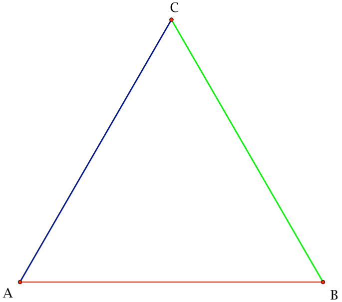
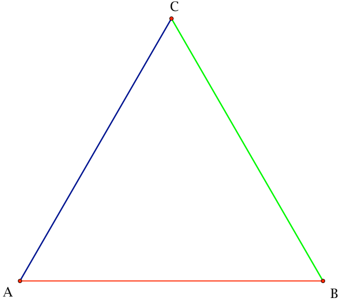
What are the symmetries of an equilateral triangle?



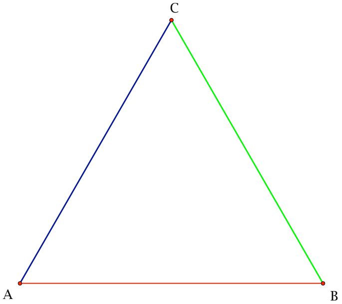
In order to answer this question precisely, we need to agree on what the word "symmetry" means.

What are the symmetries of an equilateral triangle?

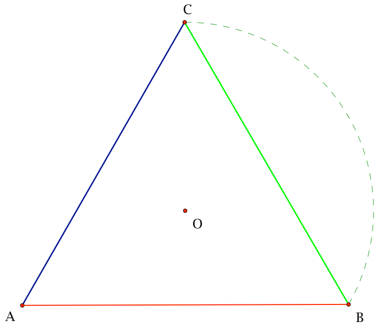


For our purposes, a symmetry of the triangle will be a rigid motion of the plane (i.e., a motion which preserves distances) which also maps the triangle to itself.
Note, a symmetry can interchange some of the sides and vertices.

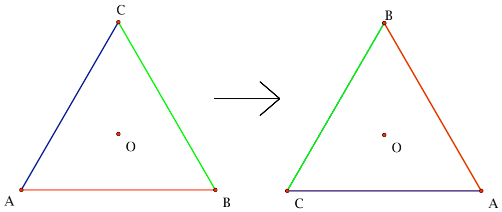
So, what are some symmetries? How can we describe them? What is good notation for them?



Rotate counterclockise, 120° about the center O.



Note this is the following map (function):

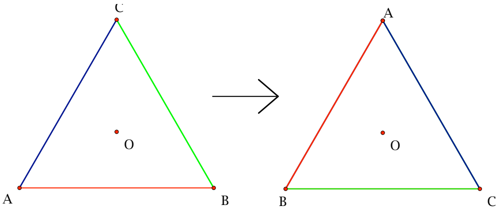


We can think of this as a function on the vertices:
$A \mapsto B, B \mapsto C, C \mapsto A$.

We might denote this by: $\begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$

We also may denote this map by R_{120}.

Rotate counterclockise, 240° about the center O. This is the map (function):

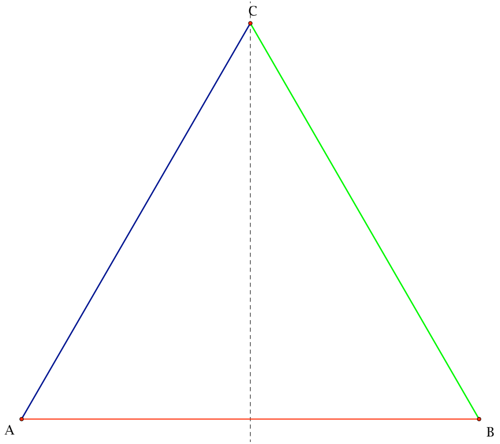


We can think of this as a function on the vertices:
$A \mapsto C, B \mapsto A, C \mapsto B.$

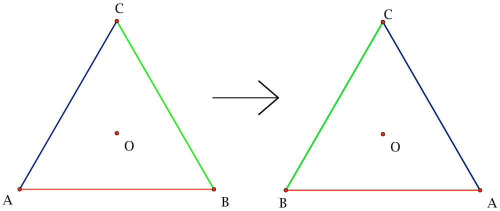
We might denote this by: $\begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$

We also may denote this map by R_{240}.

Reflect about the perpendicular bisector of AB :



Reflect about the perpendicular bisector of AB,
This is the map (function):

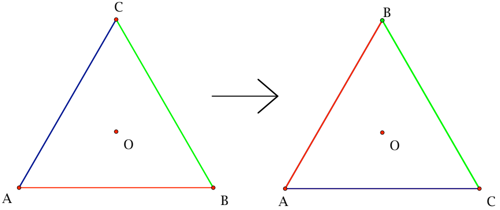


We can think of this as a function on the vertices:
$A \mapsto B, B \mapsto A, C \mapsto C$.

We might denote this by: $\begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$

We also may denote this map by F_C to indicate the reflection is the one fixing C.

Reflect about the perpendicular bisector of BC,
This is the map (function):

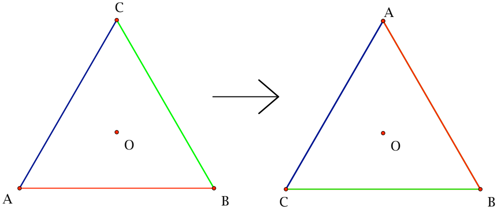


We can think of this as a function on the vertices:
$A \mapsto A, B \mapsto C, C \mapsto B.$

We might denote this by: $\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$

We also may denote this map by F_A to indicate the reflection is the one fixing A.

Reflect about the perpendicular bisector of AC,
This is the map (function):

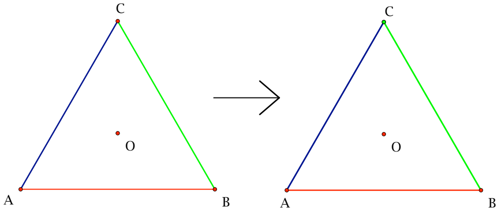


We can think of this as a function on the vertices:
$A \mapsto C, B \mapsto B, C \mapsto A.$

We might denote this by: $\begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$

We also may denote this map by F_B to indicate the reflection is the one fixing B.

The identity map of the plane: (takes every point to itself).
This is the map (function):

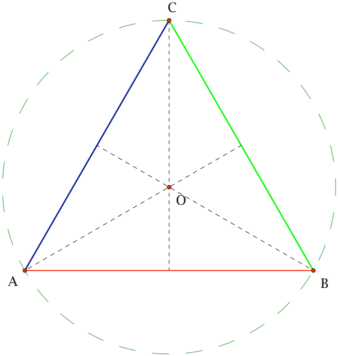


We can think of this as a function on the vertices:
$A \mapsto A, B \mapsto B, C \mapsto C$.

We might denote this by: $\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$

We also may denote this map by Id or 1.
Note, we might also denote this as R_0, since it is a rotation through 0°. However – it is **NOT** a reflection. (WHY NOT??!!)

So far we have 6 symmetries – 3 rotations, R_0, R_{120}, R_{240}, and 3 reflections, F_A, F_B, F_C.



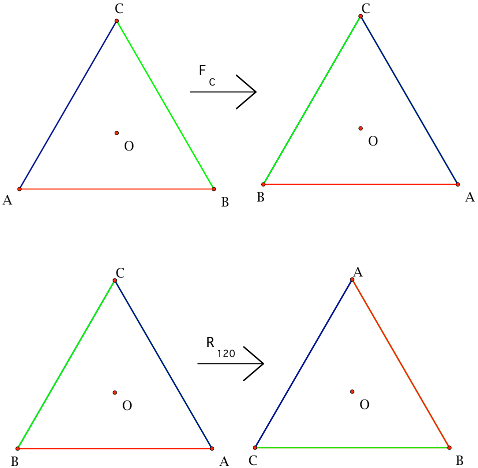
Are there any more??
Why or why not??

In fact these are all the symmetries of the triangle.
We can see this from our notation in which we write each of these maps in the form $\begin{pmatrix} A & B & C \\ X & Y & Z \end{pmatrix}$. Note there are three choices for X (i.e., X can be any of $A, B, C,$). Having made a choice for X there are two choices for Y.
Then Z is the remaining vertex. Thus there are **at most** $3 \cdot 2 \cdot 1 = 6$ possible symmetries. Since we have seen each possible rearrangement of $A, B.C$ is indeed a symmetry, we see these are all the symmetries.

Notice these symmetries are maps, i.e., functions, from the plane to itself, i.e., each has the form $f : \mathbb{R}^2 \to \mathbb{R}^2$. Thus we can compose symmetries as functions: If f_1, f_2 are symmetries then $f_2 \circ f_1(x) = f_2(f_1(x))$, is also a rigid motion. Notice, the composition must also be a symmetry of the triangle.

For example, $R_{120} \circ F_C =$?? It must be one of our 6 symmetries. Can we tell, without computing whether it is a rotation or reflection?? Why?? What about the composition of two reflections?

$R_{120} \circ F_C$, we can view this composition as follows:

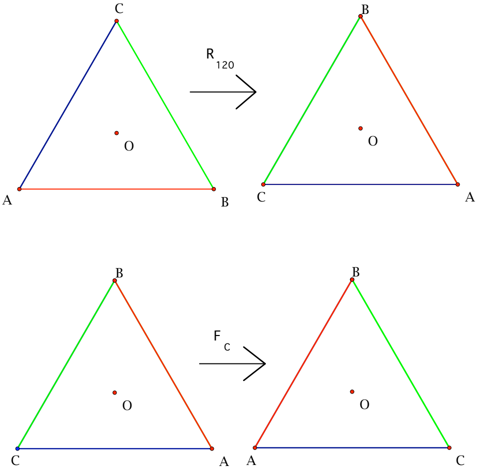


So, $R_{120} \circ F_C = F_B$.

We use our other notation:

$$R_{120} \circ F_C = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \circ \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix} = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} = F_B$$

Is $R_{120} \circ F_C = F_C = R_{120}$? Let's look: $F_C \circ R_{120}$:



So $F_C \circ R_{120} = F_A \neq F_B = R_{120} \circ F_C$.

So on our set of symmetries $S = \{R_0, R_{120}, R_{240}, F_A, F_B, F_C\}$, we get a way of combining any two to create a third, i.e., we get an **operation** on S. (Just like addition is an operation on the integers.) We will call this operation **multiplication on** S. We can make a multiplication table, or **Cayley Table**. So far we have:

$\circ$	R_0	R_{120}	R_{240}	F_A	F_B	F_C
R_0	R_0	R_{120}	R_{240}	F_A	F_B	F_C
R_{120}	R_{120}					F_B
R_{240}	R_{240}					
F_A	F_A					
F_B	F_B					
F_C	F_C	F_A				

Notice we have already seen $F_C \circ R_{120} \neq R_{120} \circ F_C$, so this operation is **non-commutative**.

Now we fill in the rest: (check)

$\circ$	R_0	R_{120}	R_{240}	F_A	F_B	F_C
R_0	R_0	R_{120}	R_{240}	F_A	F_B	F_C
R_{120}	R_{120}	R_{240}	R_0	F_C	F_A	F_B
R_{240}	R_{240}	R_0	R_{120}	F_B	F_C	F_A
F_A	F_A	F_B	F_C	R_0	R_{120}	R_{240}
F_B	F_B	F_C	F_A	R_{240}	R_0	R_{120}
F_C	F_C	F_A	F_B	R_{120}	R_{240}	R_0

We make note of several things about this table:

(i) Every symmetry appears exactly once in each row and in each column;

(ii) Every symmetry has an "opposite" or "inverse" symmetry;

(iii) Less clear from the table: If f, g, h are symmetries of our triangle $(f \circ g) \circ h = f \circ (g \circ h)$. BUT THIS IS A FACT ABOUT FUNCTIONS (and we already know it!!).

We learn in High School Algebra, and again in Calculus (and re-learned in Ch. 0)
$f(g(h(x))) = f \circ (g \circ h)(x) = (f \circ g) \circ h(x)$. We call this "Associativity".

Observations

So our set S of symmetries has the following property:

(i) There is a **binary operation** on S, i.e., a way to combine two members of S to get another one,(composition) we write $\psi\varphi$ instead of $\psi \circ \varphi$;

(ii) This operation is associative: $\psi(\sigma\varphi) = (\psi\sigma)\varphi$.

(iii) There is an identity element for the operation, i.e., an element σ so that $\sigma\psi = \psi\sigma = \psi$, for all ψ; (The identity is R_0.)

(iv) Every element has an inverse – Given $\psi \in S$ there is a $\sigma \in S$ so that $\psi\sigma = \sigma\psi = R_0$.

There are other examples of sets, say, G satisfying (i)-(iv)–
The integers $\mathbb{Z}$, with the operation $+$ (i), is associative (ii), the integer 0 is the additive identity (iii) and for any n we have $n + (-n) = 0$. (iv).

The positive real numbers $\mathbb{R}$ with multiplication $\cdot$
$GL(2, \mathbb{R})$ the set of all 2×2 invertible real matrices with the operation of matrix multiplication.
A set G with a closed binary operation, $\cdot$, satisfying (i)-(iv) is called a **group**.