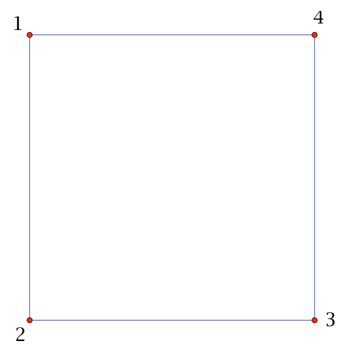
Math 453
Fall 2010
Answers to Selected Problems on Burnside's Theorem

1. Determine the number of ways in which the four corners of a square can be colored with two colors.

Solution: Number the corners of the square $1, 2, 3, 4$ in the counterclockwise direction, as in the picture.



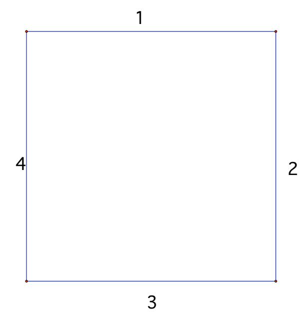
There are $2^4 = 16$ ways to arrange the colors of the corners. In order to determine the number of non-equivalent, we use Burnside's Theorem. The symmetries of the square are given by D_4. Notice that R_0 fixes all 16 arrangements. R_{90} and R_{270} only fix arrangements with all four colors the same color. Since the orbits under R_{180} are $\{1, 3\}$ and $\{2, 4\}$, the colorings fixed by R_{180} are the ones with vertices 1 and 3 are the same color, and vertices 2 and 4 are the same color. Thus, R_{180} fixes 4 colorings. The orbits under H are $\{1, 2\}$ and $\{3, 4\}$, so H (and similarly V) fixes 4 arrangements. Note the diagonal reflection D has orbits $\{1\}, \{3\}$ and $\{2, 4\}$, so fixes $2^3 = 8$ arrangements. Similarly D' fixes 8 arrangements. Thus, by Burnside's Theorem there are

$$\frac{1}{|D_4|} \sum_{\phi \in D_4} |\operatorname{fix}(\phi)| = \frac{1}{8}\left(16 + 2 \cdot 2 + 4 + 2 \cdot 4 + 2 \cdot 8\right) = \frac{1}{8}(48) = 6$$

non-equivalent colorings.

8. Determine the number of ways in which the four edges of a square can be colored with six colors, with no restrictions.

Proof: Let the edges be numbered 1,2,3, and 4 as in the picture.



As in problem 1, the symmetry group is D_4. There are 6^4 arrangements of colorings of the edge of the square. We make the following table of orders of fixed points of elements in D_4.

Type of element	Number of elements	Number of fixed colorings
Identity	1	6^4
Rotation of order 4	2	6
Rotation of order 2	1	6^2
Edge reflection	2	6^3
Diagonal reflection	2	6^2

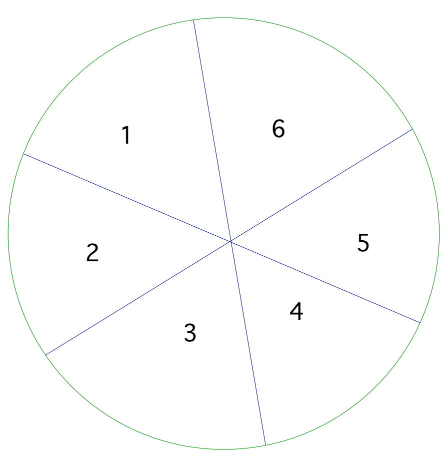
Thus, by Burnside's Theorem there are

$$\frac{1}{|D_4|}\sum_{\phi\in D_4} |\operatorname{fix}(\phi)| = \frac{1}{8}\left(6^4 + 2\cdot 6 + 6^2 + 2\cdot 6^3 + 2\cdot 6^2\right) = 2310$$

ways to color the edges of the square.

11. Suppose we cut a cake into 6 identical pieces. How many ways can we color the cake with n colors if each piece gets one color.

Proof: Number the slices of the cake 1 through 6 as in the picture.



There are n^6 arrangements of the colors of slices on the cake. The symmetry group of our cake is $G = \{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}\}$, the subgroup of rotations in D_6. (We do not allow reflections, since we wouldn't want to turn the cake upside down.) Look at the orbits of each type of element. element. R_0 fixes every element, and hence fixes all n^6 colorings. For a rotation of order 6 there is one orbit $\{1, 2, 3, 4, 5, 6\}$. So these two elements fix n colorings. The two rotations of order 3 have two orbits $\{1, 3, 5\}$ and $\{2, 4, 6\}$. Thus, these elements fix n^2 elements. Finally, the element R_{180} has three orbits: $\{1, 4\}$, $\{2, 5\}$, $\{3, 6\}$. Thus this element fixes n^3 elements. Applying Burnside's Theorem we a have the number of colorings of the cake is

$$\frac{1}{|G|} \sum_{\phi \in G} |\operatorname{fix}(\phi)| = \frac{1}{6}\left(n^6 + 2n + 2n^2 + n^3\right).$$