Math 460: Homework # 10. Due Thursday November 7

1. (Use Geometer's Sketchpad.) For most quadrilaterals, the four angle bisectors are not concurrent.

   (a) Find an equation that the sides of the quadrilateral have to satisfy if the four angle bisectors are concurrent.

   (b) Make a quadrilateral in which all four sides have different lengths and the four angle bisectors are concurrent. Verify that the equation you found in (a) holds for this example.

2. (In this problem we prove the fact that you discovered in Problem 1 of Assignment 9.) Given a triangle ABC with orthocenter H, prove that C is the orthocenter of ABH.

3. (In this problem we complete the proof of the fact that you discovered in Problem 1 of Assignment 7). See Figure 1. Given: M, N and P are midpoints; $DE = AP$, $DF = BN$, and $EF = CM$. To prove: the area of $\triangle DEF$ is 3/4 of the area of $\triangle ABC$. (Hint: The triangle $\triangle DEF$ turns out to be congruent to a triangle that appears in the Figures illustrating Problem 3 of Assignment 8 and Problem 7 of Assignment 9. You may use what you proved in those problems.)

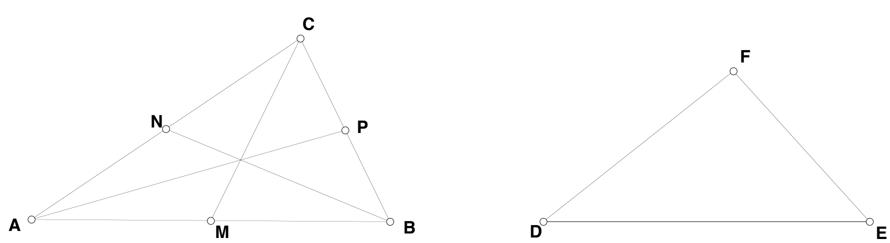

Figure 1

4. (10 points) One of these three statements is very hard to prove. Prove the other two.

   (i) A triangle is isosceles $\Longleftrightarrow$ it has two equal altitudes.

   (ii) A triangle is isosceles $\Longleftrightarrow$ it has two equal angle bisectors.

   (iii) A triangle is isosceles $\Longleftrightarrow$ it has two equal medians.

   Note: since we have defined altitudes, angle bisectors, and medians to be lines, not line segments, the statements require some explanation. In the first statement, "altitude" means the part of the altitude that goes from the vertex to the opposite side, and similarly for the other two statements.

5. (In this problem we continue to prove what you discovered in Problem 2 of Assignment 7.) See Figure 2. Given: G is the centroid of $\triangle ABC$ and U, V, W, X, Y and Z are the centroids of the six "little triangles." To prove: the area of $\triangle DEF$ is 4/9 of the area of $\triangle ABC$. (You may use anything that you have already proved about this picture in previous homework problems. In particular, the lines in Figure 2 that look concurrent are concurrent.)

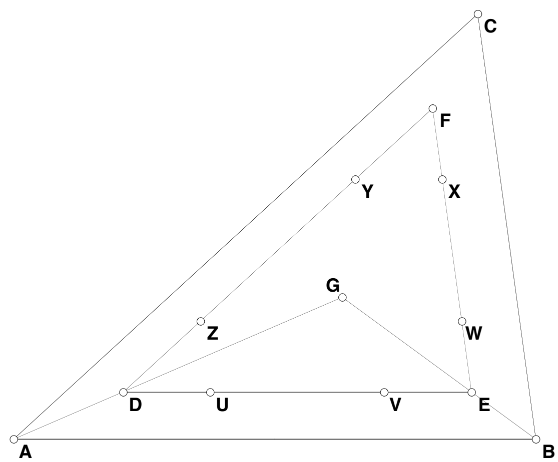

Figure 2

6. (This is a continuation of problem 5, i.e., of the proof of what you discovered in Problem 2 of Assignment 7.) See Figure 3. Given: G is the centroid of $\triangle ABC$ and U, V, W, X, Y and Z are the centroids of the six "little triangles." To prove: the area of $\triangle DZU$ is 1/16 of the area of $\triangle DEF$. (You may use anything that you have already proved about this picture in earlier homework. But be specific in saying what you are using and when it was shown.)

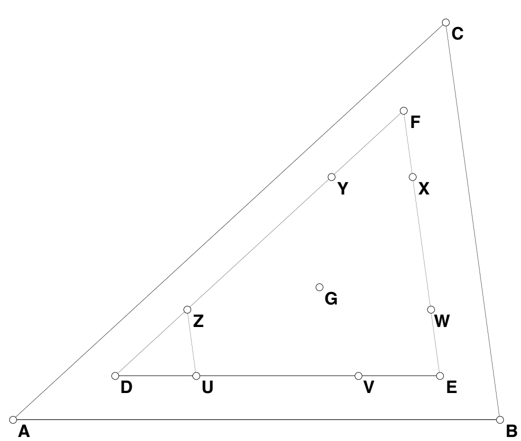

Figure 3

7. (In this problem we begin to prove the fact that you discovered in Problem 1 of Assignment 8.) See Figure 4. Given: the lines that look straight are straight. To prove:

$$\frac{XB}{XC}\frac{YC}{YA}\frac{ZA}{ZB} = 1$$

(Hint: apply Theorem 31 to $\triangle ABC$ five times. Do not draw in any extra lines.)

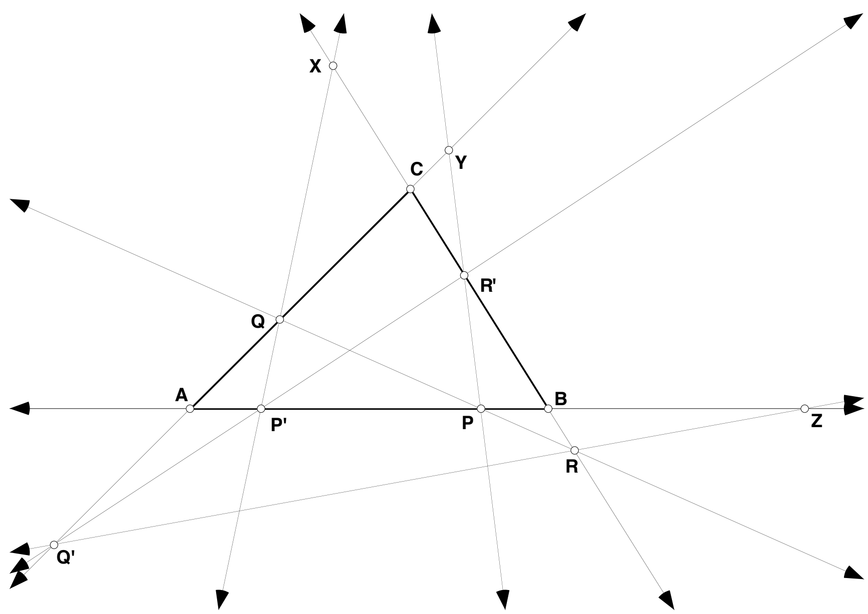

Figure 4

8. Euclid's proof of Proposition 9 uses Proposition 8 as one ingredient. Find a proof of Proposition 9 which uses only facts from Euclid that come *before* Proposition 6 (and nothing from the course notes).

9. Euclid's proof of Proposition 11 uses Proposition 8 as one ingredient. Find a proof of Proposition 11 which uses only facts from Euclid that come *before* Proposition 6 (and nothing from the course notes).

10. (See Figure 5). Given: $\triangle ABC$ is a triangle, DE is parallel to BC, M is the midpoint of BC, and J is the intersection of DC and EB. Prove that A, J, and M are collinear (that is, they lie along the same line).

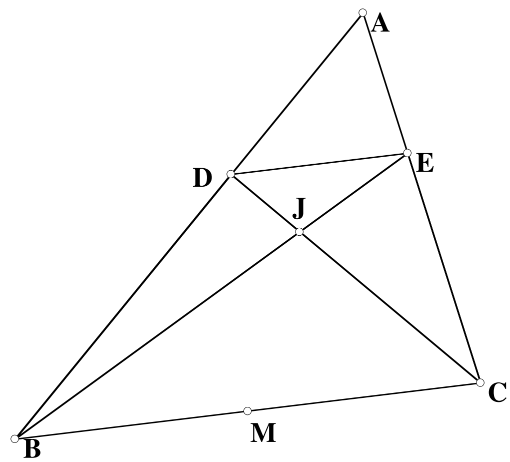

Figure 5

11. (See Figure 6.) Given: $\angle ABC$ is a right angle, and $ABDE$ and $ACFG$ are squares. To prove: $BG = CE$ and $BG \perp CE$.

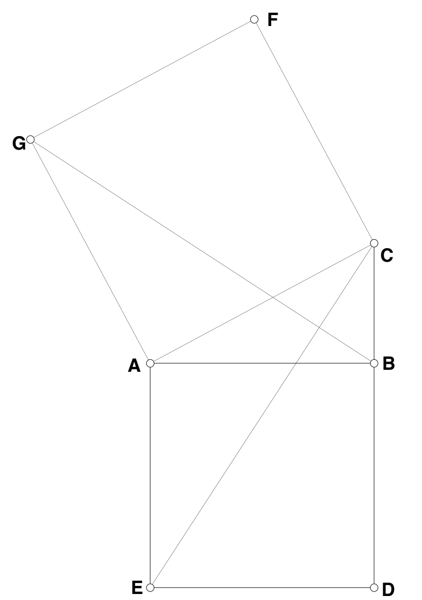

Figure 6

12. See Figure 7. Given: A, B, P and Q are in the indicated plane, RPS and AQB are straight lines, $RS \perp PA$, $RS \perp PB$, $SP = RP$. To prove: $RS \perp PQ$.

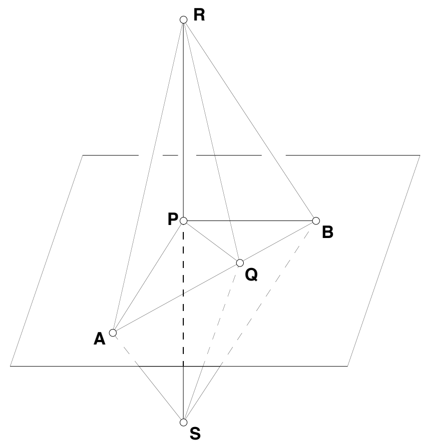

Figure 7

Note: by doing this problem you will be proving a theorem in 3-dimensional geometry:

Theorem. Let Π be a plane, let P be a point on the plane Π, and let ℓ be any line through P. If ℓ is perpendicular to two different lines through P that lie in the plane Π, then ℓ is perpendicular to *every* line through P that lies in the plane Π.

13. (See Figure 8.) Given: $ABCD$ is a trapezoid with AB parallel to DC, M,N,P,Q are the midpoints of the obvious segments. Prove that M, N, P, and Q all lie on the same line.

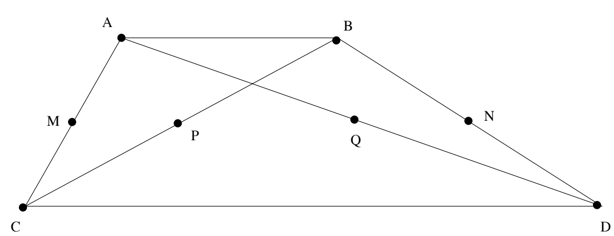

Figure 8