zariski closure of the PGL$_3(K)$-orbit of the Fermat cubic; see Dolgachev and Kanev [10] or Ottaviani [16]. Furthermore $\sec_1(V_{2,3}) \setminus V_{2,3}$ consists of two PGL$_3(K)$-orbits, the orbit of the following polynomials:

$$F_1 = y_1 y_2^2, \qquad F_2 = y_1^3 + y_2^3$$

while $\sec_2(V_{2,3}) \setminus \sec_1(V_{2,3})$ consists of three orbits, the orbits of:

$$F_3 = y_1 y_0^2 + y_2 y_1^2, \qquad F_4 = y_2^2 y_1 + y_0^3, \qquad F_5 = y_0^3 + y_1^3 + y_2^3;$$

see [10, 5.13.2]. We have the following.

Theorem 4.1. *Let $F \in K[y_0, y_1, y_2]_3$ and let A_F be the coordinate ring of the projection of $V_{2,3}$ to $\mathbf{P}^8$ from the point $[F]$. We have the following.*

(1) *If $[F] \in V_{2,3}$ then A_F is Koszul. More precisely, it is G-quadratic.*
(2) *If $[F] \in \sec_2(V_{2,3}) \setminus V_{2,3}$ then A_F is not quadratic (and hence not Koszul).*
(3) *If $[F] \notin \sec_2(V_{2,3})$ then A_F is Koszul.*

Proof. For (1) we may assume that $F = y_2^3$ and then A_F is the toric ring generated by all the monomials of degree 3 in $K[x_0, x_1, x_2]$ different from x_2^3. That such a ring is defined by a Gröbner basis of quadrics can be proved by direct computations but follows also from the result [9, 2.13] of De Negri because the vectors space U_F is a lexicographic segment.

For (2) we have to check that for each F_i (with $i = 1, \ldots, 5$) described above the corresponding ring A_{F_i} is not quadratic. One can compute explicitly a presentation of A_{F_i} by elimination using, for instance, CoCoA [6] or Macaulay 2 [12]. And then one checks that there is a cubic form the among generators of the defining ideal. Indeed there is exactly one cubic generator in each case, while the number of quadrics is either 17 (for $i = 3, 5$) or 18 (for $i = 1, 2, 4$). That there is a cubic generator in the case of the Fermat cubic F_5 has been verified also by Alzati and Russo in [1, Example 4.5] by a theoretical argument. Also the case

F_1 and F_2 can be treated easily because they are "cones". For them it is enough to prove that projections of the rational normal curve $V_{1,3} \in \mathbf{P}^3$ from the corresponding points are cubic hypersurfaces in $\mathbf{P}^2$, and this is well-known. One obtains the cuspidal cubic with F_1 and the nodal cubic with F_2. Finally for (3) one notes that if $[F] \notin \sec_2(V_{2,3})$ then the ideal I of all the forms $g \in K[x_0, x_1, x_2]$ such that $g \circ f = 0$ is a complete intersection of three quadrics; see Conca, Rossi and Valla [8, Corollary 6.12]. Since $A_F = K[I_3]$, we may apply Corollary 3.3 and conclude that A_F is Koszul. $\square$

Remark 4.2. In view of Theorem 4.1(3) one can ask whether for $F \notin \sec_2(V_{2,3})$ the ring A_F is G-quadratic. It is known that the pinched Veronese does not have a Gröbner basis of quadrics in the toric presentation but we do not know how to exclude a Gröbner basis of quadrics in every other possible coordinate system. Some of the well-known necessary conditions that a monomial ideal $U \subset K[z] = K[z_1, \ldots, z_9]$ must satisfy to be the initial ideal of the defining ideal of A_F include:

(1) $K[z]/U$ must have the same Hilbert series as A_F and the graded Betti numbers at least as big as those of A_F,
(2) the radical of U must be pure and connected in codimension 1;

see Kalkbrener and Sturmfels [14] or Varbaro [20]. But there are plenty of quadratic monomial ideals satisfying those conditions and we do not know how to exclude that they are the initial ideal of the defining ideal of the pinched Veronese, not to mention other A_F.

Remark 4.3. For $F \notin \sec_2(V_{2,3})$ another quite natural question is whether the ring A_F can be deformed, via a Sagbi deformation, to the pinched Veronese. As it follows from [8, Corollary 6.12], the answer is positive if and only if F is singular. So the "general" A_F does not have a Sagbi deformation to the pinched Veronese.

Remark 4.4. An interesting question suggested by the proof of Theorem 4.1 is the following. Suppose F is a form of degree d in $K[y_0, \ldots, y_n]$ that does not involve the variable y_n. We may consider the coordinate ring A of the projection of $V_{n,d}$ from F and also the coordinate ring B of the projection of $V_{n-1,d}$ from F. Is it true that A is Koszul or quadratic if B is so? That the opposite implication holds true follow easily from the fact that B is an algebra retract of A.

Acknowledgments

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