REDUCIBILITY OF INDUCED REPRESENTATIONS FOR $SP(2N)$ AND $SO(N)$

By David Goldberg

Introduction. Let G be a connected reductive p-adic group. Let $P = MN$ be a parabolic subgroup of G, and let σ be an irreducible admissible representation of M. One important step in the classification of the irreducible admissible representations of G, is to decompose the unitarily induced representation $\mathrm{Ind}_P^G(\sigma)$ when σ is supercuspidal. If we assume that the discrete series of G and all M are known, then decomposing the $\mathrm{Ind}_P^G(\sigma)$, with σ a discrete series representation, classifies the tempered spectrum of G. As a first approach we ask the following question: Can we determine when $\mathrm{Ind}_P^G(\sigma)$ is irreducible?

Let A be the split component of M, and suppose σ is a discrete series representation of M. Then Bruhat theory reduces us to the case where σ is fixed by some nontrivial element of the Weyl group, $W(A)$. In the case of real groups, with P a minimal parabolic subgroup, and σ a unitary character of M, Knapp and Stein [21] studied a collection of standard intertwining operators. They used these operators to give a sufficient condition for $\mathrm{Ind}_P^G(\sigma)$ to be reducible. Harish-Chandra proved that these operators spanned the commuting algebra of $\mathrm{Ind}_P^G(\sigma)$. Knapp and Stein, [17,18,22], were able to construct a finite group, R, whose structure completely determines the number of components of $\mathrm{Ind}_P^G(\sigma)$. They called this group the R-group attached to σ. We sometimes denote R by $R(\sigma)$. That this construction generalizes to arbitrary parabolic subgroups is due to Knapp [19]. Silberger, [36], showed that, for p-adic groups, we can construct R-groups in a similar fashion, and they have the same significance. The R-group determines the structure of the commuting algebra $C(\sigma)$ of $\mathrm{Ind}_P^G(\sigma)$ [16, 36].

If $G = GL(n)$ over a p-adic field, the work of Ol'šanskiĭ, [26], Bernstein and Zelevinsky, [1], and Jacquet, [14], shows that R is trivial. Based on the results of Winarsky, [39], Keys, [15], was able to compute R-groups when G is a p-adic Chevalley group, and P is a minimal parabolic subgroup of G. We will study the R-groups for intermediate parabolic subgroups for certain classical groups. To compute R-groups explicitly, we must know when $\mathrm{Ind}_P^G(\sigma)$ is reducible for certain maximal parabolic subgroups P, and certain classical groups G. From this data the computation of R-groups is purely combinatorial in nature. We often

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assume that we know some of these criteria explicitly, in order to show what possible R-groups might arise. When G is quasi-split and P is maximal, Shahidi, [33], has related the reducibility of $\mathrm{Ind}_P^G(\sigma)$ to special values of certain conjectural Langlands L-functions attached to σ. His recent work, [31,35], has significantly broadened the cases for which we can compute these special values.

We study the split groups $SO(2n+1)$, $Sp(2n)$, and $SO(2n)$. These groups have the advantage that their Levi subgroups are easily computed. If $P = MN \subset G$ is a parabolic subgroup, then, there are integers $m_1 > m_2 \ldots > m_r > 0$, $m' \geq 0$, and $n_i \geq 0$, with $(\sum m_i n_i) + m' = n$, such that

$$M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_r)^{n_r} \times G'(m'). \tag{1}$$

Here

$$G'(m') = \begin{cases} SO(2m'+1) & \text{if } G = SO(2n+1), \\ Sp(2m') & \text{if } G = Sp(2n), \\ SO(2m') & \text{if } G = SO(2n). \end{cases}$$

We call a parabolic subgroup P of G *basic* if $r = 1$ in the decomposition (1). If σ is a discrete series representation of M, then, by (1), $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho$, where each σ_i is a discrete series representation of $GL(m_i)^{n_i}$, and ρ is a discrete series representation of $G'(m')$.

Our first step is to show that, for $G = SO(2n+1)$ or $Sp(2n)$, every R-group can be decomposed as a product

$$R \simeq R_1 \times \cdots \times R_r. \tag{2}$$

Here R_i is the R-group of the representation $\sigma_i \otimes \rho$, with respect to induction from the basic parabolic subgroup with Levi component

$$GL(m_i)^{n_i} \times G'(m') \subset \begin{cases} SO(2(m_i n_i + m') + 1) \\ Sp(2(m_i n_i + m')). \end{cases} \tag{3}$$

For $G = SO(2n)$, the R-group does not always decompose according to (2). However, if all of the integers m_i are even, then (2) and (3) are valid with $G'(m') = SO(2m')$. We prove that we can find integers, k_1 and k_2, parabolic subgroups, $P_i = M_i N_i$ of $SO(2k_i)$, and irreducible discrete series representations, τ_i of M_i, such that

$$R(\sigma) = R(\tau_1) \times R(\tau_2).$$

Moreover, in the decomposition of M_1 given by (1), all the integers m_i are even. Therefore, $R(\tau_1)$ further decomposes as a product of the form (2), with each R_i an R-group attached to a basic parabolic subgroup. In the decomposition (1) for

M_2, all the integers m_i are odd. In some cases the R-group $R(\tau_2)$ does decompose according to (2).

We further reduce the computation of each possible R-group. We show that for any basic parabolic subgroup of $SO(2n+1)$ or $Sp(2n)$, the R-group is of the form $R \simeq \mathbb{Z}_2^d$. Assuming we understand reducibility criteria for maximal parabolic subgroups, we can compute the integer d explicitly. If $G = SO(2n)$, and $M \simeq GL(m_1)^{n_1} \times SO(2m')$, with m_1 even, then the same statements are true. Therefore, we can determine the R-groups for parabolic subgroups of the form $P = M_1N_1$, with M_1 as above. For the case $M = M_2$, we can show $R \simeq \mathbb{Z}_2^d$. This is carried out in two cases, according to whether (2) is valid or not. If $R(\tau_2)$ does not decompose by (2), then we can compute the integer d explicitly from the structure of the inducing representation. If the inducing representation σ is *generic*, i.e., possesses a Whittaker model, [28], then we obtain the following theorem.

THEOREM. *Let $G = SO(2n+1)$, $Sp(2n)$, or $SO(2n)$, and let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let σ be any irreducible generic discrete series representation of M. Then* $\mathrm{Ind}_P^G(\sigma)$ *decomposes with multiplicity one.*

Remark. One can remove the condition that σ is generic. This result is due to Herb [10]. □

Once we have reduced ourselves to resolving the case of maximal parabolic subgroups, we discuss some maximal parabolic subgroups for which the reducibility criteria are explicitly known. Suppose G is one of the split groups $SO(2k+1)$, $Sp(2k)$, or $SO(2k)$ with k odd. Further suppose that $M \simeq GL(k)$, and σ an irreducible unitary supercuspidal representation of M. Then the criteria for reducibility can be easily described [35].

In order to better describe the problems we intend to discuss, we need to introduce some notation and vocabulary. This is the subject of Section 1. In Section 2, we discuss the structure of the parabolic subgroups of the split classical groups we will consider. In Section 3, we review some results for the group $GL(n)$. This will be a a key ingredient in many of our arguments. In Section 4, we prove a product formula for the R-groups of $Sp(2n)$ and $SO(2n+1)$. This reduces the computation of R-groups to the case where P is a basic parabolic subgroup. These R-groups are computed in Section 6. In Section 5, we reduce computation of R-groups for $SO(2n)$ to four cases. Two of these cases involve basic parabolic subgroups, and the other two cases are slightly different. These four cases are computed in Section 6. In Section 7, we discuss some cases where the reducibility criteria for maximal parabolic subgroups is known.

We make a short remark about real groups. Our combinatorial argument for p-adic groups uses the theorems of Ol'šanskiĭ, [26], Bernstein and Zelevinsky,

[1], and Jacquet, [14], for $GL(n)$. The results of Section 3 can be translated to $G = GL(n, \mathbb{R})$ as follows. Suppose $P = MAN$ is a maximal cuspidal parabolic subgroup of G, π a discrete series representation of M, and $\nu \in i\mathfrak{a}^*$. First note that in order for G to have such a parabolic subgroup, it is necessary that n be 2, 3, or 4. Then $MA \simeq GL(1) \times GL(1)$, $GL(1) \times GL(2)$, or $GL(2) \times GL(2)$ respectively. If $n = 3$, then $W(A) = \{1\}$, and thus, $\mathrm{Ind}_P^G(\pi \otimes e^{i\nu} \otimes 1_N)$ is irreducible. If $n = 4$, then $\mathrm{Ind}_P^G(\pi \otimes e^{i\nu} \otimes 1_N)$ is a unitary *fundamental series* representation, and is therefore irreducible [9, Theorem 40.1]. If $n = 2$, the explicit formula for Plancherel factors shows that, if (π, ν) is fixed by $W(A)$, then $\mu(\pi, \nu) = 0$. Therefore, $\mathrm{Ind}_P^G(\pi \otimes e^{i\nu} \otimes 1_N)$ is always irreducible in this case as well. Therefore, the analogue of Lemma 3.4 is valid for reductive groups over $\mathbb{R}$ as well. With this in hand, the combinatorial arguments of Sections 4, 5, and 6 carry over to the groups $SO(2n+1, \mathbb{R})$, $Sp(2n, \mathbb{R})$, and $SO(2n, \mathbb{R})$. To compute reducibility criteria for the maximal parabolic subgroups that arise, we can use the explicit Plancherel formula of Harish-Chandra [11]. These Plancherel factors are explicitly computed by Knapp [20]. This gives a method of explicitly computing R-groups for these split groups which is a bit different than the one given in [20].

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1. Preliminaries. Let F be a locally compact, non discrete, nonarchimedean local field of characteristic zero. Let $\mathcal{R}$ be its ring of integers, $\mathfrak{p}$ the maximal ideal in $\mathcal{R}$. Let $q = |\mathcal{R}/\mathfrak{p}|$ be the order of the residue class field of F. Then $q = p^k$, for some prime p, and some $k > 0$. The prime p is called the residual characteristic of F.

Let G be the F-rational points of a connected reductive quasi-split algebraic group over F. By a representation of G on a complex vector space V, we mean a homomorphism $\pi : G \longrightarrow GL(V)$. A representation is *smooth* if, for every $v \in V$, $\{g \in G \mid \pi(g)v = v\}$ is open. A smooth representation is *admissible* if, for every open subgroup $H \subset G$, $\{v \mid \pi(h)v = v, \ \forall h \in H\}$ is finite dimensional.

Let (π, V) be a smooth representation of G, and let $(\widetilde{\pi}, \widetilde{V})$ be the smooth contragredient representation on the space of smooth linear forms on V. Then π is called *supercuspidal* if, for any $v \in V$ and $\tilde{v} \in \widetilde{V}$, the function $\varphi : g \mapsto \langle \pi(g)v, \tilde{v}\rangle$ has compact support modulo the center of G. An admissible representation (π, V), with unitary central character, is called *square integrable* if all its matrix coefficients are square integrable modulo $Z(G)$, i.e.,

$$\int_{G/Z(G)} |\varphi_{v,\tilde{v}}(\bar{g})|^2 \, d\bar{g} < \infty,$$

for all $v \in V, \tilde{v} \in \widetilde{V}$. The irreducible, square integrable representations are called the *discrete series* of G. It is clear that every admissible irreducible unitary supercuspidal representation is discrete series. For more details on the theory of admissible representations of G, one should consult [3,4,8,37].

Suppose that $\mathbf{G}$ is a connected reductive quasi-split algebraic group defined over F. A parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is a subgroup such that $\mathbf{G}/\mathbf{P}$ has the structure of a complete variety [2,12]. Every parabolic subgroup has a *Levi decomposition* $\mathbf{P} = \mathbf{MN}$, with $\mathbf{M}$ a reductive group, and $\mathbf{N}$ a unipotent subgroup of $\mathbf{G}$. The product is semidirect, i.e., $\mathbf{M}$ normalizes $\mathbf{N}$. Notice that $\mathbf{G}$ is a parabolic subgroup of $\mathbf{G}$. By a *maximal* parabolic subgroup, we mean a parabolic subgroup which is maximal among proper parabolic subgroups.

Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$. Denote the maximal F-split sub-torus of $\mathbf{T}$ by $\mathbf{A}_0$. Let $\mathbf{\Phi}$ be the set of restricted roots of $\mathbf{G}$ with respect to $\mathbf{A}_0$. Suppose Δ is a choice of simple roots. Let $\mathbf{\Phi}^+$ be the positive roots, $\mathbf{\Phi}^-$ the negative roots, and $\mathbf{B} = \mathbf{TU}$ the Borel subgroup of $\mathbf{G}$ with respect to this choice of simple roots [2,37,38]. For each $\alpha \in \mathbf{\Phi}$, let χ_α be the associated root character. Let $\alpha \in \mathbf{\Phi}^+$, and let $\mathbf{U}_\alpha \subset \mathbf{U}$ be the associated root subgroup. Recall that if $\mathfrak{g}$ is the Lie algebra of $\mathbf{G}$, then $\mathbf{U}_\alpha$ is the subgroup of $\mathbf{U}$ with Lie algebra $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$. A standard parabolic subgroup, $\mathbf{P} \subset \mathbf{G}$, is any subgroup containing $\mathbf{B}$. The standard parabolic subgroups are in one to one correspondence with the subsets of the simple roots Δ. This correspondence can be described as follows [37, pg. 10]. Let $\theta \subset \Delta$, and let $\Sigma^+(\theta)$ be the positive roots spanned by θ. Let $\mathbf{A}_\theta \subset \mathbf{A}_0$ be defined by

$$\mathbf{A}_\theta = \left(\bigcap_{\alpha \in \theta} \ker \chi_\alpha \right)^\circ ,$$

where the superscript $^\circ$ means the connected component of the identity. Let $\mathbf{M}_\theta$ be the centralizer of $\mathbf{A}_\theta$ in $\mathbf{G}$, and let

$$\mathbf{N}_\theta = \prod_{\alpha \in \mathbf{\Phi}^+ \setminus \Sigma^+(\theta)} \mathbf{U}_\alpha .$$

Then $\mathbf{P}_\theta = \mathbf{M}_\theta \mathbf{N}_\theta$ is the parabolic subgroup corresponding to θ. Moreover, if $\theta_1 \subset \theta_2$, then $\mathbf{P}_{\theta_1} \subset \mathbf{P}_{\theta_2}$. If $\mathbf{P}$ is any parabolic subgroup of $\mathbf{G}$, then $\mathbf{P}$ is conjugate to $\mathbf{P}_\theta$, for some $\theta \subset \Delta$. If $G = \mathbf{G}(F)$, then we call $P = \mathbf{P}(F)$ a parabolic subgroup of G. Note that $P = MN$, with $M = \mathbf{M}(F)$, and $N = \mathbf{N}(F)$.

If $P = MN$ is a proper parabolic subgroup of G, then both M and N are unimodular, but P is not. Let δ_P be the modular function of a right Haar measure on P. Let (σ, V) be a smooth representation of M. Let

$$V(\sigma) = \left\{ f \in C^\infty(G, V) | f(mng) = \sigma(m) \delta_P^{1/2}(m) f(g), \ \forall\, g \in G, m \in M, n \in N \right\}.$$

Then G acts on $V(\sigma)$ by right translations, and this is the *unitarily induced representation* $\operatorname{Ind}_P^G(\sigma)$. If σ is unitary, then so is $\operatorname{Ind}_P^G(\sigma)$.

Following Harish-Chandra, [8, pp. 170, 172, 175], we let $\mathcal{E}_c(G)$ be the equivalence classes of irreducible admissible representations of G. We make no distinction between a class $[\sigma] \in \mathcal{E}_c(G)$, and its representative σ. We let $\mathcal{E}(G) \subset \mathcal{E}_c(G)$ be the irreducible unitary equivalence classes, and let $\mathcal{E}_2(G) \subset \mathcal{E}(G)$ be the square integrable classes. Denote by $\,^\circ\mathcal{E}_c(G)$ the irreducible supercuspidal classes in $\mathcal{E}_c(G)$, and let $^\circ\mathcal{E}(G) = \,^\circ\mathcal{E}_c(G) \cap \mathcal{E}(G)$. Jacquet's theorem, [4,8,13], shows that any $\pi \in \mathcal{E}_c(G)$ is either supercuspidal, or can be realized as a subrepresentation of $\operatorname{Ind}_P^G(\sigma)$, for some parabolic subgroup $P = MN$, and some $\sigma \in \,^\circ\mathcal{E}_c(M)$.

Let $\theta \subset \Delta$, and let $P = P_\theta$. Suppose A_θ is the split component of P. Consider the Weyl group

$$\begin{aligned} W_\theta &= W(G/A_\theta) \subset W(G/A_0), \\ W_\theta &= N_G(A_\theta)/Z_G(A_\theta) = N_G(A_\theta)/M_\theta. \end{aligned}$$

There is a natural action of W_θ on the representations of M_θ. If $\sigma \in \mathcal{E}_c(M_\theta)$, and $w \in W_\theta$, then we let $\sigma^w(m) = \sigma(w^{-1}mw)$. We say that σ is *singular*, or *ramified*, if there is some $w \in W_\theta$, $w \neq 1$, such that $\sigma^w \simeq \sigma$. Otherwise, σ is said to be *nonsingular*, or *unramified*. Let $W(\sigma) = \{w \in W_\theta \mid \sigma^w \simeq \sigma\}$. Let $C(\sigma)$ be the commuting algebra of the representation $\operatorname{Ind}_P^G(\sigma)$.

Theorem 1.1. (Bruhat) [8, pg. 177]. *Let $P = MN$ be a proper parabolic subgroup of G. Let $\sigma \in \mathcal{E}_2(M)$. Then* $\dim_{\mathbb{C}} C(\sigma) \leq |W(\sigma)|$.

Corollary 1.2. *If $\sigma \in \mathcal{E}_2(M)$ is unramified, then* $\operatorname{Ind}_P^G(\sigma)$ *is irreducible.*

Thus, if σ is a discrete series representation, the study of reducibility of the induced representations reduces to the ramified case. The method of study is to construct, for any admissible irreducible representation, σ of M_θ, and any $w \in W_\theta$, a standard intertwining operator between $\operatorname{Ind}_{P_\theta}^G(\sigma)$ and $\operatorname{Ind}_{P_\theta}^G(\sigma^w)$, and use the behavior of these operators to decompose $\operatorname{Ind}_P^G(\sigma)$.

Let $X(M)_F$ be the F-rational characters of M. Let

$$\begin{aligned} \mathfrak{a}_\theta &= \operatorname{Hom}(X(M)_F, \mathbb{R}), \\ \mathfrak{a}_\theta^* &= X(M)_F \otimes_{\mathbb{Z}} \mathbb{R}, \end{aligned}$$

and

$$(\mathfrak{a}_\theta)_{\mathbb{C}}^* = \mathfrak{a}_\theta^* \otimes_{\mathbb{R}} \mathbb{C}.$$

Then $\mathfrak{a}_\theta$ is the real Lie algebra of A_θ, and $(\mathfrak{a}_\theta)_{\mathbb{C}}^*$ is its complexified dual. There is a homomorphism, [8], $H_P : M \longrightarrow \mathfrak{a}_\theta$, such that

$$q^{\langle \chi, H_P(m) \rangle} = |\chi(m)|_F, \ \forall \chi \in X(M)_F.$$

Let $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$. Consider the representation $I(\nu, \sigma) = \mathrm{Ind}_P^G \left(\sigma \otimes q^{\langle \nu, H_P(-) \rangle} \right)$. Let $w \in W_\theta$, and let $N_w = U \cap w^{-1}\overline{N}w$, where $\overline{N}$ is the unipotent radical opposed to N. Let $f \in I(\nu, \sigma)$, and $g \in G$. Consider the operator defined (only formally) by

$$A(\nu, \sigma, w)f(g) = \int_{N_w} f(w^{-1}ng)\, dn. \tag{1.1}$$

We say that $A(\nu, \sigma, w)$ converges if, for every $\tilde{v} \in \widetilde{V}$, and every $f \in I(\nu, \sigma)$, $\langle \tilde{v}, A(\nu, \sigma, w)f(g) \rangle$ converges for all $g \in G$.

LEMMA 1.3. [30,32,39] *If $A(\nu, \sigma, w)$ converges then it defines an intertwining operator between $I(\nu, \sigma)$ and $I(\nu^w, \sigma^w)$.*

THEOREM 1.4. (Harish-Chandra [32]) *If $\sigma \in \mathcal{E}_2(M)$, then $A(\nu, \sigma, w)$ converges for Re $\nu >> 0$. Moreover, $\nu \mapsto A(\nu, \sigma, w)$ is holomorphic as an operator valued function, and has a meromorphic continuation to all of $(\mathfrak{a}_\theta)^*_{\mathbb{C}}$.*

Suppose $\theta \subset \Delta$. We let ρ_θ be half the sum of the positive roots in N_θ. If $P = P_\theta$, then we may denote ρ_θ bt ρ_P.

THEOREM 1.5. (Harish-Chandra [8, pg. 182]) *Let $w \in W_\theta$, and let $\sigma \in \mathcal{E}_2(M)$. Then there is a complex number, $\mu(\nu, \sigma, w)$, so that, if Haar measure is chosen appropriately,*

$$A(\nu, \sigma, w)A(\nu^w, \sigma^w, w^{-1}) = \mu(\nu, \sigma, w)^{-1}\gamma_w^2(G/P),$$

where

$$\gamma_w(G/P) = \int_{\overline{N}_w} q^{\langle 2\rho_P, H_P(\bar{n}) \rangle}\, d\bar{n}.$$

Moreover, $\nu \mapsto \mu(\nu, \sigma, w)$ is meromorphic on $(\mathfrak{a}_\theta)^*_C$, and holomorphic and nonnegative on the unitary axis $i\mathfrak{a}^*_\theta$.

We call $\mu(\nu, \sigma, w)$ is called the *Plancherel measure* associated to σ, ν, w. If w is the longest element of the Weyl group W_θ, then we write $\mu(\nu, \sigma)$ for $\mu(\nu, \sigma, w)$. (The longest element of W_θ is the element which takes every member of $\Delta \setminus \theta$ to a negative root.) We write $\mu(\sigma)$ for $\mu(0, \sigma)$.

If $w \in W(\sigma)$, $w \neq 1$, and $A(\nu, \sigma, w)$ is holomorphic in a neighborhood of 0, then $I(\sigma)$ is reducible. Moreover, if $A(\nu, \sigma, w)$ is a rank one operator then the converse holds [37, Corollary 5.4.2.3]. Notice that the poles of the intertwining operators must match with the zeros of the Plancherel measures.

THEOREM 1.6. (Harish-Chandra [37, theorem 5.5.3.2.]) *Let* $\sigma \in \mathcal{E}_2(M)$. *There is a meromorphic normalizing factor,* $\gamma(\nu, \sigma, w)$, *so that, if we let* $\mathfrak{A}(\sigma, w) = \gamma(0, \sigma, w)A(0, \sigma, w)$, *then the collection* $\{\mathfrak{A}(\sigma, w) \mid w \in W(\sigma)\}$ *spans the commuting algebra* $C(\sigma)$.

Shahidi, [33], has described normalizing factors in terms of certain Langlands L-functions and root numbers, and has used this normalization to give criteria for reducibility of induced representations coming from maximal parabolic subgroups. These criteria depend on the L-functions and root numbers. Thus, the criteria are not yet explicitly known in general. However, for certain groups, parabolic subgroups, and σ, these criteria have been computed [6,7,14,26,29,31,33–35,39]. If the reducibility criteria for maximal parabolic subgroups were known in general, then we could use the following product formula to detect the zeros of the Plancherel measures for all parabolic subgroups. The product formula is related to a cocycle relation for the composition of intertwining operators [15,32,39].

Let $P = MN$ be a standard parabolic subgroup of G, with split component A, and let $\Phi(P, A)$ be the set of reduced roots of P with respect to A. Notice that since we are taking the roots of A in P, there is already an ordering defined on $\Phi(P, A)$, i.e., the elements of $\Phi(P, A)$ are positive. Let $\beta \in \Phi(P, A)$, and let A_β be the torus $(\ker \chi_\beta \cap A)^\circ$. Let $M_\beta = Z_G(A_\beta)$. Note that $M_\beta \supset M$. Let ${}^*P_\beta = M_\beta \cap P$. Then ${}^*P_\beta$ is a maximal parabolic subgroup of M_β, with split component A, unipotent radical $N_\beta = N \cap M_\beta$, and Levi decomposition ${}^*P_\beta = MN_\beta$. Let σ and ν be defined as before, and let $\mu_\beta(\nu, \sigma)$ be the Plancherel measure associated to the representation $\operatorname{Ind}_{{}^*P_\beta}^{M_\beta}\left(\sigma \otimes q^{\langle \nu, H_P() \rangle}\right)$.

THEOREM 1.7. (Harish-Chandra [8, pp. 184–185])

$$\gamma^{-2}(G/P)\mu(\nu, \sigma) = \prod_{\beta \in \Phi(P,A)} \gamma_\beta^{-2}(M_\beta/{}^*P_\beta)\mu_\beta(\nu, \sigma).$$

Following Knapp and Stein, [22], we can use the zeros of the rank one Plancherel measures to compute the *Knapp-Stein R-group*. This gives us a method to determine which $\mathfrak{A}(\sigma, w)$ are scalar operators. Then, Theorem 1.6 allows us to compute the dimension of $C(\sigma)$. We now describe the construction of the R-group. Suppose $\sigma \in \mathcal{E}_2(M)$. Let $\Delta' = \{\beta \in \Phi(P, A) \mid \mu_\beta(\sigma) = 0\}$, and let $R = \{w \in W(\sigma) \mid w\beta > 0,\ \forall \beta \in \Delta'\}$. By [8, pg. 183], $\mu_{w\beta}(\sigma) = \mu_\beta(\sigma^{w^{-1}})$, so $R = \{w \in W(\sigma) \mid w(\Delta') = \Delta'\}$.

Let $W' = W_{\Delta'}$, i.e., W' is generated by the reflections in Δ' (which along with its negative form a sub-root system of Φ).

THEOREM 1.8. (Knapp-Stein, Silberger [22,23,36])

$$W(\sigma) = R \ltimes W', \text{ and } W' = \{w \in W(\sigma) \mid \mathfrak{A}(\sigma, w) \text{ is scalar}\}.$$

We recall the notion of a generic representation. The subgroup

$$U' = \prod_{\alpha \in \Phi^+ \setminus \Delta} U_\alpha,$$

is normal in U. Furthermore,

$$U/U' \simeq \prod_{\alpha \in \Delta} U_\alpha / U_{2\alpha}.$$

Let $\alpha \in \Delta$, and let ψ_α be a character of $U_\alpha/U_{2\alpha}$. The character ψ of U, trivial on U', and given by

$$\psi = \prod_{\alpha \in \Delta} \psi_\alpha,$$

is called *nondegenerate* if ψ_α is nontrivial for each α. An admissible representation, (π, V), of G is called *nondegenerate*, or *generic*, if there is a nondegenerate character ψ of U, and a linear functional λ on V, such that

$$\lambda(\pi(u)v) = \overline{\psi(u)}\lambda(v), \quad \forall u \in U, v \in V.$$

Such a functional is called a *Whittaker functional.* For details on generic representations and Whittaker functionals see [27,28,32].

THEOREM 1.9. [15,18,23,36] *Let $\sigma \in \mathcal{E}_2(M)$ be generic. Then* $\dim C(\sigma) = |R|$. *Furthermore, if σ is generic, then the following hold.*

(1) $C(\sigma) \simeq \mathbb{C}[R]$, *the complex group algebra of R.*

(2) *The number of inequivalent irreducible components of $I(\sigma)$ is given by the dimension of the center of $\mathbb{C}[R]$.*

(3) *If $\mathbb{C}[R] \simeq M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_k, \mathbb{C})$ is the decomposition of $\mathbb{C}[R]$ into a sum of simple algebras, then the numbers $n_1, \ldots, n_k$ are the multiplicities of the components of* $\mathrm{Ind}_P^G(\sigma)$.

(4) *R is abelian if and only if* $\mathrm{Ind}_P^G(\sigma)$ *decomposes simply, i.e., with multiplicity one.*

Remark. There are variants of (1) and (2) in general [16]. A conjecture of Shahidi [33, Conjecture 9.4] would imply (1)-(4) hold in general. □

Suppose we knew explicitly all of the reducibility criteria when P is a maximal parabolic subgroup. Then computing R-groups for arbitrary parabolic subgroups becomes a purely combinatorial. This yields information on the possible R-groups that can arise. We will describe these combinatorics for the classical groups $SO(2n+1,F)$, $Sp(2n,F)$, and $SO(2n,F)$.

2. Parabolic Subgroups of $Sp(2n)$ and $SO(n)$. Let $G = SO(2n+1,F)$, $Sp(2n,F)$, or $SO(2n,F)$. Since we will only be concerned with the F-rational points of the underlying algebraic groups, we will denote these groups by $SO(2n+1)$, $Sp(2n)$, and $SO(2n)$. Let J_n be the $n \times n$ matrix given by

$$J_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix}.$$

Let $J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$. Then

$$Sp(2n) = \left\{ g \in GL(2n) \mid {}^t g J'_{2n} g = J'_{2n} \right\},$$

and

$$SO(n) = \left\{ g \in GL(n) \mid {}^t g J_n g = J_n; det(g) = 1 \right\}.$$

In each case, we let A_0 be the maximal split torus consisting of diagonal matrices in G. Then

$$A_0 = \left\{ \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & \lambda_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2^{-1} & \\ & & & & & & & \lambda_1^{-1} \end{pmatrix} \middle| \lambda_i \in F^* \right\},$$

if $G = Sp(2n)$ or $SO(2n)$, and

$$A_0 = \left\{ \begin{pmatrix} \lambda_1 & & & & & & & & \\ & \lambda_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & \lambda_n & & & & & \\ & & & & 1 & & & & \\ & & & & & \lambda_n^{-1} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_2^{-1} & \\ & & & & & & & & \lambda_1^{-1} \end{pmatrix} \;\middle|\; \lambda_i \in F^* \right\},$$

if $G = SO(2n+1)$.

Notational Remark. We often denote a block diagonal matrix,

$$\begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_k \end{pmatrix},$$

by $\operatorname{diag}\{X_1, \ldots, X_k\}$. For a block scalar matrix,

$$\begin{pmatrix} \lambda_1 I_{k_1} & & \\ & \ddots & \\ & & \lambda_j I_{k_j} \end{pmatrix},$$

we write $\operatorname{diag}\{\lambda_1, \ldots, \lambda_j\}$ if the dimensions k_i are clearly understood. □

Let $\Phi(G, A_0)$ be the roots of G with respect to A_0. We choose the ordering on the roots so that the Borel subgroup is the subgroup of upper triangular matrices in G. Let Δ be the simple roots in $\Phi(G, A_0)$ given by $\Delta = \{\alpha_j\}_{j=1}^{n}$, with $\alpha_j = e_j - e_{j+1}$ for $1 \leq j \leq n-1$, and

$$\alpha_n = \begin{cases} e_n & \text{if } G = SO(2n+1), \\ 2e_n & \text{if } G = Sp(2n), \\ e_{n-1} + e_n & \text{if } G = SO(2n). \end{cases}$$

We let $\langle\, , \,\rangle$ be the standard Euclidean inner product on $\Phi(G, A_0)$. If Φ is a root system of type B_n, C_n, or D_n, then we denote by $G(\Phi)$ the split group with root system Φ.

For $G = SO(2n+1)$ or $Sp(2n)$, the Weyl group $W(G/A_0) \simeq S_n \ltimes \mathbb{Z}_2^n$. Here S_n acts by permutations on the λ_i, $i = 1, \ldots, n$. We will use standard cycle notation for the elements of S_n. Thus, (ij) interchanges λ_i and λ_j. If c_i is the nontrivial element in the ith copy of $\mathbb{Z}_2$, then c_i takes λ_i to λ_i^{-1}. The element c_i is called a *sign change* because its action on $\Phi(G,A_0)$ takes e_i to $-e_i$. For $G = SO(2n)$, the Weyl group is given by $W(G/A_0) \simeq S_n \ltimes Z_2^{n-1}$. In this case, S_n acts by permutations on the λ_i, and $\mathbb{Z}_2^{n-1}$ acts by even numbers of sign changes. The requirement that the number of sign changes be even comes from the determinant condition in $SO(2n)$. Note that the sign change c_i is an element of $O(2n) = \{g \mid {}^t g J_{2n} g = J_{2n}\}$, and normalizes A_0. Each c_i acts on $SO(2n)$ by conjugation, and c_n induces the nontrivial graph automorphism on the Dynkin diagram of $\Phi(G,A_0)$.

Suppose $\theta \subset \Delta$. We let $\theta = \theta_1 \cup \cdots \cup \theta_k$, be the decomposition of θ as a disjoint union of connected components of the Dynkin diagram. However, if $G = SO(2n)$, and $\alpha_{n-1}, \alpha_n \in \theta$, but $\alpha_{n-2} \notin \theta$, then we assume that $\theta_k = \{\alpha_{n-1}, \alpha_n\}$, even though this is not connected. Therefore, we say that $\theta = \theta_1 \cup \cdots \cup \theta_k$ is the *component decomposition*, and hope that this will cause no confusion. If θ_i and θ_j are components of θ, then we write $\theta_i \sim \theta_j$ if, there is a $w \in W_\theta = W(G/A_\theta)$ so that $w\theta_i = \theta_j$.

If $w \in W(G/A_0)$ then, by [8, theorem 20], $\mu_\alpha(\sigma) = \mu_{w(\alpha)}(\sigma^w)$, for any $\alpha \in \Phi(P_\theta, A_\theta)$. This is because conjugating by an element of $W(G/A_0)$ does not change the associate class of P_θ, or the structure of the reduced root system $\Phi(P_\theta, A_\theta)$. Thus, the R-groups for P_θ and $P_{w\theta}$ are isomorphic. Therefore, we often assume that θ has a form which is particularly well suited to the computation we are considering.

We will usually assume that

(1) If $\alpha_n \in \theta$, then $\alpha_n \in \theta_k$;

(2) If $i < j$ with $\theta_i \sim \theta_j$, then $\theta_i \sim \theta_l$ for all $i \leq l \leq j$.

If $G = SO(2n)$, then it is no loss of generality (for computing R-groups) to assume that if $\alpha_n \in \theta$, then $\alpha_{n-1} \in \theta$. With this convention, the structure of those parabolic subgroups parameterized by subsets θ for which $\alpha_n \notin \theta$, is slightly different than those parameterized by subsets θ with $\alpha_n \in \theta$. Therefore, we adopt the convention that "case 1" refers to the case where $\alpha_n \notin \theta$, and "case 2" means $\alpha_n \in \theta$. We emphasize that, for $G = SO(2n)$, case 2 means $\alpha_n, \alpha_{n-1} \in \theta$.

Let $\theta' = \pm\{\alpha \in \Phi(P, A_\theta) \mid \langle \alpha, \beta \rangle = 0 \ \forall \beta \in \theta\}$. Note that if $\alpha \in \theta'$, $\beta \in \theta$, and $w \in W_\theta$, then

$$\langle w\alpha, \beta \rangle = \langle \alpha, w^{-1}\beta \rangle = 0,$$

and hence $w\alpha \in \theta'$.

Let $X = \{\theta_i\}_{i=1}^k$ and $X(\theta_i) = \{\theta_j \mid \theta_i \sim \theta_j\}$. Then either $\pm X(\theta_i)$ or $X(\theta_i)$ is the W_θ orbit of θ_i. If $\alpha_n \notin \theta_i$ then $\pm X(\theta_i) = W_\theta(\theta_i)$. This orbit structure determines a partition $X = X_1 \cup \cdots \cup X_r$ of X, with each $X_i = X(\theta_j)$, for some j.

In case 1, we let $m_i = |\theta_j| + 1$ for each $\theta_j \in X_i$. In case 2, we let m_i be defined as above for $i \neq r$, and let $m_r = |\theta_k|$. In case 2, we often write $m = m_r$. Let

$n_i = |X_i|$. That is, $X_i = \{\theta_{i1}, \ldots, \theta_{in_i}\}$. Sometimes we choose to think of X_i as the collection of roots in it elements. That is, we sometimes think of $X_i = \cup_{j=1}^{n_i} \theta_{ij}$, as opposed to $X_i = \{\theta_{ij}\}_{j=1}^{n_i}$. We will always try to indicate in what manner we are regarding X_i.

Let $b = \sum_{i=1}^{r} m_i n_i$. In case 2, we also let $b' = \sum_{i=1}^{r-1} m_i n_i$.

We usually assume that θ is of one of the following two forms:

$$\begin{aligned} \text{(2.1)} \quad \theta = &\{e_1 - e_2, \ldots, e_{m_1-1} - e_{m_1}\} \cup \\ &\{e_{m_1+1} - e_{m_1+2} \ldots, e_{2m_1-1} - e_{2m_1}\} \cup \cdots \cup \{\ldots e_{b-1} - e_b\}, \end{aligned}$$

or

$$\begin{aligned} \text{(2.2)} \quad \theta = &\{e_1 - e_2, \ldots, e_{m_1-1} - e_{m_1}\} \cup \{e_{m_1+1} - e_{m_1+2} \ldots\} \cup \\ &\cdots \cup \{\ldots e_{b'-1} - e_{b'}\} \cup \{e_{n-m+1} - e_{n-m+2}, \ldots, e_{n-1} - e_n, \alpha_n\}. \end{aligned}$$

That is, we assume that there are no gaps between the components of θ, except possibly between θ_{k-1} and θ_k in case 2.

Suppose θ is of the form (2.2). Then $A_\theta =$

$$\begin{aligned} \text{(2.3a)} \quad &\Big\{\operatorname{diag}\Big\{\lambda_{11} I_{m_1}, \ldots, \lambda_{rn_r} I_{m_r}, \lambda_{b+1}, \ldots, \lambda_n, \lambda_n^{-1}, \\ &\ldots, \lambda_{b+1}^{-1}, \lambda_{rn_r}^{-1} I_{m_r}, \ldots, \lambda_{11}^{-1} I_{m_1}\Big\}\Big\}, \end{aligned}$$

if $G = Sp(2n)$ or $SO(2n)$, and

$$\begin{aligned} \text{(2.3b)} \quad A_\theta = &\Big\{\operatorname{diag}\Big\{\lambda_{11}, \ldots, \lambda_{rn_r}, \lambda_{b+1}, \ldots, \lambda_n, 1, \lambda_n^{-1}, \\ &\ldots, \lambda_{b+1}^{-1}, \lambda_{rn_r}^{-1} \ldots, \lambda_{11}^{-1}\Big\}\Big\}, \end{aligned}$$

if $G = SO(2n+1)$. Here λ_{ij} means $\lambda_{ij} I_{m_i}$.

If θ is of the form (2.2), then

$$\begin{aligned} \text{(2.4a)} \quad A_\theta = &\Big\{\operatorname{diag}\Big\{\lambda_{11}, \ldots, \lambda_{(r-1)n_{r-1}}, \lambda_{b'+1}, \ldots, \lambda_{n-m}, I_{2m}, \\ &\lambda_{n-m}^{-1}, \ldots, \lambda_{b'+1}^{-1}, \lambda_{(r-1)n_{r-1}}^{-1}, \ldots, \lambda_{11}^{-1}\Big\}\Big\}, \end{aligned}$$

if $G = Sp(2n)$ or $SO(2n)$, and

$$\begin{aligned} \text{(2.4b)} \quad A_\theta = &\Big\{\operatorname{diag}\Big\{\lambda_{11}, \ldots, \lambda_{(r-1)n_{r-1}}, \lambda_{b'+1}, \ldots, \lambda_{n-m}, I_{2m+1}, \\ &\lambda_{n-m}^{-1}, \ldots, \lambda_{b'+1}^{-1}, \lambda_{(r-1)n_{r-1}}^{-1}, \ldots, \lambda_{11}^{-1}\Big\}\Big\}, \end{aligned}$$

if $G = SO(2n+1)$.

We let $n_{r+1} = n - b$. Note that, in case 2, $n_{r+1} = n - m - b'$.

In (2.3a,b) we now label $\lambda_{b+1},\dots,\lambda_n$ as $\lambda_{(r+1)1},\dots,\lambda_{(r+1)n_{r+1}}$. Similarly, in (2.4a,b) we relabel $\lambda_{b'+1},\dots,\lambda_{n-m}$ as $\lambda_{(r+1)1},\dots,\lambda_{(r+1)n_{r+1}}$.

LEMMA 2.1.

(1) *In case* 1,

$$M_\theta \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_r)^{n_r} \times GL(1)^{n_{r+1}}.$$

(2) *In case* 2,

$$M_\theta \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_{r-1})^{n_{r-1}} \times GL(1)^{n_{r+1}} \times G(\Phi_m),$$

where $\Phi_m = \begin{cases} B_m & \text{if } G = SO(2n+1), \\ C_m & \text{if } G = Sp(2n), \\ D_m & \text{if } G = SO(2n). \end{cases}$

Proof. We prove the lemma for $G = Sp(2n)$. The other cases are similar.

(1) We assume that θ has the form (2.1), and so A_θ has the form (2.3a). Let $g \in GL(2n)$. Then g centralizes A_θ if and only if

$$g = \mathrm{diag}\{g_{11}, g_{12}, \dots, g_{r+1n_{r+1}}, h_{r+1n_{r+1}}, \dots, h_{12}, h_{11}\},$$

with each $g_{ij}, h_{ij} \in GL(m_i)$. Moreover, $g \in G$, if and only if ${}^tgJ'_{2n}g = J'_{2n}$. This is equivalent to ${}^tg_{ij}J_{m_i}h_{ij} = J_{m_i}$ for each i and j. Therefore, $g \in G$ if and only if $h_{ij} = J_{m_i}{}^tg_{ij}^{-1}J_{m_i} = {}^\tau g_{ij}^{-1}$, where τ indicates transposition with respect to the off diagonal. Thus, M_θ is as claimed.

(2) We assume that θ is of the form (2.2) and thus, A_θ is of the form (2.4a). Then g centralizes A_θ if and only if $g = \mathrm{diag}\,\{D, Q, D'\}$, with

$$\begin{aligned} D &= \mathrm{diag}\,\{g_{11}, \dots, g_{(r-1)n_{r-1}}, g_{(r+1)1}, \dots, g_{(r+1)n_{r+1}}\}, \\ D' &= \mathrm{diag}\,\{h_{(r+1)n_{r+1}}, \dots, h_{(r+1)1}, h_{(r-1)n_{r-1}}, \dots, h_{11}\}, \end{aligned}$$

where each $g_{ij}, h_{ij} \in GL(m_i)$, and $Q \in GL(2m)$. Note that g centralizes J'_{2n} if and only if $h_{ij} = {}^\tau g_{ij}^{-1}$, $\forall i,j$ and ${}^tQJ'_{2m}Q = J'_{2m}$.

□

Another way to state the lemma is that if $G = SO(2n+1)$, $Sp(2n)$, or $SO(2n)$, and $P = MN \subset G$ is a parabolic subgroup of G, then there are positive integers $m_1, \dots, m_t, n_1, \dots, n_t$, and an $m \geq 0$, with $(\sum m_i n_i) + m = n$, such that

$$M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_t)^{n_t} \times G(\Phi_m). \tag{2.5}$$

Definition 2.2. Let $G = SO(2n+1)$, $Sp(2n)$, or $SO(2n)$, and let $P = MN$ be a parabolic subgroup of G. P will be called *basic* if $t = 1$ in the isomorphism (2.5).

These parabolic subgroups will play a crucial role in the computation of R-groups in Sections 4-7.

We now compute the Weyl groups of each P_θ. Recall that $W_\theta = W(G/A_\theta)$ is, by definition, $N_G(A_\theta)/Z_G(A_\theta) = N_G(A_\theta)/M_\theta$. Therefore, we can realize W_θ as those elements of $W(G/A_0)$ which normalize A_θ modulo $M_\theta \cap W(G/A_0)$.

Suppose $X_i = \{\theta_{i1}, \dots \theta_{in_i}\}$. Then, if we label things conveniently, we have the obvious correspondence $\theta_{ij} \leftrightarrow \lambda_{ij}$ from (2.3,2.4). If $w \in W_\theta$, then for each i,j $w : \lambda_{ij} \mapsto \lambda_{iw(ij)}^{\pm 1}$ for some $1 \le w(ij) \le n_i$. Then $w : \theta_{ij} \mapsto \pm\theta_{iw(ij)}$. Thus, the action of W_θ on each X_i can be realized as permutations and sign changes on the collection $\{\lambda_{ij}^{\pm 1}\}$.

Further note that the action of W_θ on $\Phi^+ \setminus \Sigma^+(\theta)$ can be realized as the permutations and sign changes on $\left\{\lambda_{(r+1)j}^{\pm 1}\right\}_{j=1}^{n_{r+1}}$. If $n_{r+1} = 1$, then there may or may not be a nontrivial element of W_θ which is the identity on each θ_i. More specifically, in case 2 such an element always exists. In case 1 such an element exists for $G = SO(2n+1)$ or $Sp(2n)$, but not for $G = SO(2n)$.

Suppose $G = SO(2n+1)$ or $Sp(2n)$. Let $w = sc \in W_\theta$, with $s \in S_n$ and $c \in \mathbb{Z}_2^n$. The discussion above notes that $s = s_1 s_2 \dots s_r s_{r+1}$, where each s_i permutes the elements $\{\lambda_{ij}\}_{j=1}^{n_i}$. Similarly, $c = \gamma_1\gamma_2 \dots \gamma_r\gamma_{r+1}$, where γ_i is a product of sign changes on the exponents of $\{\lambda_{ij}^{\pm 1}\}_{j=1}^{n_i}$. (Note that these statements are also clear from the form of A_θ given in (2.3,2.4).) In case 2, every element of W_θ acts trivially on $X_r = \theta_k$, because this component is not conjugate to any other. Therefore, in case 2, $s_r = 1$ and $\gamma_r = 1$. We state this more precisely as follows.

LEMMA 2.3. *Suppose* $G = SO(2n+1)$, *or* $Sp(2n)$.

(1) *In case* 1,

$$W_\theta \simeq \left(\prod_{i=1}^{r+1} S_{n_i}\right) \ltimes \mathbb{Z}_2^{n_1+\dots+n_{r+1}} \simeq \prod_{i=1}^{r+1} \left(S_{n_i} \ltimes \mathbb{Z}_2^{n_i}\right).$$

(2) *In case* 2,

$$\begin{aligned} W_\theta &\simeq \left(\left(\prod_{i=1}^{r-1} S_{n_i}\right) \times S_{n_{r+1}}\right) \ltimes \mathbb{Z}_2^{n_1+\dots+n_{r-1}+n_{r+1}} \\ &\simeq \prod_{i \ne r} \left(S_{n_i} \ltimes \mathbb{Z}_2^{n_i}\right). \end{aligned}$$

For $G = SO(2n)$ we need the following notion.

Definition 2.4. Let $1 \le i \le r+1$. We say that X_i is an *odd orbit* if m_i is odd and $\alpha_n \notin X_i$. We say X_i is an *even orbit* if m_i is even and $\alpha_n \notin X_i$.

Consider case 1. If $w \in W_\theta$, then we write $w = sc$ with $s \in S_n$ and $c \in \mathbb{Z}_2^{n-1}$. We can write $s = s_1 \dots s_r s_{r+1}$, with each $s_i \in S_{n_i}$. Here S_{n_i} acts on $\{\lambda_{ij}\}_{j=1}^{n_i}$. Let C_i^j be the product of the m_i sign changes (in $O(2n)$) such that $C_i^j : \lambda_{ij} \mapsto \lambda_{ij}^{-1}$. Note that, if X_i is even, then $C_i^j \in W(G/A_0)$. Since $C_i^j : A_\theta \longrightarrow A_\theta$, we have $C_i^j \in W_\theta$.

On the other hand, if X_i is odd, then $C_i^j \notin G$. However, if X_i and X_k are odd, then

$$C_i^j C_k^l : \begin{array}{c} \lambda_{ij} \mapsto \lambda_{ij}^{-1} \\ \lambda_{kl} \mapsto \lambda_{kl}^{-1} \end{array},$$

is a product of $m_i + m_k$ sign changes, and hence is in G. Thus, $C_i^j C_k^l \in W_\theta$.

Assume that $X_1, \dots, X_t$ are even orbits, and $X_{t+1}, \dots, X_r$ are odd orbits. Then we have shown that, if $c \in W_\theta \cap \mathbb{Z}_2^{n-1}$, then $c = \gamma_1 \dots \gamma_t c'$, where, for $1 \le i \le t$, γ_i is a product of sign changes on $\{\lambda_{ij}^{\pm 1}\}$, and c' is a product of an even number of sign changes on

$$\left\{ \lambda_{ij} \middle| \begin{array}{c} t+1 \le i \le r+1 \\ 1 \le j \le n_i \end{array} \right\}.$$

We make this explicit in the following lemma.

LEMMA 2.5. *Suppose $G = SO(2n)$. In case 1,*

$$\begin{aligned} W_\theta &\simeq \left(\prod_{i=1}^{r+1} S_{n_i} \right) \ltimes \left(\left\langle C_i^j \middle| 1 \le i \le t \right\rangle \times \left\langle C_i^j C_k^l \middle| t+1 \le i \le k \le r+1 \right\rangle \right) \\ &\simeq \left(\prod_{i=1}^{t} S_{n_i} \ltimes \mathbb{Z}_2^{n_i} \right) \times \left(\left(\prod_{i=t+1}^{r+1} S_{n_i} \right) \ltimes \mathbb{Z}_2^{n_{t+1} + \dots + n_{r+1} - 1} \right) \end{aligned}$$

Recall that, in case 2, we assume that $\alpha_n, \alpha_{n-1} \in \theta_k$. Then $X_r = \theta_k$, because no other θ_i can contain two roots which differ by a sign change. We again write $w = sc$, and note that, as above, $s = s_1 \dots s_{r-1} s_{r+1}$, with each $s_i \in S_{n_i}$. Note that, if X_i is even, then $C_i^j \in W_\theta$ for all j. If X_i is odd, then it is still the case that $C_i^j \notin G$. However, note that the sign change c_n interchanges α_n and α_{n-1} while fixing all other $\alpha \in \Delta$. Thus, c_n centralizes A_θ. Therefore, the sign change $C_i^j c_n \in W_\theta$, and has the effect of changing the sign of the exponent of λ_{ij}. Consequently, $c = \gamma_1 \dots \gamma_{r-1} \gamma_{r+1}$ where each γ_i is a product of sign changes on the exponents of $\{\lambda_{ij}^{\pm 1}\}_{j=1}^{n_i}$.

We now state this formally.

LEMMA 2.6. *In case* 2,

$$W_\theta \simeq \prod_{i \neq r} (S_{n_i} \ltimes \mathbb{Z}_2^{n_i}).$$

Examples.

(1) Let $G = Sp(18)$. Then $\Phi(G, A_0)$ is of type C_9. Let $\theta = \{e_1 - e_2, e_2 - e_3, e_4 - e_5, e_6 - e_7\}$. Then $\theta_1 = \{e_1 - e_2, e_2 - e_3\}$, $\theta_2 = \{e_4 - e_5\}$, $\theta_3 = \{e_6 - e_7\}$. $n_1 = 1$, $m_1 = 3$, $n_2 = 2$, $m_2 = 2$, and $b = 7$. Note that $\theta' \cap \Delta = \{e_8 - e_9, 2e_9\}$. We have

$$\begin{aligned}
A_\theta &= \left\{\operatorname{diag}\left\{\lambda_{11}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}, \lambda_{32}^{-1}, \lambda_{31}^{-1}, \lambda_{22}^{-1}, \lambda_{21}^{-1}, \lambda_{11}^{-1}\right\}\right\}, \\
M_\theta &= \left\{\operatorname{diag}\left\{g_{11}, g_{21}, g_{22}, g_{31}, g_{32}, g_{32}^{-1},, g_{31}^{-1}, g_{22}^{-1}, g_{21}^{-1}, g_{11}^{-1}\right\} \middle| g_{ij} \in GL(m_i)\right\} \\
&\simeq GL(3) \times GL(2)^2 \times GL(1)^2, \text{ and} \\
W_\theta &= \langle c_1c_2c_3, (46)(57), c_4c_5, c_6c_7, (89), c_8, c_9\rangle.
\end{aligned}$$

(2) Let $G = SO(17)$, and suppose $\theta = \{e_1 - e_2, e_2 - e_3, e_6 - e_7, e_7 - e_8, e_8\}$. Then $\theta_1 = \{e_1 - e_2, e_2 - e_3\}$, $\theta_2 = \{e_6 - e_7, e_7 - e_8, e_8\}$, $\theta' = \{\pm e_4 \pm e_5, \pm e_4, \pm e_5\}$, $n_1 = 1$, $m_1 = 3$, $n_3 = 2$, and

$$A_\theta = \left\{\operatorname{diag}\left\{\lambda_{11}I_3, \lambda_{21}, \lambda_{22}, I_7, \lambda_{22}^{-1}, \lambda_{21}^{-1}, \lambda_{11}^{-1}I_3\right\}\right\}.$$

Thus, $M_\theta \simeq GL(3) \times GL(1)^2 \times SO(7)$, and

$$W_\theta = \langle c_1c_2c_3, c_4, c_5, (45)\rangle \simeq \mathbb{Z}_2 \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_2^2).$$

(3) Let $G = SO(16)$, and suppose $\theta = \{e_1 - e_2, e_2 - e_3, e_4 - e_5, e_6 - e_7\}$. Then

$$A_\theta = \left\{\operatorname{diag}\left\{\lambda_{11}I_3, \lambda_{21}I_2, \lambda_{22}I_2, \lambda_{31}, \lambda_{31}^{-1}, \lambda_{22}^{-1}I_2, \lambda_{21}^{-1}I_2, \lambda_{11}^{-1}I_3\right\}\right\},$$

$$M_\theta \simeq GL(3) \times GL(2)^2 \times GL(1), \text{ and}$$

$$W_\theta = \langle (46)(57)\rangle \ltimes \langle c_4c_5, c_6c_7, c_1c_2c_3c_8\rangle \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_2^3.$$

(4) Let $G = SO(16)$, and $\theta = \{e_1 - e_2, e_2 - e_3, e_7 - e_8, e_7 + e_8\}$. Then,

$$A_\theta = \left\{\operatorname{diag}\left\{\lambda_{11}I_3, \lambda_{21}, \lambda_{22}, \lambda_{23}, I_4, \lambda_{23}^{-1}, \lambda_{22}^{-1}, \lambda_{21}^{-1}, \lambda_{11}^{-1}I_3\right\}\right\},$$

so $M_\theta \simeq GL(3) \times GL(1)^3 \times SO(4)$. Note that

$$W_\theta = \langle (45), (56) \rangle \ltimes \langle c_1c_2c_3c_8, c_4c_8, c_5c_8, c_6c_8 \rangle.$$

3. The results for $GL(n)$**.** Computing the R-groups for $SO(n)$ or $Sp(2n)$ involves computing the zeros of the rank 1 Plancherel factors μ_α associated to a reduced root α. In many cases this reduces to a computation in $GL(n)$. Therefore, it is worth recalling these results, and stating clearly the how we intend to use them.

Let n be a positive integer, and let $G = GL(n)$. Let B be the Borel subgroup of upper triangular matrices. Then the standard parabolic subgroups of G are in one to one correspondence with ordered partitions of n. If $\{n_1, \ldots, n_r\}, \sum n_i = n$ is such a partition, then we let $P_{n_1,\ldots n_r}$ be the corresponding parabolic subgroup. Let $M_{n_1,\ldots,n_r}$ be the Levi component of $P_{n_1,\ldots,n_r}$. Then

$$M_{n_1,\ldots,n_r} = \left\{ \left(\begin{matrix} g_1 & & \\ & \ddots & \\ & & g_r \end{matrix} \right) \middle| g_i \in GL(n_i) \right\}. \tag{3.1}$$

Let $A_{n_1,\ldots,n_r}$ be the split component of $P_{n_1,\ldots,n_r}$.

Let σ be an irreducible admissible supercuspidal representation of $M_{n_1,\ldots,n_r}$. We do not assume that σ is unitary for this discussion. By (3.1), $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r$, where each σ_i is an irreducible supercuspidal representation of $GL(n_i)$. In order to determine the reducibility criteria for $\mathrm{Ind}_{P_{n_1,\ldots,n_r}}^G(\sigma)$, it is enough to determine them when $r = 2$, i.e., when $P_{n_1,\ldots,n_r}$ is maximal. So, suppose $P = P_{n_1,n_2}$, and $M = M_{n_1,n_2}$. Note that if $n_1 \neq n_2$ then the Weyl group $W(G/A_{n_1,n_2})$ is trivial. If $n_1 = n_2 = m$ then $W(G/A_{m,m}) \simeq \mathbb{Z}_2$, and the nontrivial Weyl group element operates on $M_{m,m}$ by $w^{-1}(g_1, g_2)w = (g_2, g_1)$. Therefore, $\sigma \simeq \sigma^w$ if and only if $\sigma_1 \simeq \sigma_2$.

THEOREM 3.1. (Ol'šanskiĭ [26], Bernstein-Zelevinsky [1]) *Let* $G = GL(n)$. *Let* $n = n_1 + n_2$, $P = P_{n_1,n_2}$, *and let* $\sigma = \sigma_1 \otimes \sigma_2$ *be an irreducible admissible supercuspidal representation of* M_{n_1,n_2}. *Then* $\mathrm{Ind}_P^G(\sigma)$ *is reducible if and only if* $n_1 = n_2$ *and* $\sigma_1 \simeq \sigma_2 \otimes |det()|_F^{\pm 1}$.

Note that if $\sigma = \sigma_1 \otimes \sigma_2$ is unitary, then both σ_1 and σ_2 are unitary, and thus, $\sigma_1 \not\simeq \sigma_2 \otimes |det()|_F^{\pm 1}$. Therefore, we have the following corollary.

COROLLARY 3.2. *Let* G, P, M, *and* σ *be as in Theorem* 3.1. *If* σ *is unitary, then* $\mathrm{Ind}_P^G(\sigma)$ *is irreducible.*

Jacquet was able to broaden this results as indicated below.

THEOREM 3.3. (Jacquet [14]) *Let* $G = GL(n)$, $P = P_{n_1,n_2}$, *and* $M = M_{n_1,n_2}$. *If* $\sigma \in \mathcal{E}_2(M)$, *then* $\mathrm{Ind}_P^G(\sigma)$ *is irreducible.*

Remark. This shows that all the R-groups for $GL(n)$ are trivial. □

Now let G be the F-rational points of a connected reductive algebraic group defined over F. Let $\theta \subset \Delta$, and let $P_\theta = M_\theta N_\theta$ be the associated standard parabolic subgroup. Let $\sigma \in \mathcal{E}_2(M)$. Recall that $\Delta' = \{\alpha \in \Phi(P_\theta, A_\theta) \mid \mu_\alpha(\sigma) = 0\}$. Let $\alpha \in \Phi(P_\theta, A_\theta)$. Recall the definition of M_α and ${}^*P_\alpha$ given in Section 1. Since ${}^*P_\alpha$ is maximal in M_α, $\mu_\alpha(\sigma) = 0$ if, and only if, σ is ramified in M_α and $\mathrm{Ind}_{{}^*P_\alpha}^{M_\alpha}(\sigma)$ is irreducible.

The structure of the parabolic subgroups of $SO(n)$ and $Sp(2n)$ given in Lemma 2.1 indicates why the following lemma will be a key ingredient in the computation of the R-groups.

LEMMA 3.4. *Let G be the F-rational points of a connected reductive algebraic group defined over F. Suppose $P = MN$ is a parabolic subgroup of G, with split component A. Suppose that there is a reductive group M_1, and integers k and m, so that $M \simeq M_1 \times GL(k) \times GL(m)$. Let $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \in \mathcal{E}_2(M)$, with $\sigma_1 \in \mathcal{E}_2(M_1)$, $\sigma_2 \in \mathcal{E}_2(GL(k))$, and $\sigma_3 \in \mathcal{E}_2(GL(m))$. Let $\alpha \in \Phi(P, A)$, and suppose that $M_\alpha = M_1 \times GL(k + m)$. Then $\alpha \in \Delta'$ if and only if σ is ramified in M_α.*

Proof. By assumption, $\mathrm{Ind}_{{}^*P_\alpha}^{M_\alpha}(\sigma) \simeq \sigma_1 \otimes \mathrm{Ind}_{P_{k,m}}^{GL(k+m)}(\sigma_2 \otimes \sigma_3)$. Since σ_2 and σ_3 are unitary, $\mathrm{Ind}_{{}^*P_\alpha}^{M_\alpha}(\sigma)$ is always irreducible. Therefore, $\alpha \in \Delta'$ if and only if σ is ramified in M_α. □

Remark. If $m \neq k$, then $W(M_\alpha/A) = \{1\}$. Therefore, if $\alpha \in \Delta'$, then $m = k$. □

This lemma will allow us to reduce the computation of R-groups for $Sp(2n)$ and $SO(n)$ to computation of reducibility criteria for a few reduced roots.

4. Product Formula for $SO(2n + 1)$ and $Sp(2n)$. We will show that every R-group for $G = SO(2n + 1)$, or $Sp(2n)$, is a direct product of R-groups attached to basic parabolic subgroups. Suppose that $P = MN \subset G$ is a parabolic subgroup, with split component A. Suppose

$$M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_t)^{n_t} \times G(\Phi_m).$$

For $1 \leq i \leq t$, let $G_i = G(\Phi_{m_i n_i + m})$. Then G_i has a basic parabolic subgroup $P_i = M_i N_i$, with $M_i \simeq GL(m_i)^{n_i} \times G(\Phi_m)$. Let A_i be the split component of P_i, and let $W_i = W(G_i/A_i)$. By Lemma 2.3,

$$W(G/A) \simeq W_1 \times \cdots \times W_t. \tag{4.1}$$

Our goal is to make the isomorphism (4.1) explicit, and then use this isomorphism to prove a product formula. In order to be explicit we consider case 1 and case 2 separately.

Consider case 1, that is $\alpha_n \notin \theta$. Let $X = X_1 \cup \cdots \cup X_r$ as before. Assume that θ is of the form (2.1). For $1 \leq i \leq r$, let Θ_i be the unique sub-root system of $\Phi(G, A_0)$ of type $\Phi_{n_i m_i}$ containing X_i. (Here we think of X_i as a subset of Δ.) We let $\Theta_{r+1} = \theta'$ and $X_{r+1} = \emptyset$. Let $\Delta_i = \Delta(\Theta_i)$ be a choice of simple roots of Θ_i which contains $\Theta_i \cap \Delta$.

Let $G_i = G(\Theta_i)$ be the split group with root system Θ_i. That is,

$$G_i = \begin{cases} SO(2n_i m_i + 1) & \text{if } G = SO(2n+1) \\ Sp(2n_i m_i) & \text{if } G = Sp(2n). \end{cases}$$

Let $A_i \subset G_i$ be given by $A_i = A_{X_i}$, where we regard X_i as a subset of Δ_i. Let $P_i = P_{X_i}$. Then $P_i = M_i N_i$, with $M_i = Z_{G_i}(A_i)$.

By Lemma 2.3, $M_i \simeq GL(m_i)^{n_i}$, and hence P_i is a basic parabolic subgroup of G_i. Lemma 2.3 clearly gives us (2) of the following lemma. Part (1) follows from (2.3a,b).

LEMMA 4.1. *Let θ and Θ_i be as above. Then we have*

(1) $A_\theta \simeq A_1 \times \cdots \times A_{r+1}$, *and*

(2) $M_\theta \simeq M_1 \times \cdots \times M_{r+1}$.

Let $W_i = W(G_i/A_i)$. By Lemma 2.3, $W_i \simeq S_{n_i} \ltimes \mathbb{Z}_2^{n_i}$. Therefore, we get the following lemma.

LEMMA 4.2. *If $\alpha_n \notin \theta$, then $W_\theta \simeq W_1 \times \cdots \times W_{r+1}$.*

Let $\sigma \in \mathcal{E}_2(M_\theta)$. Then, by Lemma 4.1, $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_{r+1}$, with each $\sigma_i \in \mathcal{E}_2(M_i)$. Since each σ_i is a representation of the Levi factor of P_i, we can consider the induced representation $\mathrm{Ind}_{P_i}^{G_i}(\sigma_i)$. Let $W_i(\sigma_i)$ be the stabilizer of σ_i in W_i.

COROLLARY 4.3. *If $\sigma \in \mathcal{E}_2(M_\theta)$, then $W(\sigma) \simeq W_1(\sigma_1) \times \cdots \times W_{r+1}(\sigma_{r+1})$.*

Proof. Let $w \in W(\sigma)$. By Lemma 4.2, $w = w_1 w_2 \ldots w_{r+1}$, with each $w_i \in W_i$. If $g \in M_\theta$ then, by Lemma 4.1, $g = g_1 g_2 \cdots g_{r+1}$, with each $g_i \in M_i$. Moreover,

$$w^{-1} g w = (w_1 \ldots w_{r+1})^{-1}(g_1 \ldots g_{r+1})(w_1 \ldots w_{r+1}) = \prod_i (w_i^{-1} g_i w_i).$$

Therefore, $\sigma^w \simeq \bigotimes_{i=1}^{r+1} \sigma_i^{w_i}$, and consequently, $\sigma^w \simeq \sigma$ if and only if $\sigma_i^{w_i} \simeq \sigma_i$, $\forall i$. □

LEMMA 4.4. *If* $\alpha \notin \bigcup_i \Theta_i$ *then* $\alpha \notin \Delta'$

Proof. Assume that θ is of the form (2.1). Note that if α is conjugate to α_n under $W(G/A_0)$, then $\alpha \in \Theta_i$ for some i. So we assume that α is of the form $e_k \pm e_l$, for some k and l. Conjugating by an element of $W(G/A_0)$, we can assume that α is simple. Let $\alpha = \alpha_j$. Then, for some i, $\alpha_{j-1} \in \Theta_i$ and $\alpha_{j+1} \in \Theta_{i+1}$. By Lemma 2.1, we have $M \simeq M' \times GL(m_i + m_{i+1})$, where

$$M' \simeq \left(\prod_{k \neq i, i+1} M_k\right) \times GL(m_i)^{n_i - 1} \times GL(m_{i+1})^{n_{i+1} - 1}.$$

Since $m_i \neq m_{i+1}$, the remark following Lemma 3.4 implies that $\alpha \notin \Delta'$. □

Consider one orbit X_i. If θ is of the form (2.1), then we can explicitly write down X_i, as a set of roots. Let $d = \sum_1^{i-1} n_j m_j$. Then, $X_i = \bigcup_{j=0}^{n_i - 1} \{\alpha_{d+jm_i+l}\}_{l=1}^{m_i - 1}$.

Thus, if

$$\beta_{ij} = \begin{cases} e_{d+jm_i} & G = SO(2n+1) \\ 2e_{d+jm_i} & G = Sp(2n), \end{cases}$$

then we can choose $\Delta(\Theta_i) = \{\alpha_{d+l}\}_{l=1}^{n_i m_i - 1} \cup \{\beta_{in_i}\}$. From this description we can explicitly list the reduced roots $\alpha \in \Phi(P_\theta, A_\theta) \cap \Theta_i$. That is,

$$\Phi(P_\theta, A_\theta) \cap \Theta_i = \{e_{d+jm_i} \pm e_{d+km_i+1}\}_{j=0}^{n_i-1} {}_{k=j}^{n_i-1} \cup \{\beta_{ij}\}_{j=1}^{n_i}. \tag{4.2}$$

Let $\mathcal{Q} = \{e_{d+jm_i} \pm e_{d+km_i+1}\}_{j=0}^{n_i-1} {}_{k=j}^{n_i-1}$, and $\mathcal{Z} = \{\beta_{ij}\}_{j=1}^{n_i}$.

LEMMA 4.5. $\mathcal{Q}$ *and* $\mathcal{Z}$ *are the* W_θ *conjugacy classes of* $\Phi(P_\theta, A_\theta) \cap \Theta_i$.

Remark. Note that this is equivalent to saying that $\mathcal{Q}$ and $\mathcal{Z}$ are the W_i conjugacy classes of $\Phi(P_i, A_i)$. In fact this is evident from the proof given below. □

Proof.

(1) Let $1 \leq j \leq n_i$. Let C_i^j be the sign change $\lambda_{ij} \mapsto \lambda_{ij}^{-1}$. Then

$$C_i^{(j+1)}(e_{d+m_i} - e_{d+jm_i+1}) = e_{d+m_i} + e_{d+jm_i+1},$$

so these two roots are conjugate in W_θ.

(2) Let $1 < j < k < n_i$, and let $w \in W_\theta$ be the transposition in S_{n_i} given by $((j+1)\quad (k+1))$. That is, w interchanges $\lambda_{i(j+1)}$ and $\lambda_{i(k+1)}$. Then,

$$w(e_{d+m_i} - e_{d+jm_i+1}) = e_{d+m_i} - e_{d+km_i+1}.$$

(3) For $1 \leq k \leq j < n_i$, we let $w = (1\ k)$ in S_{n_i}. Then

$$w(e_{d+m_i} - e_{d+jm_i+1}) = e_{d+km_i} - e_{d+jm_i+1}.$$

(1), (2), and (3) show that all $\alpha \in \mathcal{Q}$ are W_θ conjugate. If $\alpha \in \mathcal{Q}$ and $\beta \in \mathcal{Z}$, then α and β are not $W(G/A_0)$ conjugate because they have different root lengths. Since $\alpha_n \notin \theta$ no long and short roots can be identified by W_θ. Hence, α and β are not conjugate in W_θ. Therefore, (4.2) tells us that $\mathcal{Q}$ is a W_θ conjugacy class in Θ_i.

We now show that $\mathcal{Z}$ is a W_θ conjugacy class. Since $\alpha_n = e_n$ or $2e_n$ we need only look at the action of W_θ on e_{d+m_i}. (If $G = Sp(2n)$ we can view this as the action of W_θ on the dual root system.)

Let $w = (1\ 2\ \ldots\ n_i)$ in S_{n_i}. Then $w : \lambda_{i1} \mapsto \lambda_{i2} \mapsto \ldots \lambda_{in_i} \mapsto \lambda_{i1}$. Thus, $w^k(e_{d+m_i}) = e_{d+(k+1)m_i}$, so all elements of $\mathcal{Z}$ are conjugate.

□

Let $\alpha \in \Theta_i$ for some i. We let $M_{i,\alpha}$ be the reductive group assigned to $\alpha \in \Phi(P_i, A_i)$. More precisely, $A_{i,\alpha} = (A_i \cap \ker \chi_\alpha)^\circ$, $M_{i,\alpha} = Z_{G_i}(A_{i,\alpha})$, and ${}^*P_{i,\alpha} = P_i \cap M_{i,\alpha}$. Then $\mu_\alpha(\sigma_i)$ is the Plancherel measure of σ_i with respect to $\alpha \in \Theta_i$. Since ${}^*P_{i,\alpha}$ is a maximal parabolic subgroup of $M_{i,\alpha}$, $\mu_\alpha(\sigma_i) = 0$ if and only if $\sigma_i^w \simeq \sigma_i$, for some nontrivial $w \in W(M_{i,\alpha}/A_i)$, and $\mathrm{Ind}_{{}^*P_{i,\alpha}}^{M_{i,\alpha}}(\sigma_i)$ is irreducible.

LEMMA 4.6. *Let $1 \leq i \leq r+1$, and let $\alpha \in \Theta_i$. Then*

(1) $M_\alpha \simeq M_1 \times \cdots \times M_{i-1} \times \boxed{M_{i,\alpha}} \times M_{i+1} \cdots \times M_{r+1}$;

(2) ${}^*P_\alpha \simeq M_1 \times \cdots \times M_{i-1} \times \boxed{{}^*P_{i,\alpha}} \times M_{i+1} \cdots \times M_{r+1}$.

Proof. By Lemma 4.5, we only need to check this for the roots $\alpha = e_{d+m_i} - e_{d+m_i+1}$, and $\beta = \beta_{in_i}$.

For α, we assume that θ is of the form (2.1).

$$A_\alpha = \left\{ \mathrm{diag}\left\{\lambda_{11}, \ldots, \lambda_{(r+1)n_{r+1}}, \ldots, \lambda_{11}^{-1}\right\} \,\middle|\, \lambda_{i1} = \lambda_{i2} \right\}.$$

Then

$$M_\alpha \simeq \left(\prod_{j \neq i} M_j\right) \times GL(2m_i) \times GL(m_i)^{n_i - 2}.$$

That is, if $g \in M_\alpha$, then

$$g = \begin{pmatrix} D_1 & & & & & \\ & \boxed{C} & & & & \\ & & D_2 & & & \\ & & & {}^\tau D_2^{-1} & & \\ & & & & \boxed{{}^\tau C^{-1}} & \\ & & & & & {}^\tau D_1^{-1} \end{pmatrix},$$

with

$$D_1 = \begin{pmatrix} g_{11} & & \\ & \ddots & \\ & & g_{(i-1)n_{i-1}} \end{pmatrix}, \qquad D_2 = \begin{pmatrix} g_{i3} & & \\ & \ddots & \\ & & g_{(r+1)n_{r+1}} \end{pmatrix},$$

each $g_{jk} \in GL(m_j)$, and $C \in GL(2m_i)$.

Similarly,

$$A_{i,\alpha} = \left\{ \operatorname{diag}\{\lambda_{i1}, \ldots, \lambda_{in_i}, \ldots, \lambda_{i1}^{-1}\} \middle| \lambda_{i1} = \lambda_{i2} \right\}.$$

Therefore,

$$M_{i,\alpha} \simeq GL(2m_i) \times GL(m_i)^{n_i-2}.$$

Explicitly, if $g \in M_{i,\alpha}$ then

$$g = \begin{pmatrix} \boxed{C} & & & & \\ & g_{i3} & & & \\ & & \ddots & & \\ & & & {}^\tau g_{i3}^{-1} & \\ & & & & \boxed{{}^\tau C^{-1}} \end{pmatrix}.$$

Thus, we have (1) in this case. Note that, if $g \in P_\theta$, then $g = \begin{pmatrix} D & * \\ 0 & {}^\tau D^{-1} \end{pmatrix}$, with

$$D = \begin{pmatrix} g_{11} & & \\ & \ddots & \\ & & g_{(r+1)n_{r+1}} \end{pmatrix}.$$

Thus, ${}^*P_\alpha$ consists of those $g \in G$ with

$$g = \begin{pmatrix} g_{11} & & & & & & \\ & \ddots & & & & & \\ & & \boxed{\begin{matrix} g_{i1} & * \\ 0 & g_{i2} \end{matrix}} & & & & \\ & & & \ddots & & & \\ & & & & \boxed{\begin{matrix} {}^\tau g_{i1}^{-1} & * \\ 0 & {}^\tau g_{i2}^{-1} \end{matrix}} & & \\ & & & & & \ddots & \\ & & & & & & {}^\tau g_{11}^{-1} \end{pmatrix}.$$

Similarly,

$${}^*P_{i,\alpha} = \left\{ \begin{pmatrix} \boxed{\begin{matrix} g_{i1} & * \\ 0 & g_{i2} \end{matrix}} & & & & \\ & g_{i3} & & & \\ & & \ddots & & \\ & & & {}^\tau g_{i3}^{-1} & \\ & & & & \boxed{\begin{matrix} {}^\tau g_{i1}^{-1} & * \\ 0 & {}^\tau g_{i2}^{-1} \end{matrix}} \end{pmatrix} \right\}.$$

So we have (2) in this case.

For the case $\beta = \beta_{in_i}$ we can assume, up to conjugation in $W(G/A_0)$ that $i = r + 1$. (Of course we cannot assume that $m_{r+1} = 1$.) Then $\beta = \alpha_n$, and $\theta \cup \{\alpha_n\} \subset \Delta$, so we can use Lemma 2.1. Examining the structure of $\theta \cup \{\alpha_n\}$ in the Dynkin diagram we conclude

$$M_\beta \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_r)^{n_r} \times GL(m_{r+1})^{n_{r+1}-1} \times G(\Phi_{m_{r+1}}).$$

Similarly,

$$M_{r+1,\alpha_n} \simeq GL(m_{r+1})^{n_{r+1}-1} \times G(\Phi_{m_{r+1}}),$$

and therefore, we have (1). Examining the block forms of ${}^*P_{\alpha_n}$ and ${}^*P_{r+1,\alpha_n}$, as in the last case, shows that (2) holds.

□

COROLLARY 4.7. *Let* $1 \leq i \leq r+1$, *and* $\alpha \in \Theta_i$. *Then*

(1) $W(M_\alpha/A_\theta) = W(M_{i,\alpha}/A_i)$;

(2) $\mathrm{Ind}_{*P_\alpha}^{M_\alpha}(\sigma) \simeq \sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes \left(\mathrm{Ind}_{*P_{i,\alpha}}^{M_{i,\alpha}}(\sigma_i)\right) \otimes \cdots \otimes \sigma_{r+1}.$

Proof. Result (1) follows from (1) of Lemma 4.6 and (1) of Lemma 4.1. Result (2) follows directly from Lemma 4.6. □

Let $\sigma \simeq \bigotimes_i \sigma_i \in \mathcal{E}_2(M_\theta)$. For $1 \leq i \leq r+1$, let

$$\Delta_i' = \{\alpha \in \Phi(P_i, A_i) | \mu_\alpha(\sigma_i) = 0\},$$

and

$$R_i = \{w \in W_i(\sigma_i) | w(\Delta_i') = \Delta_i'\}.$$

Then R_i is the R-group of σ_i with respect to induction from P_i to G_i.

LEMMA 4.8. *For any $\sigma \in \mathcal{E}_2(M_\theta)$, $\Delta' = \bigcup_{i=1}^{r+1} \Delta_i'$.*

Proof. Lemma 4.4 shows that, if $\alpha \in \Delta'$, then $\alpha \in \Theta_i$ for some i. Suppose $\alpha \in \Theta_i$. Then $\mu_\alpha(\sigma) = 0$ if, and only if, $\sigma^w \simeq \sigma$, for some nontrivial $w \in W(M_\alpha/A_\theta)$ and $\mathrm{Ind}_{*P_\alpha}^{M_\alpha}(\sigma)$ is irreducible. By Corollary 4.7, this is equivalent to $\sigma_i^w \simeq \sigma_i$, for a nontrivial $w \in W(M_{i,\alpha}/A_i)$, and $\mathrm{Ind}_{*P_{i,\alpha}}^{M_{i,\alpha}}(\sigma_i)$ is irreducible. This is equivalent to $\mu_\alpha(\sigma_i) = 0$. □

THEOREM 4.9. *For any $\sigma = \bigotimes_i \sigma_i \in \mathcal{E}_2(M_\theta)$, $R = R_1 \times \cdots \times R_{r+1}$.*

Proof. Let $w \in W_\theta$. Then, by Lemma 4.2, $w = w_1 \ldots w_{r+1}$, with each $w_i \in W_i$. By definition, $w \in R$ if and only if $w \in W(\sigma)$ and $w(\Delta') = \Delta'$. By Corollary 4.3, and Lemma 4.6, this is equivalent to $w_i \in W_i(\sigma_i)$, $\forall i$, and

$$(w_1 \ldots w_{r+1})\left(\bigcup_i \Delta_i'\right) = \bigcup_i \Delta_i'.$$

This last condition holds if and only if $w_i \in W_i(\sigma_i)$ and $w_i(\Delta_i') = \Delta_i'$, $\forall i$. Therefore, $w \in R$ if and only if each $w_i \in R_i$.

□

Note that what we have proved is an analogue of the result for $GL(n)$ that to determine reducibility of $\mathrm{Ind}_P^G(\sigma)$ it is enough to know the answer in the case $M = GL(m)^k$ with $mk = n$ [1].

We now consider case 2. We will again define groups G_i, and parabolic subgroups $P_i \subset G_i$, which give us a product formula similar to Theorem 4.9. Once we define G_i and P_i the steps are essentially the same as in case 1.

Let $X = X_1 \cup \cdots \cup X_r$ as before. Recall that we assume X_r is a singleton set, consisting of the component θ_k which contains α_n.

For $1 \leq i \leq r-1$, we let Θ_i be the unique sub-root system of $\Phi(G, A_0)$, of type $\Phi_{n_i m_i + m}$, containing $X_i \cup X_r$. If θ is of the form (2.2), then we let

$$\Theta_{r+1} = \begin{cases} \text{span } < \theta' \cup X_r \cup \{e_{n-m} - e_{n-m+1}\} & \text{if } n_{r+1} \geq 1 \\ \emptyset & \text{if } n_{r+1} = 0, \end{cases}$$

and $X_{r+1} = \emptyset$.

We let $\Delta_i = \Delta(\Theta_i)$ be a choice of simple roots in Θ_i containing $X_i \cup X_r$. Let $G_i = G(\Theta_i)$. For $i \neq r$, we let $A_i \subset G_i$ be given by $A_i = A_{X_i \cup X_r}$, and $P_i = P_{X_i \cup X_r}$. Then $P_i = M_i N_i$, with $M_i = Z_{G_i}(A_i)$. By Lemma 2.1, $M_i \simeq GL(m_i)^{n_i} \times G(\Phi_m)$. Thus, P_i is a basic parabolic subgroup of G_i.

Lemma 4.10. *Let θ and Θ_i be as above. Then we have*

(1) $A_\theta \simeq A_1 \times \cdots \times A_{r-1} \times A_{r+1}$, *and*

(2) $M_\theta \simeq M_1' \times \cdots \times M_{r-1}' \times M_{r+1}' \times G(\Phi_m)$, *where* $M_i' \times G(\Phi_m) \simeq M_i$.

For $i \neq r$, we let $W_i = W(G_i/A_i)$. By Lemma 2.3, $W_i \simeq S_{n_i} \ltimes \mathbb{Z}_2^{n_i}$. Thus, we get the following result.

Lemma 4.11. *If $\alpha_n \in \theta$, then $W_\theta \simeq W_1 \times \cdots \times W_{r-1} \times W_{r+1}$.*

Remark. We can choose a representative for each $w_i \in W_i$ so that, for every $g \in G(\Phi_m)$, $w_i g w_i^{-1} = g$. That is, since $w_i(\theta_k) = \theta_k$, we can choose a representative with $w_i(\alpha) = \alpha$, $\forall \alpha \in \theta_k$. For this reason, we think of W_i as acting only on X_i in this case. □

Let $\sigma \in \mathcal{E}_2(M_\theta)$. Then, by Lemma 4.10, $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_{r-1} \otimes \sigma_{r+1} \otimes \rho$, with each $\sigma_i \in \mathcal{E}_2(M_i')$, and $\rho \in \mathcal{E}_2(G(\Phi_m))$. Therefore, for $i \neq r$, we have $\sigma_i \otimes \rho \in \mathcal{E}_2(M_i)$. Let $W_i(\sigma_i)$ be the stabilizer of $\sigma_i \otimes \rho$. By the above remark $(\sigma_i \otimes \rho)^{w_i} \simeq \sigma_i^{w_i} \otimes \rho$, $\forall w_i \in W_i$. Thus, the notation $W_i(\sigma_i)$ reflects the fact that ramification of $\sigma_i \otimes \rho$ depends only on σ_i.

Corollary 4.12. *If $\sigma \in \mathcal{E}_2(M_\theta)$, then*

$$W(\sigma) \simeq W_1(\sigma_1) \times \cdots \times W_{r-1}(\sigma_{r-1}) \times W_{r+1}(\sigma_{r+1}).$$

Lemma 4.13. *If $\alpha \notin \bigcup_{i \neq r} \Theta_i$, then $\alpha \notin \Delta'$.*

Proof. As in the proof of Lemma 4.4, we can reduce to the case where α is simple, and apply Lemma 3.4. □

Suppose θ is of the form (2.2). Fix an orbit X_i, with $i \neq r$. We explicitly write down X_i. Let $d = \sum_{\substack{j<i \\ j\neq r}} n_j m_j$. Then $X_i = \cup_{j=0}^{n_i-1} \left\{\alpha_{d+jm_i+l}\right\}_{l=1}^{m_i-1}$. If we let $\beta_{ij} = e_{d+jm_i} - e_{n-m+1}$, then we can choose $\Delta_i = \left\{\alpha_{d+j}\right\}_{j=1}^{n_i m_i-1} \cup \left\{\beta_{in_i}\right\} \cup X_r$. Note that

$$\Phi(P_\theta, A_\theta) \cap \Theta_i = \left\{e_{d+jm_i} \pm e_{d+km_i+1}\right\}_{j=1}^{n_i-1}{}_{k=j}^{n_i-1} \cup \left\{\beta_{ij}\right\}_{j=1}^{n_i}.$$

Let $\mathcal{Q} = \left\{e_{d+jm_i} \pm e_{d+km_i+1}\right\}_{j=0}^{n_i-1}{}_{k=j}^{n_{i-1}}$, and $\mathcal{Z} = \left\{\beta_{ij}\right\}_{j=1}^{n_i}$.

LEMMA 4.14. *$\Phi(P_\theta, A_\theta) \cap \Theta_i = \mathcal{Q} \cup \mathcal{Z}$. Moreover, $\mathcal{Q}$ and $\mathcal{Z}$ are the W_θ conjugacy classes in $\Phi(P_\theta, A_\theta) \cap \Theta_i$.*

Proof. The proof is essentially the same as the proof of Lemma 4.5.

□

Let $i \neq r$, and $\alpha \in \Theta_i \cap \Phi(P_\theta, A_\theta)$ Then, we let $A_{i,\alpha} = (A_i \cap \ker \chi_\alpha)^\circ$, $M_{i,\alpha} = Z_{G_i}(A_{i,\alpha})$, and ${}^*P_{i,\alpha} = P_i \cap M_{i,\alpha}$.

For $\sigma_i \in \mathcal{E}_2(M_i')$, we let $\mu_\alpha(\sigma_i \otimes \rho)$ be the Plancherel measure of $\sigma_i \otimes \rho$ with respect to P_i. That is, $\mu_\alpha(\sigma_i \otimes \rho)$ is determined by $\operatorname{Ind}_{{}^*P_{i,\alpha}}^{M_{i,\alpha}} (\sigma_i \otimes \rho)$.

LEMMA 4.15. *If $i \neq r$, and $\alpha \in \Phi(P_\theta, A_\theta) \cap \Theta_i$, then*

(1) $M_\alpha \simeq M_1' \times M_2' \times \cdots \times M_{i-1}' \times \boxed{M_{i,\alpha}} \times \cdots \times M_{r-1}' \times M_{r+1}'$;

(2) ${}^*P_\alpha \simeq M_1' \times M_2' \times \cdots \times M_{i-1}' \times \boxed{{}^*P_{i,\alpha}} \times \cdots \times M_{r-1}' \times M_{r+1}'$.

Proof. By Lemma 4.14, we need only check this for the roots $\alpha = e_{d+m_i} - e_{d+2m_i}$ and $\beta = e_{d+n_i m_i} - e_{n-m+1}$. As in the proof of Lemma 4.6, each of these cases is a straightforward matrix computation. □

COROLLARY 4.16. *If $i \neq r$, and $\alpha \in \Theta_i$,*

(1) $W(M_\alpha/A_\theta) = W(M_{i,\alpha}/A_i)$;

(2) $\operatorname{Ind}_{{}^*P_\alpha}^{M_\alpha} (\sigma) \simeq \sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes \left(\operatorname{Ind}_{{}^*P_{i,\alpha}}^{M_{i,\alpha}} (\sigma_i)\right) \otimes \cdots \otimes \sigma_{r-1} \otimes \sigma_{r+1}$.

Proof. Result (1) follows from (1) of Lemma 4.15, and (1) of Lemma 4.10. Result (2) follows from Lemma 4.15.

□

Let $\sigma \simeq (\bigotimes_i \sigma_i) \otimes \rho \in \mathcal{E}_2(M_\theta)$. For $i \neq r$, let

$$\Delta_i' = \left\{\alpha \in \Phi(P_i, A_i) \middle| \mu_\alpha(\sigma_i \otimes \rho) = 0\right\},$$

and

$$R_i = \{w \in W_i(\sigma_i) | w(\Delta_i') = \Delta_i'\}.$$

Then R_i is the R-group of $\sigma_i \otimes \rho$, with respect to induction from P_i to G_i.

LEMMA 4.17. *For any* $\sigma \in \mathcal{E}_2(M_\theta)$, $\Delta' = \bigcup_{i \neq r} \Delta_i'$.

Proof. If $\alpha \in \Delta'$ then, by Lemma 4.13, $\alpha \in \Theta_i$ for some i. If $\alpha \in \Theta_i$ then, by Corollary 4.16, $\mu_\alpha(\sigma) = 0$ if and only if $\mu_\alpha(\sigma_i) = 0$. □

THEOREM 4.18. *Let* $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_{r-1} \otimes \sigma_{r+1} \otimes \rho \in \mathcal{E}_2(M_\theta)$. *Then*

$$R \simeq R_1 \times \cdots \times R_{r-1} \times R_{r+1}.$$

Proof. If $w \in W_\theta$, then $w = w_1 \ldots w_{r-1} w_{r+1}$, with each $w_i \in W_i$. By definition, $w \in R$ if and only if $w \in W(\sigma)$ and $w(\Delta') = \Delta'$. By Corollary 4.12, and Lemma 4.17, $w \in R$ if and only if, for each $i \neq r$, $w_i \in W_i(\sigma_i)$ and $w_i(\Delta_i') = \Delta_i'$. This is equivalent to $w_i \in R_i$, $\forall i \neq r$. □

5. R-groups for $SO(2n)$. We wish to prove results similar to Theorems 4.9 and 4.18 for $G = SO(2n)$.The structure of G will complicate matters a bit. Notice that corollaries 4.3 and 4.12 were crucial to the product formulas. However, for $G = SO(2n)$ the corresponding statements fail to be true. (See example (3) in Section 2.) Therefore, we take a different approach. We choose integers k_1, k_2, and parabolic subgroups $P_i = M_i N_i$ of $SO(2k_i)$ with $M_i = M_i' \times SO(2m)$, such that $M = M_1' \times M_2' \times SO(2m)$. Moreover, the reduced root system $\Phi(P_1, A_1)$ contains all the even orbits of θ, while $\Phi(P_2, A_2)$ contains all the odd orbits of θ. We write $\sigma \in \mathcal{E}_2(M)$ as $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \rho$, and show that $R(\sigma) = R(\sigma_1 \otimes \rho) \times R(\sigma_2 \otimes \rho)$. This reduces us to the cases where all the GL factors have the same parity. We then further reduce certain cases via a product formula. The remaining cases are computed explicitly in Section 6. Again we argue cases 1 and 2 separately.

Consider case 1. Assume that θ has the form (2.1). We assume that, in the decomposition $X = X_1 \cup \cdots \cup X_r$, the orbits $X_1, \ldots, X_t$ are even, and $X_{t+1}, \ldots, X_r$ are odd. Our assumption on the parity of the X_i can be phrased as follows: Suppose θ_j is in an even orbit, $\theta_{j'}$ is in an odd one. If $\alpha_i \in \theta_j$, and $\alpha_{i'} \in \theta_{j'}$, then $i < i'$.

Let $a = \sum_{j=1}^t n_j m_j$. Let $\Theta_1 \subset \Phi(G, A_0)$ be the unique sub-root system of type D_a containing $\bigcup_{i=1}^t X_i$. Let $\beta = e_{a-1} + e_a$. We let $\Delta_1 = \{\alpha_j\}_{j=1}^{a-1} \cup \{\beta\}$ be our choice of simple roots. Let Θ_2 be the sub-root system of type D_{n-a} with simple roots $\Delta_2 = \{\alpha_j \mid j \geq a+1\}$.

Let $G_1 = G(\Theta_1) = SO(2a)$, and $G_2 = G(\Theta_2) = SO(2(n-a))$. Let $\Omega_i = \Delta_i \cap \theta$.

We let $A_i = A_{\Omega_i} \subset G_i$, $M_i = Z_{G_i}(A_i)$, and $P_i = P_{\Omega_i} \subset G_i$. By Lemma 2.1,

$$M_1 \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_t)^{n_t},$$

and

$$M_2 \simeq GL(m_{t+1})^{n_{t+1}} \times \cdots \times GL(1)^{n_{r+1}}.$$

We let $W_i = W(G_i/A_i)$. By Lemma 2.5,

$$W_1 \simeq \prod_{i=1}^{t} (S_{n_i} \ltimes \mathbb{Z}_2^{n_i}),$$

and

$$W_2 \simeq \left(\prod_{i=t+1}^{r+1} S_{n_i} \right) \ltimes \mathbb{Z}_2^{n_{t+1}+\cdots+n_{r+1}-1}.$$

We summarize the discussion of the structure of M_θ and W_θ.

LEMMA 5.1. *Let θ be of the form* (2.1). *Suppose that Θ_i, P_i, M_i, A_i, and W_i are defined as above. Then,*

(1) $A_\theta = A_1 \times A_2$;

(2) $M_\theta = M_1 \times M_2$;

(3) $W_\theta = W_1 \times W_2$.

Let $\sigma \in \mathcal{E}_2(M_\theta)$. Then, by Lemma 5.1, $\sigma \simeq \sigma_1 \otimes \sigma_2$, with each $\sigma_i \in \mathcal{E}_2(M_i)$. Let $W_i(\sigma_i)$ be the stabilizer of σ_i in W_i.

COROLLARY 5.2. *If $\sigma \in \mathcal{E}_2(M_\theta)$, then $W(\sigma) = W_1(\sigma_1) \times W_2(\sigma_2)$.*

Proof. The proof of this follows the same lines as the proof of Corollary 4.3. □

LEMMA 5.3. *If $\alpha \in \Delta'$ then $\alpha \in \Theta_1 \cup \Theta_2$.*

Proof. Note that any root not in $\Theta_1 \cup \Theta_2$ is of the form $e_i \pm e_j$, with $i \leq a < j$. By conjugation in $W(G/A_0)$ we can assume that $i = a$ and $j = i + 1$. Since $m_t \neq m_{t+1}$, the result then follows from Lemma 3.4. □

Note that for each $i \le r$, there is an integer b_i such that

$$X_i = \bigcup_{j=0}^{n_i-1} \{\alpha_{b_i+jm_i+l}\}_{l=1}^{m_i-1}.$$

In fact, since we chose θ of the form (2.1), $b_i = \sum_{j=1}^{i-1} n_j m_j$. Let $X_{r+1} = \{\alpha_j \mid b \le j \le n-1\}$, and $b_{r+1} = b$. Let $i \le t$, and let

$$\mathcal{Q}_i = \bigcup_{j=1}^{n_i-1} \bigcup_{k=j}^{n_i-1} \{e_{b_i+jm_i} \pm e_{b_i+km_i+1}\},$$

$$\mathcal{C}_i = \bigcup_{j=1}^{n_i} \bigcup_{k=i+1}^{t} \{e_{b_i+jm_i} \pm e_{b_k+lm_k+1}\}_{l=0}^{n_k-1},$$

and

$$\mathcal{Z}_i = \bigcup_{j=1}^{n_i} \{e_{b_i+jm_i-1} + e_{b_i+jm_i}\}.$$

If $i \ge t+1$, then we let $\mathcal{Q}_i$ and $\mathcal{Z}_i$ be defined as above, but

$$\mathcal{C}_i = \bigcup_{j=1}^{n_i} \bigcup_{k=i+1}^{r+1} \{e_{b_i+jm_i} \pm e_{b_k+lm_k+1}\}_{l=0}^{n_k-1}.$$

LEMMA 5.4.

(1) *If $i \le t$, then*

$$\{\alpha \in \Phi(P_\theta, A_\theta) \cap \Theta_1 \mid \langle \alpha, \beta \rangle \ne 0 \textit{ for some } \beta \in X_i\} = \mathcal{Q}_i \cup \mathcal{Z}_i \cup \mathcal{C}_i.$$

(2) *If $i > t$, then*

$$\{\alpha \in \Phi(P_\theta, A_\theta) \cap \Theta_2 \mid \langle \alpha, \beta \rangle \ne 0 \textit{ for some } \beta \in X_i\} = \mathcal{Q}_i \cup \mathcal{Z}_i \cup \mathcal{C}_i.$$

In each of these sets, the W_θ conjugacy classes are $\mathcal{Q}_i$, $\mathcal{C}_i$, and $\mathcal{Z}_i$.

Proof. The proof is essentially the same as the proof of Lemma 4.5. □

For $i = 1$ or 2, and $\alpha \in \Theta_i$, we let $A_{i,\alpha} = (\ker \chi_\alpha \cap A_i)^\circ$, $M_{i,\alpha} = Z_{G_i}(A_{i,\alpha})$, and ${}^*P_{i,\alpha} = P_i \cap M_{i,\alpha}$.

LEMMA 5.5. *If $i = 1$ or 2, $\alpha \in \Theta_i$, and $j = 3 - i$, then*

(1) $M_\alpha \simeq M_{i,\alpha} \times M_j$;

(2) ${}^*P_\alpha \simeq {}^*P_{i,\alpha} \times M_j$.

Proof. Suppose that $\alpha \in \Theta_1$. Then, by Lemma 5.4, there are three cases to consider. Up to W_θ conjugacy they are:

(i) $\alpha = e_{b_i+m_i} - e_{b_i+m_i+1} \quad 1 \leq i \leq t$;

(ii) $\alpha = e_{b_i} - e_{b_i+1} \quad 2 \leq i \leq t$;

(iii) $\alpha = e_{b_i-1} + e_{b_i} \quad 2 \leq i \leq t+1$.

We consider each case. If α is of type (i), then

$$M_\alpha \simeq \left(\prod_{j\neq i} GL(m_j)^{n_j}\right) \times GL(2m_i) \times GL(m_i)^{n_i-2},$$

and

$$M_{1,\alpha} \simeq \left(\prod_{\substack{1\leq j\leq t\\ j\neq i}} GL(m_j)^{n_j}\right) \times GL(2m_i) \times GL(m_i)^{n_i-2}.$$

Therefore,

$$M_\alpha \simeq M_{1,\alpha} \times M_2$$

in this case.

If $g \in P_\theta$, then $g = \begin{pmatrix} D & * \\ 0 & {}^\tau D^{-1} \end{pmatrix}$, with $D = \operatorname{diag}\{g_{11}, \ldots, g_{(r+1)n_{r+1}}\}$. Therefore, ${}^*P_\alpha$ consists of those $g \in G$ of the form

$$\operatorname{diag}\left\{g_{11}, \ldots, \begin{pmatrix} g_{i1} & * \\ 0 & g_{i2} \end{pmatrix}, g_{i3} \ldots, {}^\tau g_{i3}^{-1}, \begin{pmatrix} {}^\tau g_{i2}^{-1} & * \\ 0 & {}^\tau g_{i1}^{-1} \end{pmatrix}, \ldots, {}^\tau g_{11}^{-1}\right\}.$$

Similarly,

$${}^*P_{i,\alpha} = \left\{\operatorname{diag}\left\{\begin{pmatrix} g_{i1} & * \\ 0 & g_{i2} \end{pmatrix}, g_{i3}, \ldots, {}^\tau g_{i3}^{-1}, \begin{pmatrix} {}^\tau g_{i2}^{-1} & * \\ 0 & {}^\tau g_{i1}^{-1} \end{pmatrix}\right\}\right\}.$$

So (2) holds in this case. If α is of type (ii), then

$$\begin{aligned} A_\alpha &= \{\lambda_{jk} \mid \lambda_{in_i} = \lambda_{(i+1)1}\} = \left\{\operatorname{diag}\{D, {}^\tau D^{-1}\}\right\}, \text{ with} \\ D &= \operatorname{diag}\{\lambda_{11}, \ldots, \lambda_{i,n_{i-1}}, \boxed{\eta I_{m_i+m_{i+1}}}, \lambda_{(i+1)2}, \ldots \lambda_{(r+1)(n_{r+1})}\}. \end{aligned}$$

Thus,

$$M_\alpha \simeq \left(\prod_{j\neq i,i+1} GL(m_j)^{n_j}\right) \times GL(m_i)^{n_i-1} \times GL(m_i+m_{i+1}) \times GL(m_{i+1})^{n_{i+1}-1}.$$

Note that

$$A_{1,\alpha} = \left\{\operatorname{diag}\{D, {}^\tau D^{-1}\}\right\}, \text{ with } D = \operatorname{diag}\{\lambda_{11}, \ldots, \boxed{\eta I_{m_i+m_{i+1}}}, \ldots, \lambda_{tn_t}\}.$$

Thus,

$$M_{1,\alpha} \simeq \left(\prod_{\substack{j\leq t\\ j\neq i,i+1}} GL(m_j)^{n_j}\right) \times GL(m_i)^{n_i-1} \times GL(m_i+m_{i+1}) \times GL(m_{i+1})^{n_{i+1}-1}.$$

Therefore, $M_\alpha \simeq M_{1,\alpha} \times M_2$. Again, we can examine the block forms of $^*P_\alpha$ and $^*P_{1,\alpha}$ to conclude that $^*P_\alpha \simeq {}^*P_{1,\alpha} \times M_2$.

If α is of type (iii), we can conjugate by an appropriate element of $W(G/A_0)$ so that $i = r+1$, and $\alpha = \alpha_n$. (We cannot assume that $m_{r+1} = 1$, nor can we assume that X_i is even only for $i \leq t$.) Then

$$\begin{aligned} M_\alpha &\simeq GL(m_1)^{n_1} \times \cdots \times GL(m_r)^{n_r} \times GL(m_{r+1})^{n_{r+1}-1} \times SO(2m_{r+1}); \\ M_{1,\alpha_n} &\simeq \left(\prod_{\substack{X_j \text{ even}\\ j\neq r+1}} GL(m_j)^{n_j}\right) \times GL(m_{r+1})^{n_{r+1}-1} \times SO(2m_{r+1}). \end{aligned}$$

and therefore, (1) holds. An examination of the block form of M_α and $^*P_\alpha$ yields (2).

For $\alpha \in \Theta_2$, the arguments are similar. If $\Theta_2 \cap \theta$ consists of two odd components, then one cannot affect a single sign change on the components of $\Theta_2 \cap \theta$. However, the arguments can proceed as above by use of a properly chosen outer automorphism.

□

COROLLARY 5.6. *Let $i = 1$ or 2, $j = 3-i$, and $\alpha \in \Theta_i$. Then,*

(1) $W(M_\alpha/A_\theta) = W(M_{i,\alpha}/A_i)$;

(2) $\operatorname{Ind}_{^*P_\alpha}^{M_\alpha}(\sigma) \simeq \operatorname{Ind}_{^*P_{i,\alpha}}^{M_{i,\alpha}}(\sigma_i) \otimes \sigma_j$.

Proof. Result (1) follows from Lemma 5.1(1), and (1) of Lemma 5.5. Result (2) follows from Lemma 5.5. □

For $i = 1$ or 2, we let

$$\Delta_i' = \{\alpha \in \Phi(P_i, A_i) \mid \mu_\alpha(\sigma_i) = 0\}, \text{ and } R_i = \{w \in W_i(\sigma_i) \mid w(\Delta_i') = \Delta_i'\}.$$

LEMMA 5.7. *If $\sigma \in \mathcal{E}_2(M_\theta)$, then $\Delta' = \Delta_1' \cup \Delta_2'$.*

Proof. If $\alpha \in \Delta'$ then, by Lemma 5.3, $\alpha \in \Theta_1 \cup \Theta_2$. If $\alpha \in \Theta_i$, then $\alpha \in \Delta'$ if and only if, $\sigma^w \simeq \sigma$ for some nontrivial $w \in W(M_\alpha/A_\theta)$ and $\mathrm{Ind}_{*P_\alpha}^{M_\alpha}(\sigma)$ is irreducible. By Corollary 5.6, this is equivalent to $\sigma_i^w \simeq \sigma_i$, for some nontrivial $w \in W(M_{i,\alpha}/A_i)$, and $\mathrm{Ind}_{*P_{i,\alpha}}^{M_{i,\alpha}}(\sigma_i)$ is irreducible. Therefore, $\alpha \in \Delta'$ if and only if $\alpha \in \Delta_i'$. □

THEOREM 5.8. *If $\sigma \in \mathcal{E}_2(M_\theta)$, then $R \simeq R_1 \times R_2$.*

Proof. Let $w \in W_\theta$. Then, by Lemma 5.1, $w = w_1w_2$ with each $w_i \in W_i$. By definition $w \in R$ if and only if $w \in W(\sigma)$ and $w(\Delta') = \Delta'$. Lemmas 5.2 and 5.7 imply $w \in R$ if and only if, for each i, $w_i \in W_i(\sigma_i)$ and $w_i(\Delta_i') = \Delta_i'$. Therefore, $w \in R$ if and only if each $w_i \in R_i$. □

This theorem reduces us to computing the R-groups in the cases $M_\theta = M_1$, or $M_\theta = M_2$. That is, M_θ is isomorphic to a product of $GL(m_i)$'s and all the m_i have the same parity. The R-group for the case $M_\theta = M_2$ will be computed in Section 6.

We now consider the case $M_\theta = M_1$. Let $X = X_1 \cup \cdots \cup X_r$, as before. Note that $n_{r+1} = 0$ since we are assuming that $M_\theta = M_1$. For $1 \leq i \leq r$, we let $\Theta_i \subset \Phi(G, A_0)$ be the unique sub-root system of $\Phi(P_\theta, A_\theta)$ of type $D_{n_im_i}$, containing X_i.

Let $G_i = SO(2n_im_i)$, with root system Θ_i. Let $A_i = A_{X_i} \subset G_i$, $M_i = Z_{G_i}(A_i) \simeq GL(m_i)^{n_i}$, and $P_i = P_{X_i} \subset G_i$. Then P_i is a basic parabolic subgroup of G_i. Note that $M \simeq M_1 \times \cdots \times M_r$. Let $W_i = W(G_i/A_i)$, and $\sigma \in \mathcal{E}_2(M)$. Then $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$, with $\sigma_i \in \mathcal{E}_2(M_i)$. Denote the stabilizer of σ_i in W_i by $W_i(\sigma_i)$, and let R_i be the R-group.

THEOREM 5.9. *Suppose θ, Θ_i, G_i, P_i, and W_i are as above. Then the following hold:*

(1) $A_\theta \simeq A_1 \times \cdots \times A_r$;

(2) $W_\theta \simeq W_1 \times \cdots \times W_r$;

(3) $W(\sigma) \simeq W_1(\sigma_1) \times \cdots \times W_r(\sigma_r)$;

(4) $R = R_1 \times \cdots \times R_r$.

Proof. That (1) holds is clear. Since the m_i are even, (2) follows from Lemma 2.6. Using the argument in the proof of Corollary 4.3, (3) follows from (2).

In the proof of Lemma 5.5, we have actually shown that, for each $\alpha \in \Phi(P_\theta, A_\theta) \cap \Theta_i$,

$$M_\alpha \simeq \left(\prod_{j\neq i} M_j\right) \times M_{i,\alpha}, \tag{5.1}$$

and

$$^*P_\alpha \simeq \left(\prod_{j\neq i} M_j\right) \times \ ^*P_{i,\alpha}. \tag{5.2}$$

Then (1), (2), (5.1), and (5.2) imply $W(M_\alpha/A_\theta) \simeq W(M_{i,\alpha}/A_i)$. Thus, for such a root, $\alpha \in \Delta'$ if and only if $\alpha \in \Delta_i'$, where $\Delta_i = \{\alpha \in \Phi(P_i, A_i) \mid \mu_\alpha(\sigma) = 0\}$. That no other reduced roots can be in Δ' follows from Lemma 3.4. Then (4) follows as in Theorem 4.9. □

Now consider case 2. We will prove the analogues of Theorems 5.7 and 5.9. This will reduce R_1 to the case of a basic parabolic subgroup. For R_2 there are two cases. In one instance R_2 will decompose as a product. The second form of R_2 will be explicitly computed in Section 6.

Recall that we are assuming that $\alpha_{n-1}, \alpha_n \in \theta_k$, $X_r = \{\theta_k\}$, and $m = |\theta_k|$. We assume that θ is of the form (2.2). We let $X_{r+1} = \{\alpha_j \mid b' < j \leq n-m\}$. As in case 1, we assume that $X_1, \dots, X_t$ are even and $X_{t+1}, \dots, X_{r-1}$ are odd. It is also convenient to assume that if X_i is even, $X_{i'}$ odd, $\alpha_j \in X_i$, and $\alpha_{j'} \in X_{i'}$, then $j < j'$.

Let $a = \sum_{i=1}^{t} n_i m_i$. We let $\Theta_1 \subset \Phi(G, A_0)$ be the unique sub-root system of type D_{a+m} containing

$$\left(\bigcup_{i=1}^{t} X_i\right) \cup X_r. \tag{5.3}$$

Let Θ_2 be the unique sub-root system of type D_{n-a} spanned by

$$\{\alpha_j \mid j \geq a+1\}. \tag{5.4}$$

We let $G_i = G(\Theta_i)$. Thus, $G_1 = SO(2(a+m))$, and $G_2 = SO(2(n-a))$. We consider Θ_i to be the root system for G_i. Let $\beta = e_a - e_{n-m+1}$. We let Δ_1 be the set of simple roots of Θ_1, which includes the set (5.3) and β. We let Δ_2 be the set of simple roots given in (5.4). For $i = 1$ or 2, we let $A_i = A_{\Theta_i \cap \theta}$, $M_i = Z_{G_i}(A_i)$, and $P_i = P_{\Theta_i \cap \theta}$. Then, by Lemma 2.1,

$$M_1 \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_t)^{n_t} \times SO(2m),$$

and

$$M_2 \simeq GL(m_{t+1})^{n_{t+1}} \times \cdots \times GL(1)^{n_{r+1}} \times SO(2m).$$

Let

$$M_1' = GL(m_1)^{n_1} \times \cdots \times GL(m_t)^{n_t},$$

and

$$M_2' \simeq GL(m_{t+1})^{n_{t+1}} \times \cdots \times GL(1)^{n_{r+1}}.$$

If $\sigma \in \mathcal{E}_2(M_\theta)$ then we write $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \rho$, with each $\sigma_i \in \mathcal{E}_2(M_i')$, and $\rho \in \mathcal{E}_2(SO(2m))$. Note that $\sigma_1 \simeq \bigotimes_{i=1}^{t} \bigotimes_{j=1}^{n_i} \sigma_{ij}$, and $\sigma_2 \simeq \bigotimes_{\substack{i \geq t+1 \\ i \neq r}} \bigotimes_{j=1}^{n_i} \sigma_{ij}$, where each $\sigma_{ij} \in \mathcal{E}_2(GL(m_i))$. For $i = 1$ or 2, we let $W_i = W(G_i/A_i)$. Then, by Lemma 2.6,

$$W_1 \simeq \prod_{i=1}^{t} (S_{n_i} \ltimes \mathbb{Z}_2^{n_i}),$$

and

$$W_2 \simeq \prod_{\substack{t+1 \leq i \\ i \neq r}} (S_{n_i} \ltimes \mathbb{Z}_2^{n_i}).$$

The next lemma follows from the above computations and those of Section 2.

LEMMA 5.10. *With θ, Θ_i, M_i, and W_i as above,*

(1) $A_\theta \simeq A_1 \times A_2$;

(2) $M_\theta \simeq M_1' \times M_2' \times SO(2m)$, *where* $M_i' \times SO(2m) \simeq M_i$;

(3) $W_\theta \simeq W_1 \times W_2$.

Remark. If $g \in M_\theta$, then $g = g_{11} \ldots g_{(r+1)n_{r+1}} \cdot g'$, with $g_{ij} \in GL(m_i)$ and $g' \in SO(2m)$. If X_i is even then we can choose representatives for W_i which act trivially on the $SO(2m)$ component g'. However, for the odd X_i, the Weyl group W_i acts on g' by the outer automorphism of $SO(2m)$ induced by c_n. If X_i is odd, then

$$(5.5) \qquad (C_i^j c_n) g (C_i^j c_n)^{-1} = g_{11} \ldots g_{i(j-1)} {}^t g_{ij}^{-1} g_{i(j+1)} \ldots (c_n g' c_n^{-1}).$$

Consequently,

$$(5.6) \qquad C_i^j c_n(\sigma) = \sigma_{11} \otimes \cdots \otimes \sigma_{i(j-1)} \otimes \widetilde{\sigma}_{ij} \otimes \sigma_{i(j+1)} \otimes \cdots \otimes c_n(\rho).$$

Therefore, if $c_n(\rho) \not\simeq \rho$, then $C_i^j c_n \notin W(\sigma)$, for all i with X_i odd. However, from (5.5) and (5.6), it is clear that if X_i and X_k are odd, and σ_{ij} and σ_{kl} are self contragredient, then $C_i^j C_k^l = (C_i^j c_n)(C_k^l c_n) \in W(\sigma)$. On the other hand, if $c_n(\rho) \simeq \rho$, we will get a product decomposition for $W(\sigma)$. We choose to proceed as in case 1 and then describe any further reduction of R_2, in the case $c_n(\rho) \simeq \rho$, at the end of this section. □

For $i = 1$ or 2, let $W_i(\sigma_i \otimes \rho)$ be the stabilizer of $\sigma_i \otimes \rho$ in W_i.

LEMMA 5.11. *If $\sigma \in \mathcal{E}_2(M_\theta)$, then $W(\sigma) \simeq W_1(\sigma_1 \otimes \rho) \times W_2(\sigma_2 \otimes \rho)$.*

Proof. Let $g \in M_\theta$ with $g = g_1 g_2 g'$, where $g_i \in M_i'$, and $g' \in SO(2m)$. Let $w = w_1 w_2 \in W_\theta$. Then $\sigma^w(g) = \sigma(w^{-1}(g_1 g_2 g')w) = \sigma_1(w_1^{-1} g_1 w_1) \otimes (\sigma_2 \otimes \rho)(w_2^{-1} g_2 g' w_2) = \sigma_1^{w_1}(g_1) \otimes (\sigma_2 \otimes \rho)^{w_2}(g_2 g')$. Therefore, $w \in W(\sigma)$ if and only if $\sigma_1^{w_1} \simeq \sigma_1$ and $(\sigma_2 \otimes \rho)^{w_2} \simeq \sigma_2 \otimes \rho$. This is equivalent to $w_i \in W_i(\sigma_i \otimes \rho)$ for $i = 1, 2$. □

We now proceed as in case 1. The following result is a consequence of Lemma 3.4.

LEMMA 5.12. *If $\alpha \in \Delta'$, then $\alpha \in \Theta_1 \cup \Theta_2$.*

For θ of form (2.2), there are integers b_i such that

$$X_i = \left\{\alpha_{b_i + jm_i + l}\right\}_{j=0}^{n_i - 1} {}_{l=1}^{m_i - 1}.$$

Let $i \leq t$, and let

$$\mathcal{Q}_i = \bigcup_{j=1}^{n_i - 1} \bigcup_{l=j}^{n_i - 1} \{e_{b_i + jm_i} \pm e_{b_i + lm_i + 1}\},$$

$$\mathcal{C}_i = \bigcup_{j=1}^{n_i} \bigcup_{k=i+1}^{t} \{e_{b_i + jm_i} \pm e_{b_k + lm_k + 1}\}_{l=0}^{n_k - 1},$$

and

$$\mathcal{Z}_i = \bigcup_{j=1}^{n_i} \{e_{b_i + jm_i - 1} + e_{b_i + jm_i}\}.$$

If $i \geq t + 1$, and $i \neq r$, then we let $\mathcal{Q}_i$ and $\mathcal{Z}_i$ be defined as above, but

$$\mathcal{C}_i = \bigcup_{j=1}^{n_i} \bigcup_{\substack{k > i \\ k \neq r}} \{e_{b_i + jm_i} \pm e_{b_k + lm_k + 1}\}_{l=0}^{n_k - 1}.$$

LEMMA 5.13.

(1) *If* $i \leq t$, *then*

$$\{\alpha \in \Phi(P_\theta, A_\theta) \cap \Theta_1 \mid \langle \alpha, \beta \rangle \neq 0 \text{ for some } \beta \in X_i\} = \mathcal{Q}_i \cup \mathcal{Z}_i \cup \mathcal{C}_i.$$

(2) *If* $i > t$, *and* $i \neq r$ *then*

$$\{\alpha \in \Phi(P_\theta, A_\theta) \cap \Theta_2 \mid \langle \alpha, \beta \rangle \neq 0 \text{ for some } \beta \in X_i\} = \mathcal{Q}_i \cup \mathcal{Z}_i \cup \mathcal{C}_i.$$

As in case 1, *the* W_θ *conjugacy classes in each of these sets are* $\mathcal{Q}_i$, $\mathcal{C}_i$, *and* $\mathcal{Z}_i$.

Proof. The proof is essentially the same as that of Lemma 4.5. □

For $\alpha \in \Theta_i$, with $i = 1$ or 2, we set $A_{i,\alpha} = (\ker \chi_\alpha \cap A_i)^\circ$, $M_{i,\alpha} = Z_{G_i}(A_{i,\alpha})$, and ${}^*P_{i,\alpha} = P_i \cap M_{i,\alpha}$.

LEMMA 5.14. *If* $\alpha \in \Theta_i \cap \Phi(P_\theta, A_\theta)$, *with* $i = 1$ *or* 2, *and* $j = 3 - i$, *then*

(1) $M_\alpha \simeq M_{i,\alpha} \times M_{j'}$;

(2) ${}^*P_\alpha \simeq {}^*P_{i,\alpha} \times M_{j'}$.

Proof. The argument is essentially the same as the proof of Lemma 5.5. □

COROLLARY 5.15. *If* $\alpha \in \Theta_i$ *and* $j = 3 - i$, *then*

(1) $W(M_\alpha/A_\theta) = W(M_{i,\alpha}/A_i)$;

(2) $\mathrm{Ind}_{{}^*P_\alpha}^{M_\alpha}(\sigma) \simeq \mathrm{Ind}_{{}^*P_{i,\alpha}}^{M_{i,\alpha}}(\sigma_i \otimes \rho) \otimes \sigma_j$.

For $i = 1$ or 2, we let

$$\Delta_i' = \{\alpha \in \Phi(P_i, A_i) \mid \mu_\alpha(\sigma_i \otimes \rho) = 0\}, \text{ and } R_i = \{w \in W_i(\sigma_i \otimes \rho) \mid w(\Delta_i') = \Delta_i'\}.$$

THEOREM 5.16. *If* $\sigma \in \mathcal{E}_2(M_\theta)$, *then*

(1) $\Delta' = \Delta_1' \cup \Delta_2'$;

(2) $R \simeq R_1 \times R_2$.

Proof. The first statement is a consequence of Lemma 5.15. The second part follows from (1), and Lemma 5.10(3). □

In case 2, we have again reduced ourselves to considering the cases $M_\theta = M_1$, or $M_\theta = M_2$. We will further resolve the case $M_\theta = M_1$, and one instance of $M_\theta = M_2$.

For the moment, we assume that the m_i all have the same parity. For $1 \leq i \leq r-1$, we let $\Theta_i \subset \Phi(G, A_0)$ be the unique sub-root system of type $D_{n_i m_i + m}$ containing $X_i \cup X_r$. We let $\Theta_{r+1} = \{\alpha_j \mid j > b'\}$, and $X_{r+1} = \emptyset$. We take a choice of simple roots, $\Delta_i = \Delta(\Theta_i)$, in Θ_i which contains $X_i \cup X_r$. Let $G_i = G(\Theta_i)$, so, $G_i = SO(2(n_i m_i + m))$, with root system Θ_i. For $i \neq r$, let $A_i \subset G_i$ be given by $A_i = A_{X_i \cup X_r}$, and $P_i = P_{X_i \cup X_r}$. Then $P_i = M_i N_i$, with $M_i = Z_{G_i}(A_i)$. By Lemma 2.1, $M_i \simeq GL(m_i)^{n_i} \times SO(2m)$. Thus, P_i is a basic parabolic subgroup of G_i. For $i \neq r$, we let $W_i = W(G_i/A_i)$. By Lemma 2.6, $W_i \simeq S_{n_i} \times \mathbb{Z}_2^{n_i}$. Thus, we get the following result.

LEMMA 5.17. *If all the integers m_i have the same parity, then*

(1) $A_\theta \simeq A_1 \times \cdots \times A_{r-1} \times A_{r+1}$,

(2) $M_\theta \simeq M_1' \times \cdots \times M_{r-1}' \times M_{r+1}' \times SO(2m)$, *where* $M_i' \times SO(2m) \simeq M_i$, *and*

(3) $W_\theta \simeq W_1 \times \cdots W_{r-1} \times W_{r+1}$.

An argument similar to the proof of Lemma 4.4 yields the following result.

LEMMA 5.18. *Suppose all the integers m_i have the same parity. Let $\alpha \in \Phi(P_\theta, A_\theta)$. If $\alpha \in \Delta'$, then $\alpha \in \bigcup_{i \neq r} \Theta_i$.*

Let $\sigma \in \mathcal{E}_2(M_\theta)$. Then, by Lemma 5.17, $\sigma \simeq \sigma_1 \otimes \sigma_2 \cdots \otimes \sigma_{r-1} \otimes \sigma_{r+1} \otimes \rho$, with each $\sigma_i \in \mathcal{E}_2(M_i')$. Let $W_i(\sigma_i \otimes \rho)$ be the stabilizer of $\sigma_i \otimes \rho$ in W_i. Let $\Delta_i' = \{\alpha \in \Phi(P_i, A_i) \mid \mu_\alpha(\sigma_i) = 0\}$, and $R_i = \{w \in W_i(\sigma_i \otimes \rho) \mid w_i(\Delta_i') = \Delta_i'\}$. For $\alpha \in \Phi(P_i, A_i)$, let $A_{i,\alpha}$, $M_{i,\alpha}$, and ${}^*P_{i,\alpha}$ be defined as before.

THEOREM 5.19. *Suppose that $G = SO(2n)$, and $P \subset G$ is a parabolic subgroup. Let $P = MN$, and suppose $M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_{r-1})^{n_{r-1}} \times SO(2m)$, with each m_i even. Let $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_{r-1} \otimes \rho \in \mathcal{E}_2(M_\theta)$ and $\alpha \in \Phi(P_i, A_i)$. Then*

(1) $W(\sigma) \simeq W_1(\sigma_1 \otimes \rho) \times \cdots \times W_{r-1}(\sigma_{r-1} \otimes \rho)$;

(2) $M_\alpha \simeq M_1' \times \cdots \times M_{i-1}' \times M_{i,\alpha} \times \cdots \times M_{r-1}'$;

(3) ${}^*P_\alpha \simeq M_1' \times \cdots \times M_{i-1}' \times {}^*P_{i,\alpha} \times \cdots \times M_{r-1}'$;

(4) $W(M_\alpha/A_\theta) = W(M_{i,\alpha}/A_i)$;

(5) $\Delta' = \bigcup_{i \neq r} \Delta_i'$;

(6) $R \simeq R_1 \times \cdots \times R_{r-1} \times R_{r+1}$.

Moreover, (2), (3), *and* (4) *also hold if m_i is odd for each i.*

Proof. The proofs of (2) and (3) follow from matrix considerations. This is similar to the proof of Lemma 5.5. Since each X_i is even, and $n_{r+1} = 0$, each W_i acts trivially on ρ. Therefore, (1) follows from Lemma 5.17(3). Now, (4) follows from (2) and Lemma 5.17(1). Result (5) then follows from Lemma 5.18 and (1)-(4). Finally, (6) then follows from (1),(4) and (5). □

Remark. Recall that if all the m_i are odd, the obstruction to being able to write $R = R_1 \times \cdots \times R_{r-1} \times R_{r+1}$ is that, if $c_n(\rho) \not\simeq \rho$, then the analogue of Theorem 5.19(1) is false. However, if $c_n(\rho) \simeq \rho$ then the corresponding statement is true. $\square$

THEOREM 5.10. *Suppose that θ is of the form* (2.2), *with each m_i odd. Let $\sigma \in \mathcal{E}_2(M_\theta)$, with $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_{r-1} \otimes \sigma_{r+1} \otimes \rho$, as above. If $c_n(\rho) \simeq \rho$, then*

(1) $W(\sigma) \simeq W_1(\sigma_1 \otimes \rho) \times \cdots \times W_{r-1}(\sigma_{r-1} \otimes \rho) \times W_{r+1}(\sigma_{r+1} \otimes \rho)$;

(2) $\Delta' = \bigcup_{i \neq r} \Delta'_i$;

(3) $R = R_1 \times \cdots \times R_{r-1} \times R_{r+1}$.

Proof. Since $c_n(\rho) \simeq \rho$, Lemma 5.17(3) and (5.6) show that (1) holds. Result (2) then follows from (1), and Theorem 5.19(2,3). Finally, (3) is a consequence of (1) and (2). $\square$

6. Basic Parabolic Subgroups and Remaining Cases for $SO(2n)$. In Section 4, we reduced the computation of R-groups for $SO(2n+1)$ and $Sp(2n)$ to the case of basic parabolic subgroups. In Section 5, we reduced the computation of R-groups for $SO(2n)$ to the computation of four cases. Namely,

(1) $M \simeq GL(k)^r$, with k even;

(2) $m > 0$, $M \simeq GL(k)^r \times SO(2m)$, $\sigma \simeq \sigma_1 \otimes \rho$, with k even, or k odd and $c_n(\rho) \simeq \rho$;

(3) $M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_r)^{n_r} \times GL(1)^{n_{r+1}}$, with each m_i odd;

(4) $M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_{r-1})^{n_{r-1}} \times GL(1)^{n_{r+1}} \times SO(2m)$, with each m_i odd, $m > 0$, and $c_n(\rho) \not\simeq \rho$.

In this section we compute each of these R-groups. We show that every R-group is an elementary 2-group. This will yield multiplicity 1 results.

We begin with the case of a basic parabolic subgroup. We study $Sp(2n)$,but the statements and proofs (with proper substitutions) are valid for $SO(2n+1)$, and cases (1) and (2) for $SO(2n)$ given above.

Let $k > 0$, and $m \geq 0$. Let $n = k + m$, and $G = Sp(2n)$. Consider the maximal parabolic subgroup P, with Levi component $GL(k) \times Sp(2m)$. For this parabolic subgroup, we take $\theta = \Delta \setminus \{e_k - e_{k+1}\}$. Let $\sigma \in \mathcal{E}_2(GL(k))$, and $\rho \in \mathcal{E}_2(Sp(2m))$. Then

$$A_\theta = \left\{ \begin{pmatrix} \lambda I_k & & \\ & I_{2m} & \\ & & \lambda^{-1} I_k \end{pmatrix} \right\}.$$

Thus, $W_\theta \simeq \mathbb{Z}_2$, with the nontrivial element given by $C : \lambda \mapsto \lambda^{-1}$. For $g \in GL(k)$, and $g' \in Sp(2m)$, the action of C on M_θ is given by $C(gg')C^{-1} = {}^t g^{-1} g'$. Therefore, $(\sigma \otimes \rho)^C \simeq \tilde{\sigma} \otimes \rho$. So, $\sigma \otimes \rho$ ramifies in G if and only if $\sigma \simeq \tilde{\sigma}$.

Definition 6.1. We say that the condition $\mathcal{X}_{k,m}(\sigma\otimes\rho)$, holds if $\mathrm{Ind}_P^G(\sigma\otimes\rho)$ is reducible. Note that, if $\mathcal{X}_{k,m}(\sigma\otimes\rho)$ holds, then $\sigma\simeq\tilde{\sigma}$. We use this notation to remind ourselves that the reducibility of $\mathrm{Ind}_P^G(\sigma\otimes\rho)$ is determined by the poles of certain L-functions, [33,35], and thus, is arithmetic in nature.

We now assume that $r>1$, $kr<n$, and let $m=n-kr$. We intend to describe the possible R-groups when $P=MN$ and $M\simeq GL(k)^r\times Sp(2m)$. Sally, [29], showed that $\mathcal{X}_{1,0}$ is the condition that $\sigma^2=1$, and $\sigma\not\equiv 1$. Keys, [15, Theorem C_n], showed that, if $k=1$, then $R\simeq\mathbb{Z}_2^m$, where m is the number of inequivalent σ_i such that $\mathcal{X}_{1,0}(\sigma_i)$ holds. We intend to prove the generalization of this result.

Let

$$\begin{aligned}\theta = {} & \{e_1-e_2,\dots,e_{k-1}-e_k\}\cup\{e_{k+1}-e_{k+2},\dots,e_{2k-1}-e_{2k}\}\\ & \cup\cdots\cup\{e_{(r-1)k+1}-e_{(r-1)k+2},\dots,e_{n-m-1}-e_{n-m}\}\\ & \cup\{e_{n-m+1}-e_{n-m+2},\dots,e_{n-1}-e_n,2e_n\}=\Delta\setminus\{e_{jk}-e_{jk+1}\}_{j=1}^r.\end{aligned}$$

Then P_θ is the parabolic subgroup in question. By Lemma 2.3,

$$W_\theta\simeq S_r\ltimes\mathbb{Z}_2^r. \tag{6.1}$$

This is also obvious from the form of $A=A_\theta$;

$$A=\left\{\mathrm{diag}\left\{\lambda_1,\dots,\lambda_r,I_{2m},\lambda_r^{-1},\dots,\lambda_1^{-1}\right\}\right\}, \tag{6.2}$$

where λ_i stands for $\lambda_i I_k$. Then, from (6.2), the explicit form of the isomorphism in (6.1) can be written

$$\begin{aligned}(i\ \ i+1):&\ \ \lambda_i\longleftrightarrow\lambda_{i+1};\\ C_i:&\ \ \lambda_i\longmapsto\lambda_i^{-1}.\end{aligned}$$

Let $g\in M$, with $g=(g_1,\dots,g_r,g')$. Then, the actions of (ij) and C_i on M are given by

$$(ij)g(ij)^{-1}=(g_1,\dots,g_{i-1},g_j,\dots,g_{j-1},g_i,\dots,g_r,g'), \tag{6.3}$$

and

$$C_igC_i^{-1}=(g_1,\dots,g_{i-1},{}^tg_i^{-1},g_{i+1},\dots,g_r,g'). \tag{6.4}$$

If $\sigma\in\mathcal{E}_2(M)$, then $\sigma\simeq\sigma_1\otimes\cdots\otimes\sigma_r\otimes\rho$, with each $\sigma_i\in\mathcal{E}_2(GL(k))$, and $\rho\in\mathcal{E}_2(Sp(2m))$. Therefore, (6.3) and (6.4) show that σ ramifies in G if, and only if, at least one of the following conditions holds:

(6.5a) $$\sigma_i \simeq \sigma_j \text{ for some } i < j \ (\iff (ij) \in W(\sigma));$$

(6.5b) $$\sigma_i \simeq \tilde{\sigma}_j \text{ for some } i < j \ (\iff (ij)C_iC_j \in W(\sigma));$$

(6.5c) $$\sigma_i \simeq \tilde{\sigma}_i \text{ for some } i \ (\iff C_i \in W(\sigma)).$$

Recall that if $m > 0$, then $e_{jk} - e_{kr+1}$ gives the same reduced root as $2e_{jk}$. Thus, by Lemma 4.5 or 4.14,

(6.6) $$\Phi(P, A) = \{e_{jk} \pm e_{lk+1}\}_{j=1}^{r-1}{}_{l=j}^{r-1} \cup \{2e_{jk}\}_{j=1}^{r}.$$

Moreover, $\mathcal{Q} = \{e_{jk} \pm e_{lk+1}\}_{j=1}^{r-1}{}_{l=j}^{r-1}$, and $\mathcal{Z} = \{2e_{jk}\}_{j=1}^{r}$ form the W_θ conjugacy classes in $\Phi(P, A)$.

LEMMA 6.2.

(1) $e_{jk} - e_{lk+1} \in \Delta'$ *if and only if* $\sigma_j \simeq \sigma_{l+1}$.

(2) $e_{jk} + e_{lk+1} \in \Delta'$ *if and only if* $\sigma_j \simeq \tilde{\sigma}_{l+1}$.

(3) $2e_{jk} \in \Delta'$ *if and only if* $\sigma_j \simeq \tilde{\sigma}_j$ *and* $\mathcal{X}_{k,m}(\sigma \otimes \rho)$ *does not hold.*

Proof. We compute the zeros of the rank one Plancherel measures. Let $\beta = e_k - e_{k+1}$. Then, by Lemma 2.1,

$$M_\beta \simeq GL(2k) \times GL(k)^{r-2} \times Sp(2m).$$

Note that ${}^*P_\beta \simeq P_{k,k} \times GL(k)^{r-2}$, where $P_{k,k} \subset GL(2k)$ is as in Section 3. Therefore, Lemma 3.4 implies $\mu_\beta(\sigma) = 0$ if and only if $\sigma_1 \simeq \sigma_2$. Consequently, $\beta \in \Delta'$ if and only if $\sigma_1 \simeq \sigma_2$. By (6.6), $\beta_{jl} = e_{jk} - e_{lk+1}$ is S_r-conjugate to β. Therefore, $\mu_{\beta_{jl}}(\sigma) = 0$ if and only if $\sigma_j \simeq \sigma_{l+1}$. This proves (1). Conjugating by the sign change C_{l+1}, we see that $\mu_{e_{jk}+e_{lk+1}}(\sigma) = 0$ if and only if $\sigma_j \simeq \tilde{\sigma}_{l+1}$, proving (2).

Let $\alpha = \begin{cases} 2e_n & \text{if } m = 0, \\ e_{n-m} - e_{n-m+1} & \text{if } m > 0. \end{cases}$ Then, by Lemma 2.1,

$$M_\alpha \simeq GL(k)^{r-1} \times Sp(2(k+m)).$$

Note that $M_\alpha \cap W_\theta = \langle C_r \rangle$, and ${}^*P_\alpha \simeq GL(k)^{r-1} \times P'$, where $P' \subset Sp(2(k+m))$ is the $GL(k) \times Sp(2m)$ parabolic subgroup. Let $G' = Sp(2(k+m))$. Then,

$$\mathrm{Ind}_{{}^*P_\alpha}^{M_\alpha}(\sigma) \simeq \bigotimes_{i=1}^{r-1} \sigma_i \otimes \mathrm{Ind}_{P'}^{G'}(\sigma_r \otimes \rho)$$

is reducible if and only if, $\mathcal{X}_{k,m}(\sigma_r \otimes \rho)$ holds. Thus, $\mu_\alpha(\sigma) = 0$ if and only if $\sigma_r \simeq \widetilde{\sigma}_r$ and $\mathcal{X}_{k,m}(\sigma_r \otimes \rho)$ does not hold. Let $\alpha_j = 2e_{jk}$. Then, by the conjugacy of α_j with α, each $\mu_{\alpha_j}(\sigma) = 0$ if and only if $\sigma_j \simeq \widetilde{\sigma}_j$ and $\mathcal{X}_{k,m}(\sigma_j \otimes \rho)$ does not hold. □

Remark. The proof of the next Lemma is essentially the same as Keys's proof of an analogous statement for the case of the minimal parabolic subgroup [15]. For this reason we call such an argument a *Keys argument.* □

LEMMA 6.3. *If $w = sc \in R$, with $s \in S_r$ and $c \in \mathbb{Z}_2^r$, then $s = 1$.*

Proof. Without loss of generality, we suppose that, under the isomorphism (6.1), $s = (12\ldots j+1)$. We can make this assumption because any permutation is a product of disjoint cycles, and the sign changes act independently on these cycles. Note that, up to conjugation by sign changes in W_θ, we have $c|_{\{1,\ldots,j+1\}} = 1$, or C_{j+1}. To see this, note that if both C_1 and C_{j+1} appear in c, then $C_1 w C_1$ has two fewer sign changes than w, and we continue by induction.

Suppose that $c|_{\{1,2,\ldots,j+1\}} = 1$. Then $w \in W(\sigma)$ implies $\sigma_1 \simeq \sigma_2 \simeq \ldots \simeq \sigma_{j+1}$. By Lemma 6.2, $e_k - e_{jk+1} \in \Delta'$. Note that $w(e_k - e_{jk+1}) = e_{2k} - e_1 < 0$. This contradicts our assumption that $w \in R$.

Suppose $c|_{\{1,2,\ldots,j+1\}} = C_{j+1}$. Then $\sigma_1 \simeq \sigma_2 \simeq \cdots \simeq \sigma_{j+1} \simeq \widetilde{\sigma}_1$. Therefore, $e_k + e_{jk+1} \in \Delta'$, while $w(e_k + e_{jk+1}) = e_{2k} - e_1 < 0$, which is a contradiction. □

THEOREM 6.4. *Let $k > 0$, and $m \geq 0$. Suppose that $G = Sp(2n)$, $P = MN$, and $M \simeq GL(k)^r \times Sp(2m)$. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho \in \mathcal{E}_2(M)$. Then the R-group, R, associated to σ is given by $R \simeq \mathbb{Z}_2^d$, where d is the number of inequivalent σ_i satisfying the condition $\mathcal{X}_{k,m}(\sigma_i \otimes \rho)$.*

Proof. By Lemma 6.3, $R \subset \mathbb{Z}_2^r$. Thus, $R \simeq \mathbb{Z}_2^d$ for some d. Let $I \subset \{1,\ldots,r\}$ be such that $w = \prod_{j\in I} C_j \in R$, and I is maximal with respect to this property. Since $w \in W(\sigma)$, we have $\sigma_j \simeq \widetilde{\sigma}_j$ for each $j \in I$. Therefore, by (6.5c), $C_j \in W(\sigma)$. If $R(w) = \{\alpha \in \Phi(P,A) \mid w\alpha < 0\}$, then $R(C_j) \subset R(w)$ for each $j \in I$. Thus, $w\alpha > 0\ \forall \alpha \in \Delta'$, implies $C_j\alpha > 0\ \forall\alpha \in \Delta'$. Therefore, for each $j \in I$, we have $C_j \in R$. Since I is maximal, $R = \langle C_j \mid j \in I \rangle$.

Let $\alpha_j = 2e_{jk}$. Since $C_j(\alpha_j) < 0$ and $C_j \in R$, we conclude that $\alpha_j \notin \Delta'$. Since $\sigma_j \simeq \widetilde{\sigma}_j$, Lemma 6.2, implies $\mathcal{X}_{k,m}(\sigma_j \otimes \rho)$ must be satisfied. Suppose $\sigma_l \simeq \sigma_j$ for some $l > j$. Let $\alpha = e_{jk} - e_{(l-1)k+1}$. Then, by Lemma 6.2, $\alpha \in \Delta'$. On the other hand, $C_j\alpha < 0$, which contradicts our assumption that $w \in R$. Thus, if $C_j \in R$, then $\sigma_l \not\simeq \sigma_j$, $\forall l > j$.

Conversely, suppose that $\sigma_j \simeq \tilde{\sigma}_j$, and $\sigma_l \not\simeq \sigma_j$, $\forall l > j$. Then (6.5c) implies that $C_j \in W(\sigma)$. Note that, $R(C_j) = \{e_{jk} \pm e_{lk+1}\}_{l \geq j} \cup \{\alpha_j\}$. Therefore, if $\mathcal{X}_{k,m}(\sigma \otimes \rho)$ holds, then Lemma 6.2 implies $\Delta' \cap R(C_j) = \emptyset$. Hence, $C_j \in R$. Since I is maximal, $j \in I$. Consequently, I is in one to one correspondence with the equivalence classes of σ_i such that $\mathcal{X}_{k,m}(\sigma \otimes \rho)$ holds. □

As we remarked at the outset of this section the statements and proofs of Lemmas 6.1-6.3, and Theorem 6.4, can be extended, almost verbatim, to $SO(2n+1)$, and the basic parabolic subgroups we need to consider for $SO(2n)$. We state this explicitly.

THEOREM 6.5.

(1) *Let $G = SO(2n+1)$. Suppose $P = MN$ is a basic parabolic subgroup of G, with $M \simeq GL(k)^r \times SO(2m+1)$. Suppose $G' = SO(2(k+m)+1)$ and $P' = M'N'$, with $M' \simeq GL(k) \times SO(2m+1)$. Let $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho \in \mathcal{E}_2(M)$. Then, $R(\sigma) \simeq \mathbb{Z}_2^d$, where d is the number of inequivalent σ_i with $\mathrm{Ind}_{P'}^{G'}(\sigma_i \otimes \rho)$ reducible.*

(2) *Let $G = SO(2n)$. Suppose k is even, and $P = MN$ is a basic parabolic subgroup of G, with $M \simeq GL(k)^r \times SO(2m)$. Let $G' = SO(2(k+m))$ and $P' = M'N'$, with $M' \simeq GL(k) \times SO(2m)$. Suppose $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho \in \mathcal{E}_2(M)$. Then, $R(\sigma) \simeq \mathbb{Z}_2^d$, where d is the number of inequivalent σ_i with $\mathrm{Ind}_{P'}^{G'}(\sigma_i \otimes \rho)$ reducible.*

(3) *Let $G = SO(2n)$. Let k be odd, and $m > 0$. Suppose $P = MN$ is a basic parabolic subgroup of G, with $M \simeq GL(k)^r \times SO(2m)$. Let $G' = SO(2(k+m))$, and $P' = M'N'$, with $M' \simeq GL(k) \times SO(2m)$. Suppose $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho \in \mathcal{E}_2(M)$. We further suppose that $c_n(\rho) \simeq \rho$. Then, $R(\sigma) \simeq \mathbb{Z}_2^d$, where d is the number of inequivalent σ_i with $\mathrm{Ind}_{P'}^{G'}(\sigma_i \otimes \rho)$ reducible.*

We turn now to the two instances in which we were unable to reduce the R-group to a product of R-groups associated to basic parabolic subgroups. Namely, cases (3) and (4) for $SO(2n)$, listed at the beginning of this section. First suppose

$$M = GL(m_1)^{n_1} \times \cdots \times GL(m_r)^{n_r} \times GL(1)^{n_{r+1}},$$

with each m_i odd. In all our arguments we assume that $b = \sum n_i m_i > 2$. If $b = 2$, then the same arguments work, by replacing conjugation in W_θ, with conjugation by an appropriate element of $O(2n) \setminus SO(2n)$.

We assume that θ is of the form (2.2). Recall that the reduced roots were given in Lemma 5.4. Let $A = A_\theta$, $M = M_\theta$, and $P = P_\theta$. We recall that

$$A = \left\{\operatorname{diag}\left\{\lambda_{11}, \ldots, \lambda_{(r+1)n_{r+1}}, \lambda_{(r+1)n_{r+1}}^{-1}, \ldots, \lambda_{11}^{-1}\right\}\right\},$$

where λ_{ij} means $\lambda_{ij} I_{m_i}$.

By Lemma 2.5, $W_\theta \simeq (\prod_{i=1}^{r+1} S_{n_i}) \ltimes \mathbb{Z}_2^{n_1+\cdots+n_{r+1}-1}$. We now make this isomorphism explicit. Let $C_i^j : \lambda_{ij} \longmapsto \lambda_{ij}^{-1}$ be the product of m_i sign changes described in Section 2. Then we showed that

$$\mathbb{Z}_2^{n_1+\cdots+n_{r+1}-1} \simeq \left\langle C_i^j C_k^l \mid 1 \le i, k \le r+1 \right\rangle \subset W_\theta. \tag{6.7}$$

For the permutations, we adopt the following notation. Let

$$(j\ \ j+1)_i : \lambda_{ij} \longleftrightarrow \lambda_{i(j+1)}. \tag{6.8}$$

Suppose $g \in M$. Then $g = (g_{ij})$, with each $g_{ij} \in GL(m_i)$. Then, following the actions given in (6.7) and (6.8),

$$(jk)_i : g_{ij} \longleftrightarrow g_{ik}, \tag{6.9}$$

and

$$C_i^j C_k^l : \begin{cases} g_{ij} \longleftrightarrow {}^t g_{ij}^{-1} \\ g_{kl} \longleftrightarrow {}^t g_{kl}^{-1}. \end{cases} \tag{6.10}$$

If $\sigma \in \mathcal{E}_2(M)$, then $\sigma \simeq \otimes_{i=1}^{r+1} \sigma_i$, with each $\sigma_i \in \mathcal{E}_2(GL(m_i)^{n_i})$. We decompose this further; $\sigma_i \simeq \otimes_{j=1}^{n_i} \sigma_{ij}$, with each $\sigma_{ij} \in \mathcal{E}_2(GL(m_i))$. Making direct computations from (6.9) and (6.10), we find that

$$(jk)_i \in W(\sigma) \text{ if and only if } \sigma_{ij} \simeq \sigma_{ik}, \tag{6.11a}$$

$$C_i^j C_k^l \in W(\sigma) \text{ if and only if } \sigma_{ij} \simeq \tilde{\sigma}_{ij} \text{ and } \sigma_{kl} \simeq \tilde{\sigma}_{kl}, \tag{6.11b}$$

and

$$(jk)_i C_i^j C_i^k \in W(\sigma) \text{ if and only if } \sigma_{ij} \simeq \tilde{\sigma}_{ik}. \tag{6.11c}$$

Remark. If $(jk)_i C_i^j C_l^d \in W(\sigma)$ then $\sigma_{ij} \simeq \sigma_{ik} \simeq \tilde{\sigma}_{ij}$ and $\sigma_{ld} \simeq \tilde{\sigma}_{ld}$. Therefore, $(jk)_i$, $C_i^j C_i^k$, and $C_i^k C_l^d \in W(\sigma)$. Consequently, conditions (6.11a-c) are the necessary and sufficient conditions for σ to ramify in G. □

Recall that $b_i = \sum_{j<i} n_j m_j$. Let $\alpha = e_{b_i+jm_i} - e_{b_k+lm_k+1}$. Choose (k',l') so that $(k',l') \neq (i,j),(k,l)$. (Note that our assumption that $b > 2$ guarantees that such k',l' exist.) Then,

$$C_k^{l+1} C_{k'}^{l'+1} \alpha = e_{b_i+jm_i} + e_{b_k+lm_k+1}, \tag{6.12}$$

so these two reduced roots are W_θ conjugate. If $b = 2$, one can use an an appropriate representative of the outer automorphism to show that the Plancherel measures along these two roots are related. Therefore, up to conjugation, we need only compute the rank 1 Plancherel measures in a few cases.

LEMMA 6.6.

(1) $\alpha = e_{b_i+jm_i} - e_{b_i+km_i+1} \in \Delta'$ *if and only if* $\sigma_{ij} \simeq \sigma_{i(k+1)}$.

(2) $\alpha = e_{b_i+jm_i} + e_{b_i+km_i+1} \in \Delta'$ *if and only if* $\sigma_{ij} \simeq \tilde{\sigma}_{i(k+1)}$.

(3) $\beta = e_{b_i+jm_i} \pm e_{b_k+lm_k+1} \notin \Delta'$, $\forall i \neq k$, $\forall j$, l, σ.

(3) $\nu = e_{b_i+jm_i-1} + e_{b_i+jm_i} \notin \Delta'$, $\forall i,j,\sigma$.

Proof. Suppose $\alpha = e_{m_1} - e_{m_1+1}$. By Lemma 2.1,

$$M_\alpha \simeq GL(2m_1) \times GL(m_1)^{n_1-2} \times \prod_{j=2}^{r+1} GL(m_j)^{n_j}.$$

Therefore, by Lemma 3.4, $\alpha \in \Delta'$ if and only if $\sigma_{11} \simeq \sigma_{12}$. Conjugating in W_θ gives (1). By (6.12), (1) implies (2).

Let $\beta = e_{b_i} - e_{b_i+1}$, for some $i > 1$. Then, by the remark following Lemma 3.4, $W(M_\beta/A_\theta) = \{1\}$. Therefore, $\beta \notin \Delta'$. Then conjugation in W_θ gives (3).

For (4), we can conjugate by an appropriate permutation in $W(G/A_0)$, so that $i = r+1$. We do not assume that $m_{r+1} = 1$ in this case. Then we take $\nu = e_{n-1}+e_n$. Using Lemma 2.1, we see that

$$M_\nu = \left(\prod_{j\neq i} GL(m_j)^{n_j}\right) \times GL(m_i)^{n_i-1} \times SO(2m_i).$$

If $g \in M_\nu$, then g has the form $g = \mathrm{diag}\{D, Q, {}^\tau D^{-1}\}$, where

$$D \in \left\{ \mathrm{diag}\left\{ \{g_{jk}\}_{j\neq i}, \{g_{il}\}_{l=1}^{n_i-1} \right\} \;\middle|\; g_{jk} \in GL(m_j), g_{il} \in GL(m_i) \right\},$$

and $Q \in SO(2m_i)$. Therefore, $W(M_\nu/A_\theta) = W(SO(2m_i)/A')$, where

$$A' = \left\{ \begin{pmatrix} \lambda I_{m_i} & \\ & \lambda^{-1} I_{m_i} \end{pmatrix} \right\}.$$

Since m_i is odd, this last Weyl group is trivial, and therefore, $\nu \notin \Delta'$. Using permutations in S_{n_i} gives (4). $\square$

We now use a Keys argument.

LEMMA 6.7. *If $sc \in R$, with $s \in \prod_i S_{n_i}$, and $c \in \mathbb{Z}_2^{n_1+\cdots n_{r+1}-1}$, then $s = 1$.*

Proof. Since the sign changes C_i^j act independently on disjoint cycles, we can assume that $s \in S_{n_i}$ for some i, and that s is a cycle. We assume that $s = (1\ldots j)_i$. Up to conjugation by an even number of sign changes, we can assume that

$$c|_{\{b_i+1,\ldots,b_i+jm_i\}} = 1, C_i^j, \text{ or } C_i^{j-1}C_i^j.$$

If $c|_{\{b_i+1,\ldots b_i+jm_i\}} = 1$ or C_i^j, then, by Lemma 6.6, we can use the arguments of Lemma 6.3 . If $c|_{\{b_i+1,\ldots b_i+jm_i\}} = C_i^{j-1}C_i^j$ then $\sigma_{i1} \simeq \sigma_{i2} \simeq \cdots \simeq \sigma_{i(j-1)} \simeq \widetilde{\sigma}_{ij}$. By Lemma 6.6, $e_{b_i+m_i} + e_{b_i+(j-1)m_i+1} \in \Delta'$. Then $sc(e_{b_i+m_i} + e_{b_i+(j-1)m_i+1}) < 0$, which is a contradiction. Therefore, $s = 1$. $\square$

THEOREM 6.8. *Suppose $G = SO(2n)$, and $P = MN$ is a parabolic subgroup with*

$$M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_r)^{n_r} \times GL(1)^{n_{r+1}},$$

with each m_i odd. Let $\sigma \in \mathcal{E}_2(M)$ and write $\sigma \simeq \otimes_{i=1}^{r+1} \otimes_{j=1}^{n_i} \sigma_{ij}$, with each $\sigma_{ij} \in \mathcal{E}_2(GL(m_i))$. Let $m(\sigma)$ be the number of inequivalent σ_{ij} satisfying $\sigma_{ij} \simeq \widetilde{\sigma}_{ij}$. Then,

$$R \simeq \begin{cases} \{1\} & m(\sigma) = 0 \\ \mathbb{Z}_2^{m(\sigma)-1} & m(\sigma) \geq 1. \end{cases}$$

Proof. Let $m = n_1 + \cdots n_{r+1} - 1$. By Lemma 6.7, $R \subset \mathbb{Z}_2^m$. If $m(\sigma) = 0$, then $W(\sigma) \cap \mathbb{Z}_2^m = \{1\}$. Therefore, $R = \{1\}$. Suppose that $m(\sigma) \geq 1$. For $w \in W_\theta$, let $R(w) = \{\alpha \in \Phi(P,A) \mid w\alpha < 0\}$. Then $w \in R$ if and only if, $w \in W(\sigma)$ and $R(w) \cap \Delta' = \emptyset$. Suppose $(i,j) \neq (k,l)$, and $\sigma_{ij} \simeq \widetilde{\sigma}_{ij}$; $\sigma_{kl} \simeq \widetilde{\sigma}_{kl}$. Then, (6.11b) implies $C_i^jC_k^l \in W(\sigma)$. Note that, if $\sigma_{it} \simeq \sigma_{ij}$, for some $t > j$, then, by Lemma 6.6, $\alpha = e_{b_i+jm_i} - e_{b_i+(t-1)m_i+1} \in \Delta'$, while $C_i^jC_k^l\alpha < 0$. In this case, $C_i^jC_k^l \notin R$. On the other hand, if $\sigma_{ij} \not\simeq \sigma_{it}$ for all $t > j$, and $\sigma_{kl} \not\simeq \sigma_{ku}$ for all $u > l$, then Lemma 6.6 implies $\Delta' \cap R(C_i^jC_k^l) = \emptyset$. Thus, under these conditions, $C_i^jC_k^l \in R$. Let

$$\mathcal{B} = \left\{ (i,j) \;\middle|\; \begin{array}{c} \sigma_{ij} \simeq \widetilde{\sigma}_{ij} \text{ and} \\ \sigma_{it} \not\simeq \sigma_{ij},\ \forall t > j \end{array} \right\}.$$

Then

$$R = \left\langle C_i^j C_k^l \mid (i,j),(k,l) \in \mathcal{B} \right\rangle \cap W_\theta \simeq \mathbb{Z}_2^{m(\sigma)-1}. \qquad \square$$

Remark. Notice that, even though we were not able to reduce this R-group via a product formula, we were able to compute it explicitly. In Theorems 6.4 and 6.5, we need further information to compute the R-groups explicitly from the inducing data. $\square$

Suppose that $m \geq 1$ and $M \simeq GL(m_1)^{n_1} \times \cdots \times GL(1)^{n_{r+1}} \times SO(2m)$, with each m_i odd, $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_{r+1} \otimes \rho$, and $c_n(\rho) \not\simeq \rho$. We assume that θ is of the form (2.2), with each m_i odd. Let $A = A_\theta$, $M = M_\theta$, and $P = P_\theta$. Recall that

$$A = \left\{ \operatorname{diag}\left\{ \lambda_{11}, \ldots, \lambda_{(r+1)n_{r+1}}, I_{2m}, \lambda_{(r+1)n_{r+1}}^{-1}, \ldots, \lambda_{11}^{-1} \right\} \right\}. \tag{6.13}$$

By Lemma 2.6,

$$W_\theta \simeq \prod_{i \neq r} S_{n_i} \ltimes \mathbb{Z}_2^{n_i}. \tag{6.14}$$

This isomorphism is given by the the following actions on (6.13):

$$(j \;\; j+1)_i : \lambda_{ij} \longleftrightarrow \lambda_{i(j+1)},$$

and

$$C_i^j c_n : \lambda_{ij} \longmapsto \lambda_{ij}^{-1}.$$

Recall that the action of $C_i^j c_n$ on M is given by (5.5). Since $c_n(\rho) \not\simeq \rho$, no single sign change $C_i^j c_n \in W(\sigma)$. Therefore, conditions (6.11a-c) describe the necessary and sufficient conditions for $W(\sigma) \neq \{1\}$.

LEMMA 6.9.

(1) $e_{b_i+jm_i} - e_{b_i+km_i+1} \in \Delta'$ *if and only if* $\sigma_{ij} \simeq \sigma_{i(k+1)}$.

(2) $e_{b_i+jm_i} + e_{b_i+km_i+1} \in \Delta'$ *if and only if* $\sigma_{ij} \simeq \tilde{\sigma}_{i(k+1)}$.

(3) $e_{b_i+jm_i} \pm e_{b_k+lm_k+1} \notin \Delta'$, $\forall i \neq k$, $\forall \sigma$.

(4) $e_{b_i+jm_i} - e_{n-m+1} \notin \Delta'$, $\forall i,j \, \forall \sigma$.

Proof. The proofs of (1)-(3) are the same as in Lemma 6.6. For (4), assume $i = r+1$, and let $\nu = e_{n-m} - e_{n-m+1}$. Then, following the same line of reasoning as in (4) of Lemma 6.6, we see that $W(M_\nu/A_\theta) \simeq W(SO(2(m_i+m))/A')$, where

$$A' = \left\{ \begin{pmatrix} \lambda I_{m_i} & & \\ & I_{2m} & \\ & & \lambda^{-1} I_{m_i} \end{pmatrix} \right\}.$$

Therefore, $W(M_\nu/A_\theta) = \{1, C_i^{n_i} c_n\}$. Thus, σ ramifies in M_ν if and only if $\sigma_{in_i} \simeq \tilde{\sigma}_{in_i}$ and $c_n(\rho) \simeq \rho$. However, by assumption, this last condition does not hold. Thus, σ never ramifies in M_ν, and (4) follows. □

The Keys argument of Lemma 6.7 works verbatim to give the next lemma. We then apply the argument of Theorem 6.8, to explicitly describe R.

LEMMA 6.10. *If $sc \in R$, with $s \in \prod_i S_{n_i}$ and $c \in \mathbb{Z}_2^{n_1+\cdots+n_{r-1}+n_{r+1}}$, then $s = 1$.*

THEOREM 6.11. *Suppose $G = SO(2n)$, and $P = MN$ is a parabolic subgroup with*

$$M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_{r-1})^{n_{r-1}} \times GL(1)^{n_{r+1}} \times SO(2m),$$

with each m_i odd. Let $\sigma \in \mathcal{E}_2(M)$, and write

$$\sigma \simeq \left(\bigotimes_{i \neq r} \bigotimes_{j=1}^{n_i} \sigma_{ij} \right) \otimes \rho,$$

with each $\sigma_{ij} \in \mathcal{E}_2(GL(m_i))$, $\rho \in \mathcal{E}_2(SO(2m))$, and $c_n(\rho) \not\simeq \rho$. Let $m(\sigma)$ be the number of inequivalent σ_{ij} satisfying $\sigma_{ij} \simeq \tilde{\sigma}_{ij}$. Then

$$R \simeq \begin{cases} \{1\} & m(\sigma) = 0 \\ \mathbb{Z}_2^{m(\sigma)-1} & m(\sigma) \geq 1. \end{cases}$$

THEOREM 6.12. *Let $G = SO(2n+1)$, $Sp(2n)$, or $SO(2n)$. Let $P = MN$ be any parabolic subgroup, and let $\sigma \in \mathcal{E}_2(M)$. Suppose that σ is generic. Then* $\mathrm{Ind}_P^G(\sigma)$ *decomposes with multiplicity one.*

Proof. By Theorems 4.9, 4.18, 5.8, 5.9, 5.16, 5.19, 5.20, 6.4, 6.5, 6.8, and 6.11, the R-group $R(\sigma)$ is abelian. Since σ is generic, Theorem 1.9 implies multiplicity one holds. □

Remark. By [5,14] every discrete series representation of $GL(n)$ is generic. Therefore, the hypotheses of Theorem 6.12 are always satisfied in case 1. We write this down explicitly below. □

COROLLARY 6.13. *Let G and $P = MN$ be as in Theorem* 6.12. *Suppose that* $M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_t)^{n_t}$. *Then for any* $\sigma \in \mathcal{E}_2(M)$, $\mathrm{Ind}_P^G(\sigma)$ *decomposes with multiplicity one.*

Remark. In [10], Herb extends Theorem 6.12 to the case where σ is not generic. □

7. Some Results for Maximal Parabolics. We will describe some results for maximal parabolic subgroups. For all of these results we require σ to be a supercuspidal representation.

THEOREM 7.1. (Shahidi [34]) *Suppose* $G = Sp(4)$ *and* $\theta = \{e_1 - e_2\}$. *Let* $P = P_\theta$. *Then* $M_\theta \simeq GL(2)$. *Let* $\sigma \in {}^\circ\mathcal{E}(M_\theta)$ *and let* ω_σ *be the central character of* σ. *Then* $\mathrm{Ind}_P^G(\sigma)$ *is reducible if and only if* $\sigma \simeq \widetilde{\sigma}$ *and* $\omega_\sigma \not\equiv 1$.

Definition 7.2. Let G be a connected reductive p-adic group and let $P = MN$ be a parabolic subgroup. Let σ be a non-unitarizable irreducible admissible supercuspidal representation of M. Then $\mathrm{Ind}_P^G(\sigma)$ is said to be in the *complementary series* if it is unitarizable.

THEOREM 7.3. *Suppose* $G = SO(2n)$ *with* $n > 1$ *an odd integer. Let* $P = P_\theta$ *with* $\theta = \Delta \setminus \{e_{n-1} + e_n\}$. *Thus,* $M_\theta \simeq GL(n)$. *Let* $\sigma \in {}^\circ\mathcal{E}(M_\theta)$. *Then* $\mathrm{Ind}_P^G(\sigma)$ *is irreducible. Moreover, there are no complementary series.*

Remark. This was actually noted in the proof of Lemma 6.6. □

Proof. We have

$$A_\theta = \left\{ \begin{pmatrix} \lambda I_n & \\ & \lambda^{-1} I_n \end{pmatrix} \right\}.$$

By Lemma 2.5, $W(G/A_\theta) = \{1\}$. Therefore, σ is unramified in G. By Theorem 1.1, $\mathrm{Ind}_P^G(\sigma)$ is irreducible. Theorem 8.1 of [33] shows that there can be no complementary series. □

One can use Theorem 7.3 to prove the following results.

THEOREM 7.4. (Shahidi [35])

(1) *Let* $G = Sp(2n)$ *with* $n > 1$ *an odd integer, and let* $P = MN$ *with* $M \simeq GL(n)$. *Let* $\sigma \in {}^\circ\mathcal{E}(M)$. *Then* $\mathrm{Ind}_P^G(\sigma)$ *is reducible if and only if* $\sigma \simeq \widetilde{\sigma}$.

(2) *Let* $G = SO(2n+1)$ *with* $n > 1$ *odd, and let* $P = MN$ *with* $M \simeq GL(n)$. *Let* $\sigma \in {}^\circ\mathcal{E}(M)$. *Then* $\mathrm{Ind}_P^G(\sigma)$ *is irreducible.*

We can use Theorem 7.4 to make the R-groups of Theorems 6.4 and 6.5a explicit in some cases.

COROLLARY 7.5. *Let k be odd and let $n = kr$, with $r \geq 1$.*

(1) *Suppose $G = Sp(2n)$, and $P = MN$, with $M \simeq GL(k)^r$. Let $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \in {}^\circ\mathcal{E}(M)$, with each $\sigma_i \in {}^\circ\mathcal{E}(GL(k))$. Then, $R \simeq \mathbb{Z}_2^d$, where d is the number of inequivalent, self contragredient σ_i.*

(2) *Let $G = SO(2n+1)$, and $P = MN$, with $M \simeq GL(k)^r$. Then, for any $\sigma \in {}^\circ\mathcal{E}(M)$, $\mathrm{Ind}_P^G(\sigma)$ is irreducible.*

Remark. Shahidi, [35], has determined the reducibility criteria in the case where $G = GL(n)$, for any n. These criteria come from the theory of twisted endoscopy of Kottwitz and Shelstad [24,25]. In more recent work, [31], Shahidi has explored the general maximal parabolic subgroup of $SO(2n)$, with n even. Here, the theory of twisted endoscopy also plays a key role. We hope to extend these results to cover the most general case for classical groups. This will be the subject of future joint work with Shahidi. $\square$

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