Math 460: Homework # 7. Due Thursday, October 10

1. (Use Geometer's Sketchpad.) Consider the following algorithm for constructing a triangle with three given sides, using "Circle by center and radius" and "Segment" commands: Start with 3 line segments XY, ZW and ST.

   Step 1: Draw a circle, $\mathcal{C}_1$, with center Y and radius ZW.
   Step 2: Draw a circle $\mathcal{C}_2$, with center X and radius ST.
   Step 3: Let U be a point of intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$.
   Step 4: Draw XU and YU.

   Then XYU has side XY, side $YU = ZW$ and side $XU = ST$, so has sides equal to the three given lengths. Note, in Step 3, there may fail to be an intersection of the two circles, in which case there is no triangle with the three given lengths (more on this later in the semester). Note, by copying the segment XY to a given point P you can construct this triangle with the given three sides at any point P and along any line PQ. Let ABC be a triangle. Use this method to construct a second triangle DEF whose sides are equal to the medians of ABC. (Notice that when you change the shape of ABC, the shape of DEF changes along with it.) Do not hide the objects used in your construction. Find an equation relating the areas of ABC and DEF. Print out a picture. Then change the shape of ABC and print a second picture showing that the equation still holds. You may want to use this as an opportunity to investigate **scripts**, or **tools** in GSP. You can create a tool by selecting the objects used to construct another object, and then selecting the object. Then click on the tool icon (bottom button on the toolbox) and select "Create New Tool". If it is not highlighted (i.e. you can't select it) your objects, as selected do not constitute a tool. GSP is a little picky about this, so you may have to experiment. So, see if you can write a script to create a triangle with three given sides.

2. (Use Geometer's Sketchpad.) Construct a triangle and its three medians. The medians divide the triangle into 6 smaller triangles—use a script to construct their centroids (hide the lines you use to do this). Now connect these 6 centroids to form a hexagon. Find the ratio of the area of this hexagon and the area of the original triangle. (Sketchpad will give you a decimal approximation to this ratio: try to find the exact ratio as a fraction. It is **not** 9/25.) Print out a picture.

3. (In this problem we prove the fact that you demonstrated experimentally in Problem 2 of the fifth assignment.) Let ABC be a triangle, and let P be a point in the interior of ABC. Construct the lines connecting P to each of the vertices, and let A', B' and C' be the points where these lines meet the sides BC, AC, and AB,

respectively. Prove that

$$\frac{A'B}{A'C}\frac{B'C}{B'A}\frac{C'A}{C'B} = 1.$$

(Hint: Use Theorem 31, twice.)

4. (10 points) In this problem we prove Theorem 27. Let $\triangle ABC$ be a triangle, and draw the line ℓ through A parallel to BC, the line m through B parallel to AC, and the line n through C parallel to AB. Let D be the intersection of ℓ and m, E the intersection of ℓ and n, and F the intersection of m and n.

   (i) Prove that the altitudes of $\triangle ABC$ are the same lines as the perpendicular bisectors of $\triangle DEF$.

   (ii) Use (i) to prove that the altitudes of $\triangle ABC$ are concurrent.

5. (See Figure 1.) Given: $\angle C$ and $\angle D$ are right angles and $\triangle APR \cong \triangle BQT$. To prove $\triangle ADF \cong \triangle BCE$

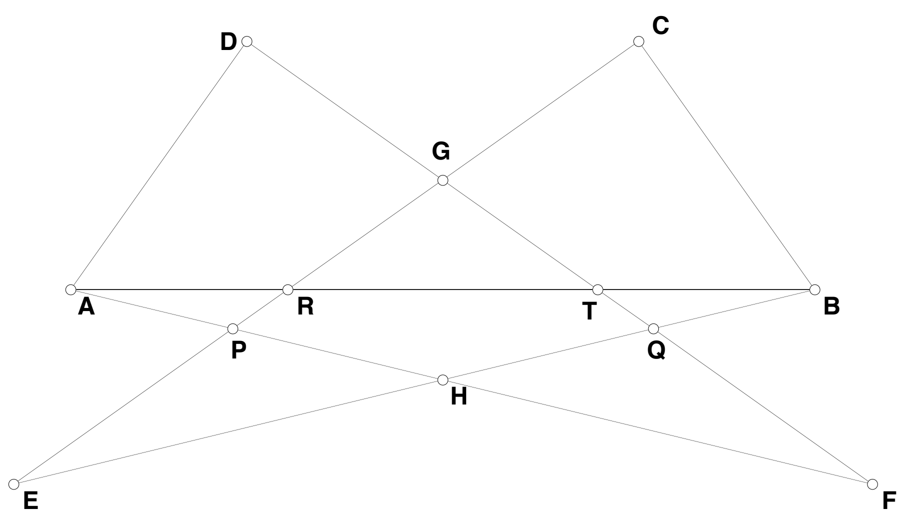

Figure 1

6. (See Figure 2) Given: DF is parallel to CB, the lines that look straight are straight, and $AE = 3 * GE$. Prove that the area of $\triangle AFD$ is 9/16 times the area of $\triangle ABC$.

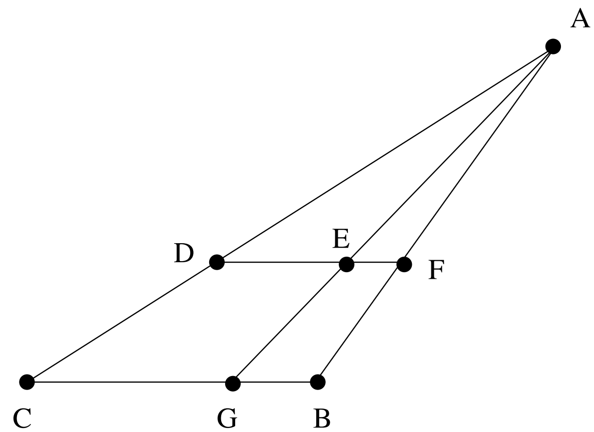

Figure 2

7. (See Figure 3.) Given: CD is parallel to AB. To prove: the triangles AED and CEB have the same area.

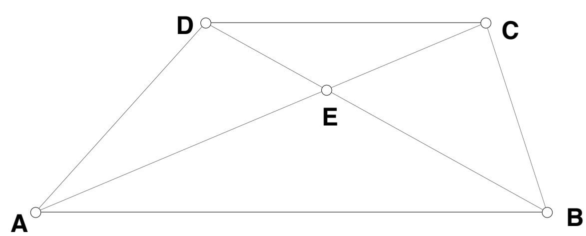

Figure 3

8. (See Figure 4.) Given: $ABCD$ is a parallelogram, M is the midpoint of BC and N is the midpoint of AD. To prove: $AQ = QP = PC$.

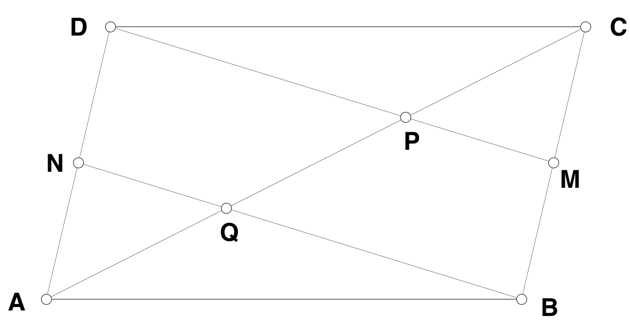

Figure 4

9. (This is a problem about three-dimensional geometry. For problems like this you are allowed to use anything in the course notes up to Theorem 5 except BF 5 and Theorem 2.) See Figure 5. Given: $\angle PMN = \angle PNM$ and $\angle MPQ = \angle NPQ$. To prove: $\angle PMQ = \angle PNQ$..

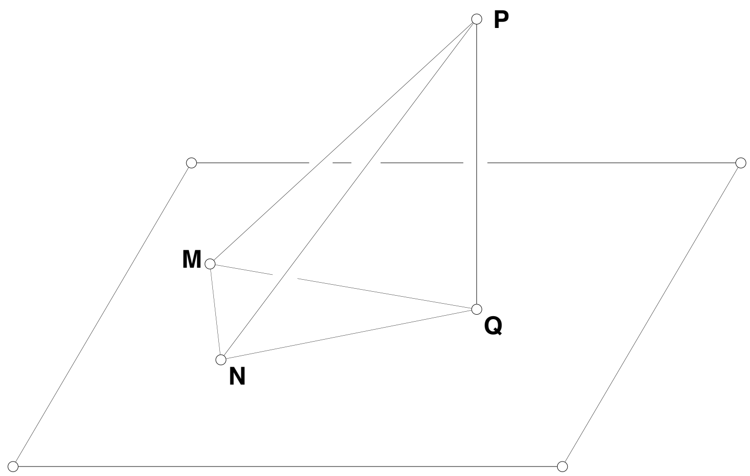

Figure 5

10. Let $ABCD$ be a parallelogram, and suppose $\triangle BCA'$ AND $\triangle ABC'$ are equilateral triangles. Prove $\triangle DA'C'$ is an equilateral. (**Be careful here, you may be tempted to use the word "similarly" in a situation where it does not apply)**

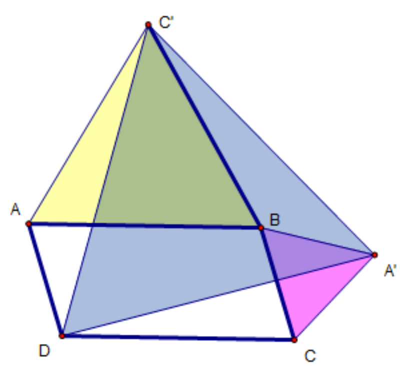

Figure 6

11. This is a continuation of problems 6 and7 from Homework #6. First, we'll say that two circles C and D are called **tangent** if they meet in exactly one point. For this problem you may assume that tangent circles share the same tangent line at their point of intersection. (This is 'obvious', but we will eventually prove it.)

Given: A circle C with center O, two points A and B on C, and the tangent line l to the circle at A. Devise a method for constructing a new circle which is tangent to C at B, and which is also tangent to the line l. Write down the list of your steps, and use Geometer's Sketchpad to make a diagram demonstrating the construction. (Remember that you may only use the basic constructions listed in problem 7 of Homework #6—except even though it doesn't say it there, you are always allowed to extend line segments into lines. Of course you may use any of the constructions from problems 7 and 8 of Homework #6. You do not need to prove anything for this problem.

(Hint: It might help to figure out the following question. Suppose you have two points X and Y on a circle C, you draw the tangent lines at both these points, and these lines intersect in the point P. What do you know about XP and YP?)