NAME:

MATH 503

Fall 2018

Midterm Exam 1

Instructions: Give a complete solution to each problem. You may use any result from class, the book, or homework **except** the statement you are asked to prove (or one whose proof relies on the given statement). Be sure to justify your statements.

1. **(13 points)** Suppose G is a finite group of even order. Show there must be some non-identity element, $x \in G$ with $x = x^{-1}$.

   Solution: Let $G = \{1 = x_1, x_2, \ldots, x_{2n}\}$ for some $n > 0$. For each i, we let $X_i = \{x_i, x_i^{-1}\}$. Then, for each i we have $1 \leq |X_i| \leq 2$. Note $X_1 = \{1\}$. Without loss of generality, we may assume $G = \coprod_{i=1}^{k} X_i$, for some k. Now

   $$2n = |G| = \sum_{i=1}^{k} |X_i| = 1 + \sum_{i=2}^{k} |X_i|,$$

   and thus $|X_i| = 1$ for some $i > 1$. So for some $x \neq 1$, we have $x = x^{-1}$. □

2. **(18 points)** Prove that a group G is abelian if an only if the map $f : G \to G$ given by $f(x) = x^{-1}$ is an isomorphism.

   Proof: Note that $f(x) = f(y)$ if and only if $x^{-1} = y^{-1}$, if an only if $x = y$. So f is injective. Also, $x = f(x^{-1})$, so f is also surjective. Suppose G is abelian, so $xy = yx$ for all $x, y \in G$. Then, for any $x, y \in G$,

   $$f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)f(y).$$

   So f is a homomorphism, and hence an isomorphism.

   Now suppose f is an isomorphism. Then, for any $x, y \in G$,

   $$yx = (x^{-1}y^{-1})^{-1} = f(x^{-1}y^{-1}) = f(x^{-1})f(y^{-1}) = xy.$$

So G is abelian. $\square$

3. Let $G = S_n$, and suppose $1 \leq i \leq n$. Set $G_i = \{\sigma \in G | \sigma(i) = i\}$.
   (a) **(10 points)** Prove G_i is a subgroup
   (b) **(6 points)** Is G_i a normal subgroup of G?. Explain.

Solution:

(a) Consider the natural action of G on $\{1, 2, \ldots, n\}$, given by $\sigma \cdot i = \sigma(i)$. Then $G_i = \text{Stab}_G(i)$, and hence is a subgroup of G.

(b) No. Consider $n = 3 = i$, so $G_3 = \{1, (12)\}$. Note $(13)(12)(13) = (23) \notin G_3$, so G_3 is not normal. $\square$

Remark: In fact, with the exception of the trivial cases $n = 1, 2$ we see G_i is never normal.

4. **(18 points)** Let G be a group and suppose A is a non-empty subset of G. Let $\langle A \rangle = \bigcap_{A \subset H \leq G} H$. Prove $\langle A \rangle$ is the smallest subgroup of G containing A.

Proof. Let $\mathcal{A} = \{H \leq G | A \subset H\}$. We proved intersection of an arbitrary collection of subgroups of G is a subgroup of G. Therefore

$$A \subset \bigcap_{H \in \mathcal{A}} H = \langle A \rangle$$

is a subgroup of G.

Alternative proof of this part

Note, $A \subset \langle A \rangle$, so $\langle A \rangle \neq \emptyset$. Suppose $x, y \in \langle A \rangle$. Then $x, y \in H$, for all $H \in \mathcal{A}$. Since each $H \leq G$, we have $xy^{-1} \in H$ for all $H \in \mathcal{A}$. Thus,

$$xy^{-1} \in \bigcap_{H \in \mathcal{A}} H = \langle A \rangle.$$

Thus, $\langle A \rangle$ is a subgroup of G.

We have shown $A \subset \langle A \rangle$. Note, if $A \subset H_1$, for some subgroup of H_1 of G, then $H_1 \in \mathcal{A}$, so

$$H_1 \supset \bigcap_{H \in \mathcal{A}} H = \langle A \rangle.$$

Thus $\langle A \rangle$ is the smallest subgroup of G containing A. $\square$

5. **True/False (3 points each)** Determine whether each of the following statements is true or false. If true, give a proof. If false, give a concrete counterexample.
   (a) If p is a prime and $\sigma \in S_n$ is an element of order p, then σ is a product of dispoint p–cycles.
   (b) If G is a group, and $a, b \in G$, then $< a > \cap < b >$ is a cyclic subgroup of G.
   (c) $\mathbb{Z} \times \mathbb{Z}$ is cyclic
   (d) If G is a group, H is a subgroup of G, and Ha and Hb are distinct right cosets, then aH and bH are distinct left cosets.
   (e) If H and K are subgroups of a group G and $H \lhd K$ and $K \lhd G$, then $H \lhd G$.

Solution:

(a) **TRUE** If $\sigma = \alpha_1 \alpha_2 \cdots \alpha_k$ is the disjoint cycle decomposition of σ, with each $\alpha_i \neq 1$, then $p = |\sigma| = \mathrm{lcm}(|\alpha_1|, \ldots, |\alpha_k|)$. Since p is prime, and $|\alpha_i| \neq 1$, we have $|\alpha_i| = p$, for each i. Since the order of a cycle is its length, each α_i is a p–cycle.

(b) **TRUE** This holds because any subgroup of a cyclic group is cyclic.

(c) **FALSE** Suppose $\mathbb{Z} \times \mathbb{Z} = \langle (a, b) \rangle$, for some $a, b \in \mathbb{Z}$. Then, for any $(n, m) \in \mathbb{Z} \times \mathbb{Z}$ we have $(n, m) = c(a, b) = (ca, cb)$ for some $c \in \mathbb{Z}$. So $(1, 0) = (ca, cb)$ for some c. Since $ca = 1$, we have $c \neq 0$, and since $cb = 0$, and $c \neq 0$, we have $b = 0$. But then $(3, 2) \notin \langle (a, b) \rangle = \mathbb{Z} \times \mathbb{Z}$, which is a contradiciton.
(Alternatively: The multiples of any non-zero (a, b) all lie on one line in the plane, and therefore cannot include every element of $\mathbb{Z} \times \mathbb{Z}$.)

(d) **FALSE** Let $G = S_3$, $H = \{1, (12)\}$, $a = (13)$ and $b = (123)$. Then $Ha \neq Hb$, but $aH = bH$.

(e) **FALSE** Let $G = D_8 \subset S_4$, let $K = \{1, (12)(34), (13)(24), (14)(23)\}$, and $H = \{1, (12)(34)\}$. Then $|G : K| = 2 = |K : H|$, so $K \lhd G$ and $H \lhd K$, but $(13)[(12)(34](13) = (14)(23) \notin H$, so H is not normal in G.

□

6. **(20 points)** Consider the following three groups, G, H, and K.

i) Let $a, b \in \mathbb{R}$, and define $T_{a,b} : \mathbb{R} \to \mathbb{R}$ be defined by $T_{a,b}(x) = ax + b$. **You may assume $G = \{T_{a,b} | a, b \in \mathbb{R}, a \neq 0\}$ is a group with the operation of composition of functions, and with $T_{1,0}$ as the identity element.**

ii) Let

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R}, a \neq 0 \right\}.$$

Then H is a subgroup of $GL_2(\mathbb{R})$, i.e. a group with the operation of matrix multiplication.

iii) Let $K = \mathbb{R}^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$. Then K is a group with the operation of vector addition.

For each pair of groups among G, H, and K determine whether or not they are isomorphic groups.

bigskip

Solution:

Note, K is abelian.

$$(T_{a,b}T_{c,d})(x) = T_{a,b}(T_{c,d}(x)) = T_{a,b}(cx+d) = a(cx+d)+b = acx+(ad+b) = T_{ac,ad+b}(x)$$

so

$$T_{a,b}T_{c,d} = T_{ac,ad+b}. \tag{1}$$

Then $T_{c,d}T_{a,b} = T_{ca,cb+d} \neq T_{a,b}T_{c,d}$. (For example, $T_{1,2}T_{2,3} = T_{2,5} \neq T_{2,7} = T_{2,3}T_{1,2}$.) So G is non-abelian. So $G \not\simeq K$.

Also

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix}. \tag{2}$$

and

$$\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ca & cb+d \\ 0 & 1 \end{pmatrix} \tag{3}$$

So H is non-abelian. Thus $H \not\simeq K$.

Let $\varphi : G \to H$ be given by $\varphi(T_{a,b}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Computations (1) and (2), above, show φ is a homomorphism. Note $\varphi(T_{a,b}) = \varphi(T_{c,d})$ if and only if

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$$

or if and only if $(a,b) = (c,d)$, which occurs if and only if $T_{a,b} = T_{c,d}$. So φ is injective. Note if $h \in H$, and $h = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \varphi(T_{a,b})$ so φ is surjective. Thus, $G \simeq H$. $\square$