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REDUCIBILITY FOR NON-CONNECTED p-ADIC GROUPS, WITH G° OF PRIME INDEX

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ABSTRACT. We determine the structure of representations induced from discrete series of parabolic subgroups of quasi-split p-adic groups G with G/G° a cyclic group of prime order. We attach to each such representation an R-group which extends the definition of the Knapp-Stein R-group. We show that this R-group has the properties predicted by Arthur. We apply our results to the case of Orthogonal groups.

Introduction. Let F be a locally compact, non-discrete, nonarchimedean local field of characteristic zero. In [7, 8, 9] we described the component structure of those parabolically induced from discrete series representations of certain connected classical groups defined over F. Our method was to compute the Knapp-Stein R-groups which can arise. We were able to compute the R-groups because the action of the Weyl groups is well understood in these cases. There is another construction of the R-group which relies on the conjectural parameterization of discrete series L-packets [19]. Arthur has given an extension of this R-group construction to the case where G is disconnected [3]. Arthur suggests that this generalized R-group should play a role in determining the reducibility of induced representations. We examine some properties of induced representations for disconnected groups, under the assumption that the connected component is of prime index. We are able to construct an R-group, on the group side, and we show that it plays a similar role to that of the Knapp-Stein R-group in the connected case. We also note that modulo some very deep conjectures, this R-group is the one predicted by Arthur in [3].

Let $\mathbf{G}$ be a reductive quasi-split group, defined over F, and assume that $\mathbf{G}/\mathbf{G}^{\circ} \simeq \mathbb{Z}/p\mathbb{Z}$, with p prime. Let $G = \mathbf{G}(F)$, and $G^{\circ} = \mathbf{G}^{\circ}(F)$. Suppose $\mathbf{P} = \mathbf{MN}$ is a parabolic subgroup of $\mathbf{G}$ (see Section 1), and σ is a discrete series representation of $M = \mathbf{M}(F)$. Let $\mathbf{P}^{\circ} = \mathbf{P} \cap \mathbf{G}^{\circ}$, and suppose that $\mathbf{A}^{\circ}$ is the split component of $\mathbf{P}^{\circ}$. Suppose σ_0 is an irreducible subrepresentation of $\sigma|_{M^{\circ}}$. Let $i_{G,M}(\sigma)$ be the representation of G, unitarily induced from σ, and let $i_{G^{\circ},M^{\circ}}(\sigma_0)$ be the representation of G°, induced from σ_0. We write $\Pi_{\sigma}(G)$ and $\Pi_{\sigma_0}(G^{\circ})$ for the collection of equivalence classes of components of $i_{G,M}(\sigma)$ and $i_{G^{\circ},M^{\circ}}(\sigma_0)$ respectively. Using Frobenius reciprocity, and the theory of Gelbart and Knapp, we are able to describe the relationship between $i_{G,M}(\sigma)$ and $i_{G^{\circ},M^{\circ}}(\sigma_0)$. This is equivalent to describing the structure of the representation $i_{G,M^{\circ}}(\sigma_0) = \mathrm{Ind}_{G^{\circ}}^{G}\big(i_{G^{\circ},M^{\circ}}(\sigma_0)\big)$. It is this representation whose component structure should be related to Arthur's R-group. We break our results into several cases. First of all, it is possible that $\mathbf{M} \neq \mathbf{M}^{\circ}$, and $\sigma|_{M^{\circ}} = \sigma_0$.

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In this case the relationship is quite easy to describe, because $i_{G^\circ,M^\circ}(\sigma_0) = [i_{G,M}(\sigma)]|_{G^\circ}$. It follows that every component of $i_{G,M}(\sigma)$ restricts to G° irreducibly (Proposition 2.2). Thus, the structure of $i_{G,M^\circ}(\sigma_0)$ can be easily stated (Corollary 2.3). If $\mathbf{M} \neq \mathbf{M}^\circ$, and $\sigma|_{M^\circ}$ is reducible, then $i_{G,M^\circ}(\sigma_0) = i_{G,M}(\sigma)$. Here relationship between $i_{G,M}(\sigma)$ and $i_{G^\circ,M^\circ}(\sigma_0)$ depends on the action of the Weyl group $W(\mathbf{G},\mathbf{A}^\circ)$ on σ_0. If $w\sigma_0 \simeq \sigma_0$ implies $w \in W(\mathbf{G}^\circ,\mathbf{A}^\circ)$, then each component of $i_{G,M}(\sigma)$ restricts to G° reducibly, and there is a one-to-one correspondence between the components of $i_{G,M}(\sigma)$, and those of $i_{G^\circ,M^\circ}(\sigma_0)$ (Proposition 2.4). If $w\sigma_0 \simeq \sigma_0$, for some $w \in W(\mathbf{G},\mathbf{A}^\circ)\setminus W(\mathbf{G}^\circ,\mathbf{A}^\circ)$, then we can easily determine the dimension of $\mathrm{Hom}_G\big(i_{G,M^\circ}(\sigma_0), i_{G,M^\circ}(\sigma_0)\big)$ (Lemma 2.5). If $\mathbf{M} = \mathbf{M}^\circ$, then the extent of our description of $i_{G,M}(\sigma) = \mathrm{Ind}_{G^\circ}^G\big(i_{G^\circ,M}(\sigma)\big)$ is similar to that of the case where $\sigma|_{M^\circ}$ is reducible (Lemma 2.6).

We construct an R-group to reflect the structure of $i_{G,M^\circ}(\sigma_0)$. We call this the *Arthur R-group*, because of its connection with the group $R_{\psi,\sigma}$ predicted in [3, 7] (see Section 4). Let W' be the subgroup of $W(\mathbf{G}^\circ,\mathbf{A}^\circ)$ generated by the root reflections in the zeros of the Plancherel measures of σ_0. We can construct a group $R_G(\sigma_0)$ so that:

$$W_G(\sigma_0) = \{w \in W(\mathbf{G},\mathbf{A}^\circ) \mid w\sigma_0 \simeq \sigma_0\} = R_G(\sigma_0) \ltimes W'$$

(Lemma 2.7). Moreover, $R_{G^\circ}(\sigma_0) \subset R_G(\sigma_0)$, where $R_{G^\circ}(\sigma_0)$ is the Knapp-Stein R-group attached to $i_{G^\circ,M^\circ}(\sigma_0)$.

There is a 2-cocycle, η, of $R_{G^\circ}(\sigma_0)$ so that the commuting algebra $C(\sigma_0)$ of $i_{G^\circ,M^\circ}(\sigma_0)$ is isomorphic to the twisted group algebra $\mathbb{C}[R_{G^\circ}(\sigma_0)]_\eta$. Let $\tilde{R}_0$ be a central extension of $R_{G^\circ}(\sigma_0)$ by an abelian group Z over which η splits. Since $R_G(\sigma_0)/R_{G^\circ}(\sigma_0)$ is cyclic, there is a central extension $\tilde{R}$ of $R_G(\sigma_0)$ by Z with $\tilde{R}/\tilde{R}_0 \simeq R_G(\sigma_0)/R_{G^\circ}(\sigma_0)$. For a character ω of Z, we let $\Pi(\tilde{R}_0,\omega)$ be the equivalence classes of irreducible representations of $\tilde{R}_0$ which have Z-central character ω^{-1}. Then, for some character ω_{σ_0} of Z, there is a one to one correspondence between $\Pi(\tilde{R}_0,\omega_{\sigma_0})$ and $\Pi_{G^\circ}(\sigma_0)$ [1]. Moreover, if this correspondence is given by $\rho \longmapsto \pi_\rho$, then $\dim \mathrm{Hom}_{G^\circ}\big(\pi_\rho, i_{G^\circ,M^\circ}(\sigma_0)\big) = \dim \rho$. Arthur writes down the projections of $i_{G^\circ,M^\circ}(\sigma_0)$ onto its isotypic components by using the character theory of $\Pi(\tilde{R}_0,\omega_{\sigma_0})$. Using these projections we are able to describe the action of $R_G(\sigma_0)/R_{G^\circ}(\sigma_0) = \tilde{R}/\tilde{R}_0$ on the elements of $\Pi_{\sigma_0}(G^\circ)$ (Theorem 2.10). This leads to our main result (Theorem 2.11), which we restate below.

THEOREM A. *There is a bijective map $\tau \longmapsto \Pi_\tau$ between $\Pi(\tilde{R},\omega_{\sigma_0})$ and $\Pi_{\sigma_0}(G)$ such that:*

(1) $\dim \mathrm{Hom}_G(\Pi_\tau,\pi) = \dim \tau$;

(2) If $\rho \in \Pi(\tilde{R}_0,\omega_{\sigma_0})$, then $\pi_\rho \subset \Pi_\tau|_{G^\circ}$ if and only if $\rho \subset \tau|_{\tilde{R}_0}$. ∎

We then turn to the orthogonal group $O_n(F)$. Here we use the explicit description of $i_{G^\circ,M^\circ}(\sigma_0)$. Suppose n is even. In [8], we showed that there are non-negative integers d_1, d_2, and $d = d_1 + d_2$, such that

$$R_{G^\circ}(\sigma_0) = \begin{cases} \mathbb{Z}_2^{d-1} & \text{if } d_2 > 0 \\ \mathbb{Z}_2^{d} & \text{if } d_2 = 0 \text{ (Theorem 3.1).} \end{cases}$$

In Theorem 3.3, we show that $R_G(\sigma_0) \simeq \mathbb{Z}_2^d$ or $\mathbb{Z}_2^{d+1}$, with the latter occurring when $\mathbf{M} \neq \mathbf{M}^\circ$, $\sigma|_{M^\circ}$ is irreducible, and $\mathbf{M}^\circ$ satisfies one additional condition. Thus, $i_{G,M^\circ}(\sigma_0)$ has either 2^d or 2^{d+1} components, and is a multiplicity one representation (Corollary 3.4). The case where $G = G^\circ \times \mathbb{Z}/p\mathbb{Z}$ is subsumed by Proposition 2.2 (Lemma 3.5). The explicit description of the reducibility and multiplicity structure for $O_{2n+1}(F)$ then follows from the results of [8] (Corollary 3.6).

Finally, we make some remarks about the connection between the group $R_G(\sigma_0)$, and Arthur's conjectural group $R_{\psi,\sigma}$. Shelstad has shown [19] that, for real groups, the Knapp-Stein R-group, $R_{G^\circ}(\sigma_0)$, can be computed in terms of the Langlands parameterization. It is conjectured that Shelstad's construction will be valid in general. Since the parameterization is understood when M^0 is a torus, [12, 16], Keys was able to confirm that Shelstad's R-group, and the Knapp-Stein R-group are isomorphic in some cases [13]. However, the parameterization is far from understood in general, and therefore, the conjecture on R-groups is also far from understood. Arthur has extended the construction of Shelstad to a group, $R_{\psi,\sigma}$ which should reflect the structure of $i_{G,M^\circ}(\sigma_0)$. If we assume that Shelstad's conjecture holds, then we can show that $R_G(\sigma_0) \simeq R_{\psi,\sigma}$ (Proposition 4.1). Thus, by Theorem A, $R_{\psi,\sigma}$ would have the properties conjectured by Arthur in [3]. Though our results are only for $\mathbf{G}/\mathbf{G}^\circ$ of prime order, we hope to be able to extend them in the near future.

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1. **Preliminaries.** Let F be a locally compact, non-discrete, nonarchimedean local field of characteristic zero and residual characteristic q. Suppose $\mathbf{G}$ is a reductive quasi-split algebraic group defined over F. Let $\mathbf{G}^\circ$ be the connected component of the identity in $\mathbf{G}$. We will assume that, if $\mathbf{G} \neq \mathbf{G}^\circ$, then $\mathbf{G}/\mathbf{G}^\circ$ is cyclic of prime order. We let $\chi\colon \mathbf{G}/\mathbf{G}^\circ :\longrightarrow \mathbb{C}^\times$ be a generator of $\widehat{\mathbf{G}/\mathbf{G}^\circ}$.

Let $\mathbf{B}^\circ = \mathbf{T}^\circ\mathbf{U}$ be a Borel subgroup of $\mathbf{G}^\circ$, and denote by $\Phi(\mathbf{G}^\circ, \mathbf{T}^\circ)$ the roots of $\mathbf{T}^\circ$ in $\mathbf{G}^\circ$. Let Δ be the simple roots with respect to this choice of Borel subgroup. If $\theta \subset \Delta$, then there is a parabolic subgroup, $\mathbf{P}_\theta^\circ \supset \mathbf{B}^\circ$, of $\mathbf{G}^\circ$ attached to θ [22]. Moreover, any parabolic subgroup of $\mathbf{G}^\circ$ containing $\mathbf{B}^\circ$ arises from this construction. A subgroup $\mathbf{P}$ of $\mathbf{G}$ is called a *parabolic subgroup* if $\mathbf{P} = N_{\mathbf{G}}(\mathbf{P}^\circ)$, for some parabolic subgroup $\mathbf{P}^\circ$ of $\mathbf{G}^\circ$. Since $\mathbf{P}^\circ$ is conjugate in $\mathbf{G}^\circ$ to some $\mathbf{P}_\theta^\circ$, $\mathbf{P}$ is conjugate in $\mathbf{G}$ to some $\mathbf{P}_\theta = N_{\mathbf{G}}(\mathbf{P}_\theta^\circ)$. Note that, by our assumption on $\mathbf{G}/\mathbf{G}^\circ$, we either have $\mathbf{P}_\theta = \mathbf{P}_\theta^\circ$, or $\mathbf{P}_\theta$ intersects every connected component of $\mathbf{G}$. Suppose $\mathbf{A}^\circ$ is the split component of $\mathbf{P}^\circ$, *i.e.*, the maximal split torus in the center of $\mathbf{M}^\circ$. Then $\mathbf{P}^\circ = \mathbf{M}^\circ\mathbf{N}$, with $\mathbf{M}^\circ = Z_{\mathbf{G}^\circ}(\mathbf{A}^\circ)$. If $\tilde{\mathbf{M}} = N_{\mathbf{G}}(\mathbf{M}^\circ)$, then $\mathbf{P} = \mathbf{M}\mathbf{N}$, with $\mathbf{M} = \tilde{\mathbf{M}} \cap \mathbf{P}$. We call the group $\mathbf{M}$ the *Levi component* of $\mathbf{P}$, and call a group a *Levi subgroup* of $\mathbf{G}$ if it is the Levi component of a parabolic subgroup of $\mathbf{G}$. Note that if $\mathbf{M} \neq \mathbf{M}^\circ$, then $\mathbf{M}/\mathbf{M}^\circ \simeq \mathbb{Z}/p\mathbb{Z}$. Let $\mathbf{A}$ be the maximal split torus in the centralizer of $\mathbf{M}$ in $\mathbf{M}^\circ$. Then $\mathbf{A} \subset \mathbf{A}^\circ$, and $\mathbf{A}$ is called the *split component* of $\mathbf{P}$.

Suppose that $g \in N_{\mathbf{G}}(\mathbf{M}^\circ) \setminus N_{\mathbf{G}^\circ}(\mathbf{M}^\circ)$. Then $g^{-1}\mathbf{T}^\circ g$ is a maximal torus of $\mathbf{M}^\circ$. Therefore, there is some $m_0 \in \mathbf{M}^\circ$, with $m_0^{-1}g^{-1}\mathbf{T}^\circ g m_0 = \mathbf{T}^\circ$. So, without loss of generality, we may assume that $g \in N_{\mathbf{G}}(\mathbf{T}^\circ)$. Note that if $a \in \mathbf{A}^\circ$, and $m_0 \in \mathbf{M}^\circ$, then, since $gm_0g^{-1} \in \mathbf{M}^\circ$, we have $(g^{-1}ag)m_0(g^{-1}a^{-1}g) = m_0$. Thus, $g^{-1}\mathbf{A}^\circ g$ is a split torus in $Z(\mathbf{M}^\circ)$, and hence $g^{-1}\mathbf{A}^\circ g = \mathbf{A}^\circ$.

We let $W(\mathbf{G}^\circ, \mathbf{A}^\circ) = N_{\mathbf{G}^\circ}(\mathbf{A}^\circ)/\mathbf{M}^\circ$ be the Weyl group of $\mathbf{G}^\circ$ with respect to $\mathbf{A}^\circ$. We define $W(\mathbf{G}, \mathbf{A}^\circ)$ to be $N_{\mathbf{G}}(\mathbf{A}^\circ)/\mathbf{M}^\circ$. We call $W(\mathbf{G}, \mathbf{A}^\circ)$ the *Weyl group* of $\mathbf{G}$ with respect to $\mathbf{A}^\circ$. We have seen that if $\mathbf{M} \neq \mathbf{M}^\circ$, then $N_{\mathbf{G}}(\mathbf{A}^\circ) \cap (\mathbf{M} \setminus \mathbf{M}^\circ) \neq \emptyset$. Let $g \in N_{\mathbf{G}}(\mathbf{A}^\circ) \cap (\mathbf{M} \setminus \mathbf{M}^\circ)$. Then g represents a class in $W(\mathbf{G}, \mathbf{A}^\circ) \setminus W(\mathbf{G}^\circ, \mathbf{A}^\circ)$. Moreover, $g^p \in \mathbf{M}^\circ$, while clearly $g^k \notin \mathbf{M}^\circ$, for $1 \leq k \leq p-1$. Thus, when $\mathbf{M} \neq \mathbf{M}^\circ$, we have $W(\mathbf{G}, \mathbf{A}^\circ) = W(\mathbf{G}^\circ, \mathbf{A}^\circ) \ltimes \mathbb{Z}/p\mathbb{Z}$.

Recall that $\mathbf{G}$ acts on $\mathbf{G}^\circ$ by conjugation, and this action is represented by a graph automorphism of the Dynkin diagram for the root system $\Phi(\mathbf{G}^\circ, \mathbf{T}^\circ)$. If $g \in W(\mathbf{G}, \mathbf{T}^\circ)$, then we also denote by g the associated action on $\Phi(\mathbf{G}^\circ, \mathbf{T}^\circ)$.

For a locally compact, totally disconnected, topological group H, we denote by $\mathcal{E}_c(H)$ the equivalence classes of irreducible admissible, representations of H. We let $\mathcal{E}_2(H)$ be the collection of equivalence classes of discrete series representations, and let $\mathcal{E}_t(H)$ be the collection of irreducible tempered representations of H.

Let $P = MN$ be a parabolic subgroup of G, *i.e.* $P = \mathbf{P}(F)$, with $\mathbf{P} \subset \mathbf{G}$. Suppose $\sigma \in \mathcal{E}_c(M)$ and assume that σ acts on a vector space V. Let δ_P be the modular function of P. Denote by $V(\sigma)$ the space of smooth functions from G to V satisfying $f(mng) = \delta_P^{1/2}(m)\sigma(m)f(g)$, for all $m \in M$, $n \in N$, and $g \in G$. Then G acts on $V(\sigma)$ by right translations, and we call this the *representation* of G unitarily induced from σ. We denote this representation by $\mathrm{Ind}_P^G(\sigma)$. If σ is unitary, the class of $\mathrm{Ind}_P^G(\sigma)$ depends only on M, and not on the choice of N, and in this case we may write $i_{G,M}(\sigma)$ for $\mathrm{Ind}_P^G(\sigma)$. Suppose σ is unitary and σ_0 is an irreducible component of $\sigma|_{M^\circ}$. We denote by $i_{G,M^\circ}(\sigma_0)$ the representation $\mathrm{Ind}_{P^\circ}^G(\sigma_0) = \mathrm{Ind}_{G^\circ}^G\left(i_{G^\circ,M^\circ}(\sigma_0)\right)$. A straightforward computation shows that, if $\mathbf{M} \neq \mathbf{M}^\circ$, then $i_{G,M}(\sigma)|_{G^\circ} \simeq i_{G^\circ,M^\circ}(\sigma|_{M^\circ})$. Thus, the structure of $i_{G,M}(\sigma)$, $i_{G^\circ,M^\circ}(\sigma_0)$, and $i_{G,M^\circ}(\sigma_0)$ are all closely related. When $\sigma \in \mathcal{E}_2(M)$, determining the structure of $i_{G,M}(\sigma)$ in terms of the number of components and multiplicities, is fundamental to understanding the representation theory of G. Moreover, understanding the commuting algebra of intertwining operators $C_G(\sigma_0)$ of $i_{G,M^\circ}(\sigma_0)$ is an important aspect of the twisted trace formula [2, 3].

We review the theory of intertwining operators and R-groups. We first concentrate on the case $\mathbf{G} = \mathbf{G}^\circ$, and then discuss the extensions of this theory which exist for non-connected groups, along with some conjectures of Arthur [3].

Suppose $\mathbf{G} = \mathbf{G}^\circ$. Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$, with split component $\mathbf{A}$. For $\sigma \in \mathcal{E}_c(M)$ and $w \in W(\mathbf{G}, \mathbf{A})$, we define $w\sigma$ by $w\sigma(m) = \sigma(\tilde{w}^{-1}m\tilde{w})$, where $\tilde{w}$ is any representative for w. This gives an action of $W(\mathbf{G}, \mathbf{A})$ on $\mathcal{E}_c(M)$. Let $W_G(\sigma) = \{w \in W(\mathbf{G}, \mathbf{A}) \mid w\sigma \simeq \sigma\}$. We may write $W(\sigma)$ for $W_G(\sigma)$ if G is implicit. We let $\mathfrak{a} = \mathrm{Hom}\big(X(\mathbf{M})_F, \mathbb{R}\big)$, where $X(\mathbf{M})_F$ is the group of F-rational characters of $\mathbf{M}$. Then $\mathfrak{a}$ is the real Lie algebra of A. Its dual $\mathfrak{a}^*$ is given by $X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R}$, and the complexified dual is $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$. There is a homomorphism $H_P\colon M \longrightarrow \mathfrak{a}$, given by $q^{\langle \nu, H_P(m)\rangle} = |\nu(m)|_F$, for each $\nu \in X(\mathbf{M})_F$, and all $m \in M$, [10].

For any $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, and $\sigma \in \mathcal{E}_2(M)$, we let $I(\nu,\sigma) = \mathrm{Ind}_P^G(\sigma \otimes q^{\langle \nu, H_P() \rangle})$. If $w \in W(\mathbf{G},\mathbf{A})$, we let $\mathbf{N}_w = \mathbf{U} \cap \tilde{w}^{-1}\bar{\mathbf{N}}\tilde{w}$, where $\bar{\mathbf{N}}$ is the unipotent radical opposed to $\mathbf{N}$. We formally define an operator by

$$A(\nu,\sigma,\tilde{w})f(g) = \int_{N_w} f(\tilde{w}^{-1}ng)\,dn. \tag{1.1}$$

If $A(\nu,\sigma,\tilde{w})$ converges for every choice of f and g, then we say that $A(\nu,\sigma,\tilde{w})$ converges. If $A(\nu,\sigma,\tilde{w})$ converges, then it defines an intertwining operator between $I(\nu,\sigma)$ and $I(w\nu, w\sigma)$.

THEOREM 1.1 (HARISH-CHANDRA). *Let $w \in W(\mathbf{G},\mathbf{A})$, and $\sigma \in \mathcal{E}_2(M)$. Let $P' = \tilde{w}^{-1}P\tilde{w}$. Then $A(\nu,\sigma,\tilde{w})$ converges for ν in the positive Weyl chamber, and can be extended to a meromorphic function of ν on $\mathfrak{a}_{\mathbb{C}}^*$. Moreover, there is a complex number $\mu(\nu,\sigma,w)$ so that*

$$A(\nu,\sigma,\tilde{w})A(w\nu,w\sigma,\tilde{w}^{-1}) = \mu(\nu,\sigma,w)^{-1}\gamma_w(G/P)\gamma_{w^{-1}}(G/P'), \tag{1.2}$$

where the constant $\gamma_w(G/P)$ is defined in [10]. Moreover, $\nu \longmapsto \mu(\nu,\sigma,w)$ is meromorphic on $\mathfrak{a}_{\mathbb{C}}^*$, holomorphic and non-negative on $i\mathfrak{a}^*$. ■

When w_0 is the longest element of the Weyl group, then we call $\mu(\nu,\sigma,w_0)$ the *Plancherel measure* attached to σ and ν. We write $\mu(\sigma)$ for $\mu(0,\sigma,w_0)$. By using Plancherel measures, one can normalize the operators $A(\nu,\sigma,\tilde{w})$ by a meromorphic (in ν) scalar factor, to obtain a family of intertwining operators which are holomorphic on the unitary axis $i\mathfrak{a}^*$ [17, 22]. Shahidi, [18], has shown that Plancherel measures and normalizing factors are related to conjectural Langlands L-functions. We denote the normalized operators by $\mathcal{A}(\nu,\sigma,\tilde{w})$ and write $\mathcal{A}(\sigma,\tilde{w})$ for $\mathcal{A}(0,\sigma,\tilde{w})$. These normalized operators satisfy the cocycle condition

$$\mathcal{A}(\sigma,\tilde{w}_1\tilde{w}_2) = \mathcal{A}(w_2\sigma,\tilde{w}_1)\mathcal{A}(\sigma,\tilde{w}_2), \tag{1.3}$$

for all $w_1, w_2 \in W(\mathbf{G},\mathbf{A})$.

Suppose $w \in W(\sigma)$. Choose an operator $T_w\colon V \longrightarrow V$ with $T_w w\sigma = \sigma T_w$. Then $\mathcal{A}'(\sigma,\tilde{w}) = T_w\mathcal{A}(\sigma,\tilde{w})$ is a self intertwining operator for $\mathrm{Ind}_P^G(\sigma)$.

THEOREM 1.2 (HARISH-CHANDRA) [22, THEOREM 5.5.3.2]. *The commuting algebra $C(\sigma)$ of $\mathrm{Ind}_P^G(\sigma)$ is spanned by $\{\mathcal{A}'(\sigma,\tilde{w}) \mid w \in W(\sigma)\}$.* ■

The theory of R-groups gives an algorithm for computing a basis of $C(\sigma)$ from among the operators $\mathcal{A}'(\sigma,\tilde{w})$. Let $\Phi(\mathbf{P},\mathbf{A})$ be the reduced roots of $\mathbf{P}$ with respect to $\mathbf{A}$, and let $\beta \in \Phi(\mathbf{P},\mathbf{A})$. Let $\mathbf{A}_\beta$ be the torus $\ker(\beta \cap \mathbf{A})^\circ$. We denote by $\mathbf{M}_\beta$ the centralizer of $\mathbf{A}_\beta$ in $\mathbf{G}$. Then $\mathbf{M}$ is a maximal proper Levi subgroup of $\mathbf{M}_\beta$. Let $\mu_\beta(\sigma)$ be the Plancherel measure attached to $i_{M_\beta,M}(\sigma)$. Then $\mu_\beta(\sigma) = 0$ if and only if $W_{M_\beta}(\sigma) \neq 1$, and $i_{M_\beta,M}(\sigma)$ is irreducible [22]. Let $\Delta' = \{\beta \in \Phi(\mathbf{P},\mathbf{A}) \mid \mu_\beta(\sigma) = 0\}$. Denote by W' the subgroup of $W(\sigma)$ generated by the reflections in the elements of Δ'. Let $R_G(\sigma) = \{w \in W(\sigma) \mid w\beta > 0, \forall \beta \in \Delta'\}$. If G and σ are implicit, we may denote $R_G(\sigma)$ by R.

THEOREM 1.3 (KNAPP-STEIN, SILBERGER [14, 20, 21]). *For $\sigma \in \mathcal{E}_2(M)$, we have $W(\sigma) = R \ltimes W'$. Moreover, $W' = \{w \in W(\sigma) \mid \mathcal{A}'(\sigma, \tilde{w})$ is scalar$\}$.* ∎

So, $\{\mathcal{A}'(\sigma, \tilde{r}) \mid r \in R\}$ is a basis for $C(\sigma)$. Note that if $r_1, r_2 \in R$, then

$$\mathcal{A}'(\sigma, \tilde{r}_1\tilde{r}_2) = \eta(r_1, r_2)\mathcal{A}'(\sigma, \tilde{r}_1)\mathcal{A}'(\sigma, \tilde{r}_2), \tag{1.4}$$

where the 2-cocycle η is given by

$$T_{r_1 r_2} = \eta(r_1, r_2)T_{r_1}T_{r_2}. \tag{1.5}$$

Thus, $C(\sigma) \simeq \mathbb{C}[R]_\eta$, the group algebra of R, twisted by the cocycle η.

We recall the role this cocycle plays in the description of $i_{G,M}(\sigma)$. Let $1 \longrightarrow Z \longrightarrow \tilde{R} \longrightarrow R \longrightarrow 1$ be a central extension over which η splits. We identify η with its pullback to $\tilde{R} \times \tilde{R}$. Choose a function $\xi: \tilde{R} \longrightarrow \mathbb{C}^\times$ splitting η, *i.e.*, $\xi(r_1)^{-1}\xi(r_2)^{-1}\xi(r_1 r_2) = \eta(r_1, r_2)$. Let ω_σ be the character of Z satisfying $\omega_\sigma(z)\xi(r) = \xi(zr)$. We get a unitary intertwining operator $\tilde{\mathcal{A}}(\sigma, \tilde{r}) = \xi(\tilde{r})^{-1}\mathcal{A}'(\sigma, \tilde{r})$, for $\tilde{r} \in \tilde{R}$. Now $\tilde{r} \longmapsto \tilde{\mathcal{A}}(\sigma, \tilde{r})$ is a homomorphism with $\tilde{\mathcal{A}}(\sigma, z\tilde{r}) = \omega_\sigma^{-1}(z)\tilde{\mathcal{A}}(\sigma, \tilde{r})$ for $z \in Z$. Let $\Pi(\tilde{R}, \omega_\sigma)$ be the set of equivalence classes of irreducible representations of $\tilde{R}$ with Z-central character ω_σ^{-1}.

THEOREM 1.4 (ARTHUR [1]). *There is a bijection $\rho \longmapsto \pi_\rho$ from $\Pi(\tilde{R}, \omega_\sigma)$ to $\Pi_\sigma(G)$ with the property that* $\dim \mathrm{Hom}_G\big(\pi_\rho, i_{G,M}(\sigma)\big) = \dim \rho$. ∎

Arthur writes down the projections of $i_{G,M}(\sigma)$ onto its isotypic components, and we use this description in Section 2.

There is a conjectural construction of the R-group, based on Langlands's conjectured parameterization. We describe this construction briefly, and refer the reader to [3, 15, 19] for more details. Let W_F be the Weil group of F. We denote by L_F the Langlands group $W_F \times \mathrm{SU}(2, \mathbb{R})$. Let $\hat{G}$ be the complex group whose root datum is dual to that of G. The L-group of $\mathbf{G}$ is given by ${}^L G = \hat{G} \ltimes W_F$, where W_F acts on $\hat{G}$ via its action on root data [5].

We assume that $\mathcal{E}_t(G)$ and $\mathcal{E}_t(M)$ can be partitioned into finite subsets, called *L-packets*, with the properties described in [5]. Suppose that $\psi: L_F \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow {}^L G$ is a parameterization for the L-packet Π of G, as conjectured by Langlands. Let $S_\psi = Z_{\hat{G}}\big(I_m(\psi)\big)$, and let S_ψ° be its connected component. Choose a maximal torus $T_\psi \subset S_\psi^\circ$, and let $N_\psi = N_{S_\psi}(T_\psi)$. We denote by $\mathbf{N}_\psi$ the group N_ψ / T_ψ, and by $\mathbf{S}_\psi$ the group S_ψ / S_ψ°. Then there is a map $\mathbf{N}_\psi \longrightarrow \mathbf{S}_\psi$ given by $nT_\psi \longmapsto nS_\psi^\circ$. Since any two maximal tori of S_ψ° are conjugate in S_ψ°, this map is surjective, with kernel $W_\psi^\circ = W(S_\psi^\circ, T_\psi)$. Similarly, there is a surjective map $\mathbf{N}_\psi \longrightarrow W(S_\psi, T_\psi) = W_\psi$. Call $\mathbf{S}_\psi^1$ the kernel of this map. Then $R_\psi = W_\psi / W_\psi^\circ \simeq \mathbf{S}_\psi / \mathbf{S}_\psi^1$ is called the *R-group* of ψ.

Suppose now that $\psi: L_F \longrightarrow {}^L M$ parameterizes a discrete series L-packet Π of M. Then $L_F \xrightarrow{\psi} {}^L M \hookrightarrow {}^L G$ should define a tempered L-packet of G. One expects that $\Pi_\psi(G) = \bigcup_{\sigma \in \Pi} \Pi_\sigma(G)$ is this L-packet.

Now, by duality, one can identify W_ψ with those elements w in $W(\mathbf{G}, \mathbf{A})$ such that $w\sigma \in \Pi$ for each $\sigma \in \Pi$. So $W_G(\sigma)$ should be isomorphic to a subgroup $W_{\psi,\sigma}$ of W_ψ.

Let $W^\circ_{\psi,\sigma} = W_{\psi,\sigma} \cap W^\circ_\psi$. Then the R-group, $R_{\psi,\sigma} = W_{\psi,\sigma}/W^\circ_{\psi,\sigma}$, should be isomorphic to the R-group given by $W_G(\sigma)/W'$ [3, 19]. When $F = \mathbb{R}$, Shelstad has confirmed this last isomorphism. If F is p-adic, then Keys has confirmed the isomorphism of $R_{\psi,\sigma}$ and $R_G(\sigma)$ in some cases [13].

Now suppose $\mathbf{G}/\mathbf{G}^\circ \simeq \mathbb{Z}/p\mathbb{Z}$. Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$ and $\mathbf{P}^\circ = \mathbf{P} \cap \mathbf{G}^\circ$. Suppose $\sigma \in \mathcal{E}_2(M)$. One can still define the operators, $A(\nu, \sigma, w)$, now for $w \in W(\mathbf{G}, \mathbf{A}^\circ)$. Arthur has studied these operators, and their normalization [2]. Many of the preliminary results on these operators match with those for connected groups, [2, Theorem 2.1]. It is not yet clear what the analogues of Theorems 1.2 and 1.3 should be. On the other hand one can attempt to extend the dual group construction of the R-group. Namely, one can define an L-group ${}^LG \supset {}^LG^\circ$ [4], and one hopes to describe the irreducible tempered representations of G (in packets) via parameters $\psi\colon L_F \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^LG^\circ$, such that $Z_{\hat{G}}\big((I_m\psi)\big) \cap (\hat{G} \setminus \hat{G}^\circ) \neq \emptyset$. In fact, Arthur points out that these maps should define those packets $\{\sigma\}$ such that $\sigma|_{G^\circ}$ is irreducible [3]. Now suppose $\sigma \in \mathcal{E}_2(M)$ is parameterized by ψ. Arthur extends the definitions of $S_\psi, \mathbf{N}_\psi, \mathbf{S}^1_\psi, W_\psi$, and R_ψ. Note that S°_ψ, T_ψ, and thus, W°_ψ depend only on ψ and $\mathbf{G}^\circ$, not on the larger group $\mathbf{G}$ [3].

If $\psi\colon L_F \longrightarrow {}^LM^\circ$ parameterizes a discrete series L-packet Π_0 of M°, then we let Π be the collection of components of $\mathrm{Ind}^M_{M^\circ}(\sigma_0)$ for $\sigma_0 \in \Pi_0$. Then Π is an L-packet of M, and every L-packet of M should arise in this way. Composing with the inclusion map, we get a parameter for the L-packet of components of $i_{G,M}(\sigma)$ for $\sigma \in \Pi$. Again, we can identify W_ψ with $\{w \in W(\mathbf{G}, \mathbf{A}^\circ) \mid w\Pi = \Pi\}$. Thus, $W_G(\sigma_0) = \{w \in W(\mathbf{G}, \mathbf{A}) \mid w\sigma_0 \simeq \sigma_0\}$, should be isomorphic to a subgroup $W_{\psi,\sigma}$ of W_ψ. Note that, since $W^\circ_\psi \subset W(\mathbf{G}^\circ, \mathbf{A}^\circ)$, we have $W^\circ_{\psi,\sigma_0} = W_{\psi,\sigma} \cap W^\circ_\psi$. We let $R_{\psi,\sigma} = W_{\psi,\sigma}/W^\circ_{\psi,\sigma_0}$. Arthur, [3, Section 7], indicates that $R_{\psi,\sigma}$ should be associated to the reducibility and component structure of $i_{G,M^\circ}(\sigma_0)$.

We recall some results of Gelbart and Knapp.

THEOREM 1.5 (GELBART-KNAPP [6]). *Suppose H is a totally disconnected group and $H_0 \subset H$ is an open normal subgroup such that H/H_0 is finite abelian. Let π be an irreducible admissible representation of H.*

(a) $\pi|_{H_0}$ is a finite direct sum of irreducible admissible representations of H_0.

(b) If $\pi|_{H_0} \simeq \sum_{i=1}^{\ell} m_i\pi_i$, with π_i irreducible, $\pi_i \not\simeq \pi_j$ for $i \neq j$, and each $m_i > 0$, then $m_1 = m_2 = \cdots = m_\ell$. We denote this integer by m.

(c) If $H_{\pi_1} = \{h \in H \mid h\pi_1 \simeq \pi_1\}$, then H/H_{π_1} permutes the classes of π_i simply and transitively.

(d) Let $X_{H_0}(\pi)$ be the collection of one dimensional characters of H, trivial on H_0, with the property that $\pi \otimes \nu \simeq \pi$. Then $|X_{H_0}(\pi)| = m^2\ell$.

(e) If $\pi|_{H_0}$ and $\pi'|_{H_0}$ are multiplicity free and have a common constituent, then $\pi|_{H^\circ} = \pi|'_{H_0}$, and $\pi' = \pi \otimes \nu$ for some character ν of H, trivial on H_0. ∎

2. **Reducibility when $G/G^\circ \simeq \mathbb{Z}/pZ$.** We assume that G is the F-points of a reductive quasi-split algebraic group $\mathbf{G}$, and $G/G^\circ \simeq \mathbb{Z}/p\mathbb{Z}$, with p prime.

LEMMA 2.1. *If π is an irreducible admissible representation of G, then $\pi|_{G^\circ}$ is a multiplicity one representation.*

PROOF. Since $|\widehat{G/G^\circ}| = p$, $|X_{G^\circ}(\pi)|$ is square free. Thus, by part (d) of Theorem 1.5, $m = 1$. ∎

We now describe the reducibility of $i_{G,M}(\sigma)$ in the case where $M \neq M^\circ$, and $\sigma|_{M^\circ}$ is irreducible.

PROPOSITION 2.2. *Suppose $P = MN$, and $M \neq M^\circ$. Suppose $(\sigma, V) \in \mathcal{E}_2(M)$, with $\sigma_0 = \sigma|_{M^\circ}$ irreducible. Let $\pi_0 = i_{G^\circ,M^\circ}(\sigma_0)$, and $\pi = i_{G,M}(\sigma)$. Suppose that, $\pi_0 = n_1\pi_1 \oplus \cdots \oplus n_s\pi_s$, with π_i irreducible, and $\pi_i \not\simeq \pi_j$, for $i \neq j$. Then $\pi \simeq n_1\Pi_1 \oplus \cdots \oplus n_s\Pi_s$, with Π_i irreducible, and $\Pi_i \not\simeq \Pi_j$, for $i \neq j$.*

PROOF. Since $\sigma|_{M^\circ} = \sigma_0$, we have $\pi|_{G^\circ} = \pi_0$. Thus, π has at most s inequivalent components, and if $\pi_i \subseteq \Pi|_{G^\circ}$, then Π appears in π with multiplicity less than or equal to n_i. In fact, since $M \neq M^\circ$, and $\sigma|_{M^\circ} = \sigma_0$, we see that restriction of functions, from the space $V(\sigma)$ of π to the space $V(\sigma_0)$ of π_0 is an isomorphism. Moreover, since $G = MG^\circ$, we see that if $T \in \mathrm{Hom}_G(\pi, \pi)$ is non-zero, then $T|_{V(\sigma_0)} \neq 0$. Therefore, restriction gives an embedding $\mathrm{Hom}_G(\pi, \pi) \hookrightarrow \mathrm{Hom}_{G^\circ}(\pi_0, \pi_0)$.

Let $w \in W_{G^\circ}(\sigma_0)$. Then $w\sigma|_{M^\circ} = \sigma_0$, so $w\sigma \simeq \sigma \otimes \chi^j$, for some j. Let T_w be a linear automorphism of V such that $T_w w\sigma = (\sigma \otimes \chi^j)T_w$. Note that T_w gives an M° isomorphism between σ_0 and $w\sigma_0$. Let $T'_w \colon V(\sigma) \longrightarrow V(\sigma)$ be given by $T'_w f(g) = T_w\big(f(g)\big) \otimes \chi^{-j}(g)$. We let $\mathcal{A}'(\sigma, \tilde{w}) = T'_w \mathcal{A}(\sigma, \tilde{w})$, where $\mathcal{A}(\sigma, \tilde{w})$ is the normalized operator given in [2]. Since $T_w w\sigma_0 = \sigma_0 T_w$, and $A(\sigma, \tilde{w})|_{V(\sigma_0)} = A(\sigma_0, w)$, we see that $\mathcal{A}'(\sigma, \tilde{w})|_{V(\sigma_0)} = \mathcal{A}'(\sigma_0, \tilde{w})$. Since $\{\mathcal{A}'(\sigma_0, \tilde{w}) \mid w \in R_{G^\circ}(\sigma_0)\}$ is a basis for $\mathrm{Hom}_{G^\circ}(\pi_0, \pi_0)$, the restriction of intertwining operators is surjective. Thus, $\mathrm{Hom}_G(\pi, \pi) = \mathrm{Hom}_{G^\circ}(\pi_0, \pi_0)$, giving the desired result. ∎

COROLLARY 2.3. *Suppose $P = MN$, and $M \neq M^\circ$. Suppose $\sigma \in \mathcal{E}_2(M)$, with $\sigma_0 = \sigma|_{M^\circ}$ irreducible. Let $\pi_0 = i_{G^\circ,M^\circ}(\sigma_0)$. Suppose that, $\pi_0 = n_1\pi_1 \oplus \cdots \oplus n_s\pi_s$, with π_i irreducible, and $\pi_i \not\simeq \pi_j$, for $i \neq j$. For each i, choose an irreducible representation Π_i of G with $\Pi_i|_{G^\circ} \simeq \pi_i$. Then*

$$i_{G,M^\circ}(\sigma_0) = \bigoplus_{i=1}^{s} n_i \bigoplus_{j=0}^{p-1} \Pi_i \otimes \chi^j.$$

Moreover, the collection $\{\Pi_i \otimes \chi^j\}_{i,j}$ are pairwise inequivalent.

PROOF. By Proposition 2.2, we can take Π_i so that $\pi = i_{G,M}(\sigma) = \bigoplus n_i\Pi_i$, and $\Pi_i|_{G^\circ} = \pi_i$. Moreover, since $\Pi_i|_{G^\circ}$ is irreducible, we know that $\Pi_i \not\simeq \Pi_i \otimes \chi$. Since $\sigma|_{M^\circ} = \sigma_0$, Theorem 1.5 implies $\mathrm{Ind}_{M^\circ}^{M}(\sigma_0) = \bigoplus_j \sigma \otimes \chi^j$. Thus,

$$\begin{aligned} i_{G,M^\circ}(\sigma_0) &= \bigoplus_{j=0}^{p-1} i_{G,M}(\sigma \otimes \chi^j) \simeq \bigoplus_{j=0}^{p-1} \pi \otimes \chi^j \\ &= \bigoplus_{i=1}^{s} n_i \bigoplus_{j=0}^{p-1} \Pi_i \otimes \chi^j, \end{aligned}$$

as claimed. ∎

Now suppose that $\sigma|_{M^\circ} = \sigma_1 \oplus \cdots \oplus \sigma_p$. Then, for each i, $\mathrm{Ind}_{M^\circ}^{M}(\sigma_i) = \sigma$. Moreover, $\sigma \simeq \sigma \otimes \chi$. Let $\pi = i_{G,M}(\sigma)$, and $\pi_i = i_{G^\circ,M^\circ}(\sigma_i)$. Then $\pi|_{G^\circ} = i_{G,M}(\sigma|_{M^\circ}) = \oplus_{i=1}^{p} \pi_i$. Note that, for each i, we have $\mathrm{Ind}_{G^\circ}^{G}(\pi_i) = \big(i_{G,M^\circ}(\sigma_i)\big) = i_{G,M}(\sigma) = \pi$. Now, by Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi,\pi) \simeq \mathrm{Hom}_{G^\circ}(\pi_1, \pi|_{G^\circ}) \simeq \mathrm{Hom}_{G^\circ}\Big(\pi_1, \bigoplus_{i=1}^{p} \pi_i\Big).$$

By [22, Theorem 2.5.8], we know that, for $i \neq 1, \mathrm{Hom}_{G^\circ}(\pi_1, \pi_i) = 0$, unless $w_0\sigma_1 \simeq \sigma_i$, for some $w_0 \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$. Note that, for such a w_0, $w_0\sigma_1 \not\simeq \sigma_1$, but, $w_0\sigma \simeq \sigma$, (see Theorem 1.5(e))

Now suppose that $w_0\sigma_1 = \sigma_2$. We further suppose that $m \in M \setminus M^\circ$ has the property that $m\sigma_i \simeq \sigma_{i+1}, i = 1, 2, \ldots, p-1$. Assume that, for $1 \leq i < j$, there is a $w_i \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$ with $w_i\sigma_1 \simeq \sigma_i$. Let $w_j = mw_{j-1}m^{-1}w_0$. Then $w_j\sigma_1 = m\sigma_{j-1} = \sigma_j$. So, in this case, for each j, there is a $w \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$ with $w\sigma_1 \simeq \sigma_j$. Note that the existence of such a $w_0 \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$ is equivalent to the condition $W_G(\sigma_0) \neq W_{G^\circ}(\sigma_0)$.

PROPOSITION 2.4. *Suppose $M \neq M^\circ$, and $\sigma|_{M^\circ}$ is reducible. Let σ_0 be any irreducible component of $\sigma|_{M^\circ}$. Let $\pi = i_{G,M^\circ}(\sigma_0)$, and $\pi_0 = i_{G^\circ,M^\circ}(\sigma_0)$. If $W_G(\sigma_0) = W_{G^\circ}(\sigma_0)$, then* $\mathrm{Hom}_G(\pi,\pi) \simeq \mathrm{Hom}_{G^\circ}(\pi_0,\pi_0)$*, and each component of π restricts to G° reducibly. Thus, if the decomposition of π_0 into irreducibles is $\pi_0 = \oplus_{i=1}^{s} n_i\tau_i$, then we have $\pi = \oplus_{i=1}^{s} n_i\Pi_i$, with $\tau_i \subsetneq \Pi_i|_{G^\circ}$.*

PROOF. We have already noted that $\mathrm{Hom}_G(\pi,\pi) = \mathrm{Hom}_{G^\circ}(\pi_0,\pi_0)$. Let $m \in M \setminus M^\circ$. Since $m\sigma_0$ is a different component of $\sigma|_{M^\circ}$, we know $m\pi_0$ and π_0 have no common constituents. Thus, for each component τ of π_0, $m\tau \not\simeq \tau$, so $\Pi = \mathrm{Ind}_{G^\circ}^{G}(\tau)$ is irreducible. Therefore, we get the result on the decomposition of π. Moreover, $\Pi|_{G^\circ} = \oplus_{j=0}^{p-1} m^j\tau$, and only τ appears in π_0. ∎

LEMMA 2.5. *Suppose $M \neq M^\circ$, and $\sigma|_{M^\circ} = \sigma_1 \oplus \cdots \oplus \sigma_p$. Let $\pi = i_{G,M^\circ}(\sigma_0)$ and $\pi_i = i_{G^\circ,M^\circ}(\sigma_i)$. Suppose that $W_G(\sigma_0) \neq W_{G^\circ}(\sigma_0)$. Then, for each i,*

$$\dim\big(\mathrm{Hom}_G(\pi,\pi)\big) = p\dim\big(\mathrm{Hom}_{G^\circ}(\pi_i,\pi_i)\big).$$

PROOF. We know $\mathrm{Hom}_G(\pi,\pi) \simeq \mathrm{Hom}_{G^\circ}(\pi_1, \oplus_{i=1}^{p} \pi_i)$. By our assumption, and the discussion preceding Proposition 2.4, we know $\pi_i \simeq \pi_j$ for all $1 \leq i \leq j \leq p$. Consequently, $\mathrm{Hom}_G(\pi,\pi) \simeq \mathrm{Hom}_{G^\circ}(\pi_1, p\pi_1)$, giving the result. ∎

We now look at the case where $M = M^\circ$. Since $i_{G,M}(\sigma) = i_{G,M^\circ}(\sigma)$, we again need to explore the reducibility of the representations $\mathrm{Ind}_{G^\circ}^{G}(\tau)$, with $\tau \subseteq i_{G^\circ,M}(\sigma)$.

LEMMA 2.6. *Suppose $M = M^\circ$, and $\sigma \in \mathcal{E}_2(M)$. Let $\pi_0 = i_{G^\circ,M}(\sigma)$, and suppose $\pi_0 = \oplus_{i=1}^{s} n_i\tau_i$. If, for some i, $\mathrm{Ind}_{G^\circ}^{G}(\tau_i)$ is reducible, then $w\sigma \simeq \sigma$ for some $w \in$*

$W(\mathbf{G},\mathbf{A}^\circ) \setminus W(\mathbf{G}^\circ,\mathbf{A}^\circ)$. *Furthermore, if such a w exists, then* $\dim\big(\mathrm{Hom}_G(\pi,\pi)\big) = p\dim\big(\mathrm{Hom}_{G^\circ}(\pi_0,\pi_0)\big)$.

PROOF. Let $g \in G \setminus G^\circ$. We may assume that $g \in N_G(\mathbf{T}^\circ)$. Let $M' = gMg^{-1}$. Let $\sigma' = g\sigma$. We know that $\mathrm{Ind}_{G^\circ}^{G}(\tau_i)$ is reducible, if and only if $g\tau_i \simeq \tau_i$. Since $g\tau_i$ is a subrepresentation of $i_{G^\circ,M'}(\sigma')$, we see that $g\tau_i \simeq \tau_i$ implies that, for some $w_0 \in W(\mathbf{G},\mathbf{A}^\circ)$, $w_0 M w_0^{-1} = M'$, and $w_0\sigma \simeq \sigma'$. Now, if $w = g^{-1}w_0$, then $w \in N_G(M) \setminus N_{G^\circ}(M)$, and $w\sigma \simeq \sigma$. We have seen that such a w lies in $N_G(\mathbf{A}^\circ) \setminus N_{G^\circ}(\mathbf{A}^\circ)$, and thus, represents an element of $W(\mathbf{G},\mathbf{A}^\circ) \setminus W(\mathbf{G}^\circ,\mathbf{A}^\circ)$.

Now suppose that $w\sigma \simeq \sigma$, for some $w \in W(\mathbf{G},\mathbf{A}) \setminus W(\mathbf{G}^\circ,\mathbf{A})$. Choose a representative $\tilde{w}$ for w. We have $\tilde{w}\pi_0 \simeq \pi_0$, and therefore, each connected component of G/G° fixes the representation π_0. By Frobenius reciprocity and Mackey theory,

$$\mathrm{Hom}_G(\pi,\pi) \simeq \mathrm{Hom}_{G^\circ}(\pi|_{G^\circ},\pi_0) \simeq \mathrm{Hom}_{G^\circ}\Big(\bigoplus_{G/G^\circ}(g\pi_0,\pi_0)\Big) \simeq \mathrm{Hom}_{G^\circ}(p\pi_0,\pi_0),$$

giving the result. ∎

Let $\Phi(\mathbf{P}^\circ,\mathbf{A})$ be the reduced roots of $\mathbf{A}^\circ$ in $\mathbf{P}^\circ$. Suppose $\sigma_0 \in \mathcal{E}_2(M^\circ)$. For $\alpha \in \Phi(\mathbf{P}^\circ,\mathbf{A}^\circ)$, let $\mu_\alpha(\sigma_0)$ be the Plancherel measure attached to σ_0 and α. We let $\Delta' = \{\alpha \mid \mu_\alpha(\sigma_0) = 0\}$ and $W' = W(\Delta')$ be the subgroup of $W(\mathbf{G}^\circ,\mathbf{A}^\circ)$ generated by the root reflections in Δ'. Then $W_{G^\circ}(\sigma_0)/W' \simeq R_{G^\circ}(\sigma_0)$, is the R-group attached to $i_{G^\circ,M^\circ}(\sigma_0)$. We let $R_G(\sigma_0) = \{w \in W_G(\sigma_0) \mid w\alpha > 0\ \forall\alpha \in \Delta'\}$. Notice that $R_{G^\circ}(\sigma_0) \subset R_G(\sigma_0)$. This should reflect the fact that $i_{G,M^\circ}(\sigma_0)$ may have more components than $i_{G^\circ,M^\circ}(\sigma_0)$.

LEMMA 2.7. *Let $P^\circ = M^\circ N$ be a parabolic subgroup of G°. Suppose $\sigma_0 \in \mathcal{E}_2(M^\circ)$. Then $W_G(\sigma_0) = R_G(\sigma_0) \ltimes W'$.*

PROOF. We first show that $W' \triangleleft W_G(\sigma_0)$. Let $\alpha \in \Delta'$ and $w \in W_G(\sigma_0)$. If $\beta = w\alpha$, then $w_\beta = ww_\alpha w^{-1} \in W_{G^\circ}(\sigma_0)$. Note that

$$i_{M^\circ_\beta,M^\circ}(\sigma_0) = i_{M^\circ_{w\alpha},M^\circ}(w\sigma_0) \simeq w\big(i_{M^\circ_\alpha,M^\circ}(\sigma_0)\big),$$

which is irreducible, since $\alpha \in \Delta'$. Thus, $\beta \in \pm\Delta'$, and consequently, $w_\beta \in W'$. Therefore, $W' \triangleleft W_G(\sigma_0)$. Moreover, since $\pm\Delta'$ is a root system, it is clear that $W' \cap R_G(\sigma_0) = \{1\}$.

Let $w \in W_G(\sigma_0)$. Define $R(w) = \{\alpha \in \Delta' \mid w\alpha < 0\}$. If $R(w) = \emptyset$, then $w \in R_G(\sigma_0)$. Suppose that, for w_1 with $|R(w_1)| < |R(w)|$, we have $w_1 = rw'$, with $r \in R_G(\sigma_0)$, and $w' \in W'$. Suppose $\alpha \in \Delta'$, and $w\alpha < 0$. Let $w_1 = ww_\alpha$. Then $w_1\alpha > 0$. Moreover, since $\pm\Delta'$ is a root system, $w_\alpha(\Delta' \setminus \{\alpha\}) = \Delta' \setminus \{\alpha\}$. Therefore, if $\beta \in \Delta'$, and $w_1\beta < 0$, then $w_\alpha\beta \in R(w) \setminus \{\alpha\}$. Consequently, $|R(w_1)| < |R(w)|$, and we can write $w_1 = rw_1'$, with $r \in R_G(\sigma_0)$, and $w_1' \in W'$. Thus, $w = rw'$, with $w' = w_1'w_\alpha$. ∎

We call $R_G(\sigma_0)$ The Arthur R-group attached to $i_{G,M^\circ}(\sigma_0)$. We will show that the structure of $i_{G,M^\circ}(\sigma_0)$ is reflected in the representation theory of $R_G(\sigma_0)$. In Section 4, we show that, if Arthur's group $R_{\psi,\sigma}$ exists, then it must be isomorphic to $R_G(\sigma_0)$.

THEOREM 2.8. *Suppose that $P^\circ = M^\circ N$, and $\sigma_0 \in \mathcal{E}_2(M^\circ)$. Choose any $\sigma \in \mathcal{E}_2(M)$ with $\sigma_0 \subset \sigma|_{M^\circ}$.*

(a) If $M \neq M^\circ$, and $\sigma_0 = \sigma|_{M^\circ}$, then $R_G(\sigma_0) = R_{G^\circ}(\sigma_0) \ltimes \mathbb{Z}/p\mathbb{Z}$.

(b) If $\sigma|_{M^\circ}$ is reducible, or $M = M^\circ$, then

$$R_G(\sigma_0)/R_{G^\circ}(\sigma_0) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } W_G(\sigma_0) \neq W_{G^\circ}(\sigma_0) \\ 1 & \text{if } W_G(\sigma_0) = W_{G^\circ}(\sigma_0). \end{cases}$$

PROOF. (a) Since $M \neq M^\circ$, and $\sigma_0 = \sigma|_{M^\circ}$, we know that $m\sigma_0 \simeq \sigma_0$, for all $m \in M \setminus M^\circ$. Fix such an m. Then m represents an element $w \in W(\mathbf{G}, \mathbf{A}^\circ) \setminus W(\mathbf{G}^\circ, \mathbf{A}^\circ)$, and thus, $w \in W_G(\sigma_0) \setminus W_{G^\circ}(\sigma_0)$. We can write $w = rw'$, with $r \in R_G(\sigma_0)$ and $w' \in W'$. Since $W' \subset W_{G^\circ}(\sigma_0)$, we have $r \in R_G(\sigma_0) \setminus R_{G^\circ}(\sigma_0)$. Note that, since $m^p \in M^\circ$, we have $w^p = 1$. Since W' is normal in $W_G(\sigma_0)$, we have $1 = w^p = (rw')^p = r^p w''$, with $w'' \in W'$. However, since $R_G(\sigma_0) \cap W' = \{1\}$, we see that $r^p = 1$. Since $M/M^\circ = \mathbb{Z}/p\mathbb{Z}$, we now have $R_G(\sigma_0) = R_{G^\circ}(\sigma_0) \ltimes \langle r \rangle$.

(b) If $W_G(\sigma_0) = W_{G^\circ}(\sigma_0)$, then we clearly have $R_G(\sigma_0) = R_{G^\circ}(\sigma_0)$. On the other hand, if $w \in W_G(\sigma_0) \setminus W_{G^\circ}(\sigma_0)$, then we have $w = rw'$, with $r \in R_G(\sigma_0) \setminus R_{G^\circ}(\sigma_0)$. Since $W(\mathbf{G}, \mathbf{A}^\circ) \simeq W(\mathbf{G}^\circ, \mathbf{A}^\circ) \ltimes \mathbb{Z}/p\mathbb{Z}$, we see that r is of order p modulo $R_{G^\circ}(\sigma_0)$. If $r_1 \in R_G(\sigma_0)$, then, in $W(\mathbf{G}, \mathbf{A}^\circ)$, we can write $r_1 = r^j r_0$, for some $0 \leq j \leq p-1$, and $r_0 \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$. Clearly $r_0 \in R_{G^\circ}(\sigma_0)$, and therefore, $R_G(\sigma_0)/R_{G^\circ}(\sigma_0) \simeq \mathbb{Z}/p\mathbb{Z}$. ■

COROLLARY 2.9. *Let $P^\circ = M^\circ N$ be a parabolic subgroup of G°. Suppose that $\sigma_0 \in \mathcal{E}_2(M^\circ)$, and let $C_G(\sigma_0)$ be the commuting algebra of $i_{G,M^\circ}(\sigma_0)$. Then* $\dim C_G(\sigma_0) = |R_G(\sigma_0)|$.

PROOF. Let $C(\sigma_0)$ be the commuting algebra of $i_{G^\circ,M^\circ}(\sigma_0)$. Then, we know that $\dim C(\sigma_0) = |R_{G^\circ}(\sigma_0)|$. From Proposition 2.2, Proposition 2.4, Lemma 2.5, and Lemma 2.6, we know that

$$\dim C_G(\sigma_0) = \begin{cases} \dim C(\sigma_0) & \text{if } W_G(\sigma_0) = W_{G^\circ}(\sigma_0) \\ p \dim C(\sigma_0) & \text{otherwise.} \end{cases}$$

By Theorem 2.8, we see that $\dim C_G(\sigma_0) = |R_G(\sigma_0)|$. ■

We wish to prove an extension of Theorem 1.4 in this case. Let $R = R_G(\sigma_0)$ and $R_0 = R_{G^\circ}(\sigma_0)$. Let η be the 2-cocycle of R_0 given by (1.5). If $r \in R \setminus R_0$, we choose an equivalence T_r between σ and $r\sigma$. This gives us an extension of η to a 2-cocycle of R which is also defined by (1.5). We also denote this extension by η. We take a central extension $\tilde{R}$ of R by Z over which η splits. Since R/R_0 is cyclic, we have the following diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 1 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & Z & \longrightarrow & \tilde{R} & \longrightarrow & R & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & Z & \longrightarrow & \tilde{R}_0 & \longrightarrow & R_0 & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

with $n = 1$ or p. Thus, we can choose $\xi: \tilde{R} \longrightarrow \mathbb{C}^\times$ which splits η, and the restriction of ξ to $\tilde{R}_0$ is a choice for the function discussed is Section 1. Let r denote both a generator of $\tilde{R}/\tilde{R}_0$, and its image in R/R_0. Then it makes sense to compare $r\rho$ and $r\pi_\rho$, for $\rho \in \Pi(\tilde{R}_0, \omega_{\sigma_0})$.

PROPOSITION 2.10. *Let $\sigma_0 \in \mathcal{E}_2(M^\circ)$. Let $R_0 = R_{G^\circ}(\sigma_0)$, and $R = R_G(\sigma_0)$. Suppose that $R/R_0 \simeq Z/p\mathbb{Z}$. Let $\rho \in \Pi(\tilde{R}_0, \omega_{\sigma_0})$, and suppose π_ρ is the irreducible component of $\pi_0 = i_{G^\circ, M^\circ}(\sigma_0)$ attached to ρ. Then, for each $r \in \tilde{R}/\tilde{R}_0 = R/R_0$, we have $r\pi_\rho \simeq \pi_{r\rho}$.*

PROOF. From [1, Section 2] we know that, if θ_ρ is the character of ρ, then

$$\Phi_\rho = \frac{\dim \rho}{|\tilde{R}_0|} \sum_{w \in \tilde{R}_0} \overline{\theta_\rho}(w) \tilde{\mathcal{A}}(\sigma_0, \tilde{w}) \tag{2.1}$$

is the orthogonal projection of π_0 onto its π_ρ-isotypic subspace. Suppose $\Phi_\rho f = f$, and let $V_\rho(f)$ be the G°-span of f. Then $\pi_0|_{V_\rho(f)} \simeq \pi_\rho$. For $h \in \pi_0$, let $\mathrm{Th}(x) = h(r^{-1}xr)$. Then we can realize $r\pi_\rho$ on the G°-span W of Tf. Note that we have $W \subset \pi_0$, if we realize π_0 as $\mathrm{Ind}_{MN'}^{G^\circ}(r\sigma_0)$, where $N' = rNr^{-1} \subset rUr^{-1}$. To show that $r\pi_\rho \simeq \pi_{r\rho}$, it is enough to show that $\Phi'_{r\rho}(Tf) = Tf$, where $\Phi'_{r\rho}$ is the operator given by (2.1), with respect to our second realization of π_0.

First note that, in this second realization of π_0, the intertwining operator $T_w: w\sigma_0 \longrightarrow \sigma_0$ is replaced by the operator $T'_w = T_{r^{-1}} T_w T_r$, and thus the cocycle here is $\eta_r(w_1, w_2) = \eta(r^{-1}w_1 r, r^{-1}w_2 r)$. Of course η_r and η define the same class in $H^2(R, \mathbb{C}^\times)$. Therefore, to get the same map $\rho \longmapsto \pi_\rho$, we need to define $\tilde{\mathcal{A}}(r\sigma_0, \tilde{w}) = \xi(r^{-1}wr)\mathcal{A}'(r\sigma_0, \tilde{w})$. Now note that

$$\begin{aligned} A(0, r\sigma_0, \tilde{w})Tf(g) &= \int_{N'_w} f(\tilde{r}^{-1}\tilde{w}^{-1}ng\tilde{r})\, dn \\ &= \int_{N'_w} f\big(\tilde{r}^{-1}\tilde{w}^{-1}\tilde{r}(\tilde{r}^{-1}n\tilde{r})(\tilde{r}^{-1}g\tilde{r})\big)\, dn. \end{aligned} \tag{2.2}$$

A straightforward calculation shows that $\tilde{r}^{-1}N'_w\tilde{r} = N_{r^{-1}wr}$. Thus, we can rewrite (2.2) as

$$\int_{N_{r^{-1}wr}} f\big(\tilde{r}^{-1}\tilde{w}^{-1}\tilde{r}n(\tilde{r}^{-1}g\tilde{r})\big)\, dn = A(0, \sigma_0, \tilde{r}^{-1}\tilde{w}\tilde{r})f(\tilde{r}^{-1}g\tilde{r}) = T\big(A(0, \sigma_0, \tilde{r}^{-1}\tilde{w}\tilde{r})f\big)(g).$$

Therefore,

$$\begin{aligned}
\Phi'_{r\rho}(Tf) &= \frac{\dim\rho}{|\tilde{R}_0|}\sum_{w\in\tilde{R}_0}\overline{\theta_{r\rho}}(w)\tilde{\mathcal{A}}(r\sigma_0,\tilde{w})Tf\\
&= \frac{\dim\rho}{|\tilde{R}_0|}\sum_{w\in\tilde{R}_0}\overline{\theta_\rho}(r^{-1}wr)\xi(r^{-1}wr)\mathcal{A}'(r\sigma_0,\tilde{w})Tf\\
&= \frac{\dim\rho}{|\tilde{R}_0|}\sum_{w\in\tilde{R}_0}\overline{\theta_\rho}(r^{-1}wr)\xi(r^{-1}wr)T\big(\mathcal{A}'(\sigma_0,\tilde{r}^{-1}\tilde{w}\tilde{r})f\big)\\
&= T(\Phi_\rho f) = Tf.
\end{aligned}$$

Thus, we have shown that $r\pi_\rho \simeq \pi_{r\rho}$. ■

We can now prove our main result.

THEOREM 2.11. *Suppose that $G/G^\circ \simeq \mathbb{Z}/p\mathbb{Z}$. Let $P^\circ = M^\circ N$ be a parabolic subgroup of G°. Suppose $\sigma_0 \in \mathcal{E}_2(M^\circ)$. Let $\pi = i_{G,M^\circ}(\sigma_0)$, and $\pi_0 = i_{G^\circ,M^\circ}(\sigma_0)$. Denote by $R = R_G(\sigma_0)$ the Arthur R-group attached to π, and $R_0 = R_{G^\circ}(\sigma_0) \subset R$ the Knapp-Stein R-group attached to π_0. Then, to each $\tau \in \Pi(\tilde{R},\omega_{\sigma_0})$, we can attach an element of $\Pi_\tau \in \Pi_{\sigma_0}(G)$ such that:*

1. *If $\tau \not\simeq \tau'$, then $\Pi_\tau \not\simeq \Pi_{\tau'}$.*
2. *The multiplicity of Π_τ in π is $\dim\tau$.*
3. *If $\rho \in \Pi(\tilde{R}_0,\omega_{\sigma_0})$, then $\rho \subset \tau|_{\tilde{R}_0}$ if and only if $\pi_\rho \subset \Pi_\tau|_{G^\circ}$, where π_ρ is the component of π_0 which is attached to ρ.*
4. *Every irreducible component of π is isomorphic to Π_τ, for some τ.*

PROOF. Suppose that $R = R_0$. Then, by Theorem 2.8, we know that $M = M^\circ$ or $\sigma_0 \not\subset \sigma|_{M^\circ}$, for some $\sigma \in \mathcal{E}_2(M)$. Moreover, Theorem 2.8 also implies that $W_G(\sigma_0) = W_{G^\circ}(\sigma_0)$. Therefore, by Proposition 2.4 and Lemma 2.6, each component π_ρ of π_0 induces irreducibly to G. We let $\Pi_\rho = \mathrm{Ind}_{G^\circ}^G(\pi_\rho)$. Then, Proposition 2.4 and Lemma 2.6 imply that $\Pi_\rho \not\simeq \Pi_{\rho'}$ for $\rho \not\simeq \rho'$. Thus, the correspondence $\rho \longmapsto \Pi_\rho$ has the desired properties.

Now suppose that $R/R_0 \simeq Z/p\mathbb{Z}$. We identify $\widehat{\tilde{R}/\tilde{R}_0}$, and $\widehat{G/G^\circ}$. Suppose $\rho \in \Pi(\tilde{R}_0,\omega_{\sigma_0})$. Let $r \in \tilde{R} \setminus \tilde{R}_0$, and also denote by r its projection to R. Then $\mathrm{Ind}_{G^\circ}^G(\pi_\rho)$ is reducible if and only if $r\pi_\rho \simeq \pi_\rho$. By Proposition 2.10, this is equivalent to $r\rho \simeq \rho$, which is equivalent to the reducibility of $\mathrm{Ind}_{\tilde{R}_0}^{\tilde{R}}\rho$.

If $r\pi_\rho \not\simeq \pi_\rho$, then there are distinct elements $\rho = \rho_1,\dots,\rho_p \in \Pi(\tilde{R}_0,\omega_{\sigma_0})$, such that $r^j\pi_\rho = \pi_{\rho_{j+1}}$, for $1 \le j \le p-1$. Of course, we then know that $\rho_{j+1} = r^j\rho$, for each $1 \le j \le p-1$. Let $\tau = \mathrm{Ind}_{\tilde{R}_0}^{\tilde{R}}\rho$. Then τ is irreducible, and $\tau = \mathrm{Ind}_{\tilde{R}_0}^{\tilde{R}}(\rho_j)$ for each j. Similarly, $\Pi_\tau = \mathrm{Ind}_{G^\circ}^G(\pi_\rho)$ is irreducible, and $\Pi_\tau = \mathrm{Ind}_{G^\circ}^G(\pi_{\rho_j})$ for each $1 \le j \le p$. Thus,

$$\dim\big(\mathrm{Hom}_G(\Pi_\tau,\pi)\big) = p\dim\big(\mathrm{Hom}_{G^\circ}(\pi_\rho,\pi_0)\big) = p\dim\rho = \dim\tau.$$

Moreover, $\Pi_\tau|_{G^\circ} = \bigoplus_{j=1}^p \pi_{\rho_j}$, and only those components of π equivalent with Π_τ contain any π_{ρ_j} upon restriction to G°.

On the other hand, if $r\rho \simeq \rho$, then $\mathrm{Ind}_{\tilde{R}_0}^{\tilde{R}}(\rho) \simeq \oplus_{j=0}^{p-1} \tau \otimes \chi^j$. Let Π_τ, be any irreducible component of $\mathrm{Ind}_{G^\circ}^{G}(\pi_\rho)$. Then

$$\mathrm{Ind}_{G^\circ}^{G}(\pi_\rho) \simeq \bigoplus_{j=0}^{p-1} \Pi_\tau \otimes \chi^j,$$

and $\Pi_\tau \not\simeq \Pi_\tau \otimes \chi$. Let $\Pi_{\tau\otimes\chi^j} = \Pi_\tau \otimes \chi^j$. Then, for each j,

$$\dim \mathrm{Hom}_G(\Pi_{\tau\otimes\chi^j}, \pi) = \dim\big(\mathrm{Hom}_{G^\circ}(\pi_\rho, \pi_0)\big) = \dim \rho = \dim \tau \otimes \chi^j.$$

Furthermore, only the representations $\Pi_{\tau\otimes\chi^j}$ contain π_ρ upon restriction to G°. Therefore, we have a correspondence $\tau \longmapsto \Pi_\tau$ exhibiting the properties (1)–(4). ∎

3. **Reducibility for O_n.** Here we use the results of Section 2 to determine the R-groups and reducibility for $O_n(F)$. Since the results for O_{2n+1} are trivial, we leave them to the end of the section.

Let $J_n = \begin{pmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & \end{pmatrix} \in \mathrm{GL}_n$. We let $\mathbf{G} = O_{2n}$, defined with respect to J_{2n}. That is, $\mathbf{G} = \{g \in \mathrm{GL}_{2n} \mid {}^t g J_{2n} g = J_{2n}\}$. Then $\mathbf{G}^\circ = SO_{2n} = \{g \in \mathbf{G} \mid \det g = 1\}$. Clearly $\mathbf{G}/\mathbf{G}^\circ$ is of order 2. We often write $\mathbf{G} = \mathbf{G}(n)$ or $\mathbf{G}^\circ = \mathbf{G}^\circ(n)$ in order to specify the rank of $\mathbf{G}$. By convention, we take $\mathbf{G}(0) = \mathbf{G}^\circ(0) = 1$. Note that

$$\mathbf{G}^\circ(1) = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\} \simeq \mathrm{GL}_1,$$

and

$$\mathbf{G}(1) = G^\circ(1) \cup \left\{\begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}\right\}.$$

We may write $\mathrm{GL}(m)$ for the group $\mathrm{GL}_m(F)$. The non-trivial character χ of $\mathbf{G}/\mathbf{G}^\circ$ is given by $\chi(g) = \mathrm{sgn}(\det g)$.

Let $\mathbf{T}^\circ$ be the maximal torus of diagonal elements in $\mathbf{G}^\circ$,

$$\mathbf{T}^\circ = \left\{ \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & \lambda_n^{-1} & & \\ & & & & \ddots & \\ & & & & & \lambda_1^{-1} \end{pmatrix} \middle| \lambda_i \in \mathrm{GL}_1 \right\}.$$

We often write $a = (\lambda_1, \ldots, \lambda_n)$ or $a = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}, \ldots, \lambda_1^{-1}\}$, for $a \in \mathbf{T}^\circ$. Let $\Phi(\mathbf{G}^\circ, \mathbf{T}^\circ)$ be the roots of $\mathbf{T}^\circ$ in $\mathbf{G}^\circ$. Then $\Phi(\mathbf{G}^\circ, \mathbf{T}^\circ)$ is of type D_n. The Weyl group $W(\mathbf{G}^\circ, \mathbf{A}^\circ)$ is isomorphic to $S_n \ltimes \mathbb{Z}_2^{n-1}$. We recall the explicit description of $W(\mathbf{G}^\circ, \mathbf{T}^\circ)$. First, let the transposition (ij) be given by

$$(ij)\colon (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_n) \longmapsto (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_n).$$

A representative for (ij) is given by $\begin{pmatrix} E_{ij} & 0 \\ 0 & E_{ij} \end{pmatrix}$, where E_{ij} is the standard permutation matrix of GL_n associated to (ij).

For each i we let the sign change c_i be given by

$$c_i\colon (\lambda_i, \ldots, \lambda_i, \ldots, \lambda_n) \longmapsto (\lambda_i, \ldots, \lambda_i^{-1}, \ldots, \lambda_n).$$

Note that c_i has no representative in $\mathbf{G}^\circ$. However,

$$\bar{c}_i = \begin{pmatrix} I_{i-1} & & & & \\ & 0 & & 1 & \\ & & I_{2n-2i} & & \\ & 1 & & 0 & \\ & & & & I_{i-1} \end{pmatrix} \in \mathbf{G} \setminus \mathbf{G}^\circ,$$

represents c_i. Thus, $W(\mathbf{G}^\circ, \mathbf{T}^\circ) = \langle (ij), c_i c_j \mid 1 \leq i < j \leq n \rangle \simeq S_n \ltimes \mathbb{Z}_2^{n-1}$. Now, it is clear that $W(\mathbf{G}, \mathbf{A}^\circ) \simeq S_n \ltimes \mathbb{Z}_2^n$, given by $\langle (ij), c_k \mid 1 \leq i < j \leq n,\ 1 \leq k \leq n \rangle$.

We take the collection of simple roots,

$$\Delta = \{e_i - e_{i+1}\}_{i=1}^{n-1} \cup \{e_{n-1} + e_n\},$$

so that the Borel subgroup $\mathbf{B}^\circ = \mathbf{T}^\circ \mathbf{U}$ is the upper triangular matrices in $\mathbf{G}^\circ$. Note that if $u = (u_{ij}) \in \mathbf{U}$, then $u_{n,n+1} = 0$. Thus, $C = \bar{c}_n \in N_{\mathbf{G}}(\mathbf{B}^\circ)$, and consequently, $\mathbf{B} = N_{\mathbf{G}}(\mathbf{B}^\circ) = \mathbf{B}^\circ \cup \mathbf{B}^\circ C$. Moreover, the Levi component of $\mathbf{B}$ is $\mathbf{T} = \mathbf{T}^\circ \cup \mathbf{T}^\circ C$.

Suppose $\mathbf{P}^\circ = \mathbf{M}^\circ \mathbf{N}$ is a parabolic subgroup of $\mathbf{G}^\circ$ containing $\mathbf{B}^\circ$. Then there are positive integers $m_1, \ldots, m_r$ and a non-negative integer k, so that

$$\mathbf{M}^\circ \simeq \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r} \times \mathbf{G}^\circ(k).$$

In fact, without loss of generality, we can suppose that the split component, $\mathbf{A}^\circ$, of $\mathbf{P}^\circ$ is of the form

$$\mathbf{A}^\circ = \left\{ \mathrm{diag}\{\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r}, I_{2k}, \lambda_r^{-1} I_{m_r}, \ldots, \lambda_1^{-1} I_{m_1}\} \right\},$$

and thus,

$$\mathbf{M}^\circ = \left\{ \begin{pmatrix} g_1 & & & & & & \\ & \ddots & & & & & \\ & & g_r & & & & \\ & & & h & & & \\ & & & & {}^{\tau}g_r^{-1} & & \\ & & & & & \ddots & \\ & & & & & & {}^{\tau}g_1^{-1} \end{pmatrix} \middle| \begin{array}{l} g_i \in \mathrm{GL}_{m_i} \\ h \in \mathbf{G}^\circ(k) \end{array} \right\}.$$

Here ${}^{\tau}g$ means the transpose of g with respect to the second diagonal. We make the further assumption that $m_1 \geq m_2 \geq \cdots \geq m_r$. Notice that there is one ambiguity in our notation. Namely, if $\mathbf{M}^\circ = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_{r-1}} \times \mathrm{GL}_1$, then we can also write $\mathbf{M}^\circ = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_{r-1}} \times \mathbf{G}^\circ(1)$. We take the convention that $k \neq 1$, *i.e.*, we choose the first notation. This makes our description of $\mathbf{M}^\circ$ consistent with that of $\mathbf{A}^\circ$. Moreover, it is consistent with the notation in [8]. However, with this convention, if $m_r = 1$ and $k = 0$, then $\mathbf{P} = N_{\mathbf{G}}(\mathbf{P}^\circ) = \mathbf{MN}$, with

$$\mathbf{M} = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_{r-1}} \times \mathbf{G}(1).$$

In all other cases, $\mathbf{P} = \mathbf{MN}$, with

$$\mathbf{M} = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r} \times \mathbf{G}(k).$$

While this may cause some confusion, it seems to be the less confusing of the two options. If $\mathbf{M}^\circ = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r} \times \mathbf{G}^\circ(k)$, then we may write $\mathbf{M} = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_t} \times \mathbf{G}(k')$, with the understanding that $t = r$ or $r - 1$, and $k' = k$ or 1, as appropriate.

Suppose that $\mathbf{M} = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_t} \times \mathbf{G}(k')$. If $\sigma \in \mathcal{E}_2(M)$, then $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_t \otimes \rho$, with $\sigma_i \in \mathcal{E}_2\big(\mathrm{GL}(m_i)\big)$, and $\rho \in \mathcal{E}_2\big(G(k')\big)$. Note that if $k' > 0$, then $G(k')$ acts on $\mathcal{E}_2(M^\circ)$ by conjugation. We take C to be the representative of this action.

Suppose that $M^\circ \simeq \mathrm{GL}(m_1) \times \cdots \times \mathrm{GL}(m_r) \times G^\circ(k)$. Then $W(\mathbf{G}, \mathbf{A}^\circ)$ is a subgroup of $S_r \ltimes \mathbb{Z}_2^r$. More precisely, $W(\mathbf{G}, \mathbf{A}^\circ)$ is generated by the elements,

$$C_i\colon (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_r) \longmapsto (\lambda_1, \ldots, \lambda_i^{-1}, \ldots, \lambda_r),$$

for $1 \leq i \leq r$, and the permutations

$$w_{ij}\colon (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_r) \longmapsto (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_r),$$

for those $1 \leq i < j \leq r$, with $m_i = m_j$. Note that if $g = (g_1, \ldots, g_r, h) \in M$, then $C_i g = (g_1, \ldots, {}^t g_i^{-1}, \ldots, g_r, h)$, and $w_{ij} g = (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_r, h)$. If $k = 0$, then $C_i \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$ if and only if m_i is even. If $k = 0$, and both m_i and m_j are odd, then $C_j C_j \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$.

Suppose $\sigma_0 = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho_0 \in \mathcal{E}_2(M^\circ)$. Then we have

$$C_i\sigma_0 \simeq \sigma_1 \otimes \cdots \otimes \tilde{\sigma}_i \otimes \cdots \otimes \sigma_r \otimes \rho_0, \quad \text{and} \tag{3.1}$$

$$w_{ij}\sigma_0 \simeq \sigma_1 \otimes \cdots \otimes \sigma_j \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes \sigma_r \otimes \rho_0. \tag{3.2}$$

Thus,

$$C_i\sigma_0 \simeq \sigma_0 \quad \text{if and only if } \sigma_i \simeq \tilde{\sigma}_i; \tag{3.3}$$

$$w_{ij}\sigma_0 \simeq \sigma_0 \quad \text{if and only if } \sigma_i \simeq \sigma_j; \tag{3.4}$$

$$w_{ij}C_iC_j\sigma_0 \simeq \sigma_0 \quad \text{if and only if } \sigma_i \simeq \tilde{\sigma}_j. \tag{3.5}$$

Note that $w\sigma_0 \simeq \sigma_0$ for some non-trivial $w \in W(\mathbf{G}, \mathbf{A}^\circ)$ if and only if at least one of the conditions (3.3)–(3.5) holds.

If $k > 0$, then (3.3)–(3.5) also give the conditions for $w\sigma_0 \simeq \sigma_0$ for some non-trivial $w \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$. However, if $k = 0$, then $C_i \notin W(\mathbf{G}^\circ, \mathbf{A}^\circ)$ if m_i is odd. Thus, if both m_i and m_j are odd, then $C_iC_j\sigma_0 \simeq \sigma_0$ if and only if both $\sigma_i \simeq \tilde{\sigma}_i$, and $\sigma_j \simeq \tilde{\sigma}_j$. This, along with (3.3) for m_i even, (3.4), and (3.5), give the conditions for $w\sigma_0 \simeq \sigma_0$ for some $w \in W(\mathbf{G}^\circ, \mathbf{A}^\circ)$ [8].

Let $I_1(M) = \{i \mid m_i \text{ is even}\}$, and

$$I_1(\sigma_0) = \begin{cases} \{1,2,\ldots,r\} & \text{if } k > 0 \text{ and } C\rho_0 \simeq \rho_0, \\ I_1(M) & \text{otherwise}. \end{cases}$$

Let $I_2(\sigma_0) = \{1,2,\ldots,r\} \setminus I_1(\sigma_0)$. Now take

$$J_1(\sigma_0) = \{i \in I_1(\sigma_0) \mid i_{G^\circ(m_i+k),\mathrm{GL}(m_i)\times G^\circ(k)}(\sigma_i \otimes \rho_0) \text{ is reducible}\},$$

and $J_2(\sigma_0) = \{i \in I_2(\sigma_0) \mid \sigma_i \simeq \tilde{\sigma}_i\}$. For $j = 1$ or 2, we let d_j be the number of equivalence classes of σ_i with $i \in J_j(\sigma_0)$, and let $d = d_1 + d_2$.

THEOREM 3.1 (SEE [8]). *The R-group, R_0, attached to $i_{G^\circ,M^\circ}(\sigma_0)$ is given by*

$$R_0 = \begin{cases} \mathbb{Z}_2^d & \text{if } d_2 = 0 \\ \mathbb{Z}_2^{d-1} & \text{if } d_2 \neq 0\,. \end{cases}$$

In either case R_0 is a subgroup of $\langle C_iC_j \mid i \neq j\rangle$. ∎

COROLLARY 3.2. *For any $\sigma_0 \in \mathcal{E}_2(M)$, $i_{G^\circ,M^\circ}(\sigma_0)$ is a multiplicity one representation with $|R_0|$ components. Moreover, $C(\sigma) \simeq \mathbb{C}[R_0]$, i.e. the cocycle η splits.* ∎

The last statement of this corollary was proved by Herb [11].

THEOREM 3.3. *Let $\mathbf{G} = \mathbf{G}(n) = O_{2n}$, and suppose that $\mathbf{P}^\circ = \mathbf{M}^\circ\mathbf{N}$ is a parabolic subgroup of $\mathbf{G}^\circ$. Let $\mathbf{M}^\circ = \mathrm{GL}_{m_1} \otimes \cdots \otimes \mathrm{GL}_{m_r} \otimes \mathbf{G}^\circ(k)$. We denote by $C = c_n$ the odd sign change in both $W(\mathbf{G}, \mathbf{A}^\circ)$, and $W\big(\mathbf{G}(k), \mathbf{S}^\circ\big)$, where $\mathbf{S}^\circ$ is the maximal torus of $\mathbf{G}^\circ(k)$. Let $\sigma_0 = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho_0 \in \mathcal{E}_2(M^\circ)$. Suppose $d = d_1 + d_2$ is as in Theorem 3.1. Then*

$$R_G(\sigma_0) \simeq \begin{cases} \mathbb{Z}_2^{d+1} & \textit{if } k > 1 \textit{ and } C\rho_0 \simeq \rho_0 \\ \mathbb{Z}_2^d & \textit{otherwise}. \end{cases}$$

PROOF. Let $R = R_G(\sigma_0)$. Suppose that $k > 1$, and $C\rho_0 \simeq \rho_0$. Then $M \neq M^\circ$, and $\sigma|M^\circ$ is irreducible. Thus, by Theorem 2.8(a), $R_G(\sigma_0) \simeq R_0 \ltimes \mathbb{Z}/2\mathbb{Z}$. Moreover, since $\{\alpha \in \Phi(\mathbf{G}^\circ, \mathbf{T}^\circ) \mid C\alpha < 0\} = \emptyset$, and $C \in W_G(\sigma_0)$, we clearly have $C \in R$. Thus $R \simeq \mathbb{Z}_2^{d+1}$.

We now consider all the other cases, except for the case where $m_r = 1$ and $k = 0$. Suppose that $d_2 = 0$. Then, from (3.2)–(3.5), we see that $W_G(\sigma_0) = W_{G^\circ}(\sigma_0)$, so $R = R_0 = \mathbb{Z}_2^d$. Now suppose that $d_2 > 0$. For $i = 1$ or 2, let

$$B_i(\sigma_0) = \{j \in J_i(\sigma_0) \mid \sigma_\ell \not\simeq \sigma_j,\ \forall \ell > j\}.$$

Our assumption that $d_2 > 0$ implies that $B_2(\sigma_0) \neq \emptyset$. Now, from [8] we have

$$R_0 = \langle C_j \mid j \in B_1(\sigma_0)\rangle \times \langle C_jC_\ell \mid \ell, j \in B_2(\sigma_0)\rangle \simeq \mathbb{Z}_2^{d-1}.$$

Now, for $j \in B_2(\sigma_0)$, we have $C_j \in W_G(\sigma_0) \setminus W_{G^\circ}(\sigma_0)$. Moreover, from Lemmas 3.4, 6.6, and 6.9 of [8], we have $C_j \in R$. Thus,

$$R = \langle C_j \mid j \in B_1(\sigma_0) \cup B_2(\sigma_0) \rangle \simeq \mathbb{Z}_2^d.$$

Finally assume that $m_r = 1$ and $k = 0$. Then

$$M = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_{r-1}} \times G(1).$$

Note that $\sigma_r \in \widehat{F^\times}$, so $C\sigma_r = \tilde{\sigma}_r = \sigma_r^{-1}$. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_{r-1} \otimes \rho$. Note that $\dim \rho = 1$ or 2. If $\dim \rho = 1$, then $\rho|_{\mathrm{GL}(1)} = \sigma_r$, and thus $\sigma_r^2 = 1$. Therefore, $d_2 > 0$, so $R_0 \simeq \mathbb{Z}_2^{d-1}$. Since $C \in W_G(\sigma_0) \setminus W_{G^\circ}(\sigma_0)$, we have $C \in R$, so $R = \mathbb{Z}_2^d$. On the other hand, if $\dim \rho = 2$, then $C \notin W_G(\sigma_0)$. If $d_2 = 0$, then $W_G(\sigma_0) = W_{G^\circ}(\sigma_0)$, and Theorems 3.1 and 2.8(b) imply $R = R_0 \simeq \mathbb{Z}_2^d$. If $d_2 \neq 0$, then $W_G(\sigma_0) \neq W_{G^\circ}(\sigma_0)$, and in particular $C_j \in W_G(\sigma_0)$, for some j with m_j odd. By Lemmas 3.4, 6.6, and 6.9 of [8], we have $C_j \in R$, so $R \simeq \mathbb{Z}_2^d$. ∎

COROLLARY 3.4. *Let* $\mathbf{G}$, $\mathbf{M}$, $\mathbf{M}^\circ$, *and* σ_0 *be as in Theorem 3.3. Then* $i_{G,M^\circ}(\sigma)$ *is a multiplicity one representation. Moreover, if d is an in Theorem 3.1, then* $i_{G,M^\circ}(\sigma_0)$ *has* 2^d *components, unless* $k > 1$ *and* $C\rho_0 \simeq \rho_0$, *in which case* $i_{G,M^\circ}(\sigma_0)$ *has* 2^{d+1} *components.*

PROOF. This follows immediately from Corollary 3.2, Theorem 3.3, and Theorem 2.11. ∎

We now consider the case where $\mathbf{G} = \mathbf{G}^\circ \times \mathbb{Z}/p\mathbb{Z}$. This case is quite simple, and includes the case $\mathbf{G} = O_{2n+1}$. Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of G, with split component $\mathbf{A}$. Then $\mathbf{A} = \mathbf{A}^\circ$, the split component of $\mathbf{M}^\circ$, and $\mathbf{M} = \mathbf{M}^\circ \times \mathbb{Z}/p\mathbb{Z}$. Let $\sigma \in \mathcal{E}_2(M)$. If σ_0 is a subrepresentation of $\sigma|_{M^\circ}$, then, for $m \in M \setminus M^\circ$, $m\sigma_0 \simeq \sigma_0$. Thus, $\sigma|_{M^\circ}$ is irreducible. By Proposition 2.2 and Theorem 2.11, we get the following result.

LEMMA 3.5. *Suppose* $\mathbf{G} = \mathbf{G}^\circ \times \mathbb{Z}/p\mathbb{Z}$. *Let* $\mathbf{P}^\circ = \mathbf{M}^\circ\mathbf{N}$ *be a parabolic subgroup of* $\mathbf{G}^\circ$, *and* $\sigma \in \mathcal{E}_2(M)$. *Let* $\sigma_0 = \sigma|_{M^\circ}$. *Then* $R_G(\sigma_0) = R_{G^\circ}(\sigma_0) \times \mathbb{Z}/p\mathbb{Z}$. *Suppose* $i_{G^\circ,M^\circ}(\sigma_0) = \bigoplus_{i=1}^s n_i\pi_i$. *For each i, choose an irreducible admissible representation* Π_i *of G, with* $\pi_i = \Pi_i|_{G^\circ}$. *Then*

$$i_{G,M^\circ}(\sigma_0) = \bigoplus_{i=1}^{s} n_i \bigoplus_{j=0}^{p-1} \Pi_i \otimes \chi^j. \quad \blacksquare$$

We apply this result to $O_{2n+1} = SO_{2n+1} \times \{\pm I\}$. We write $\mathbf{G} = \mathbf{G}(n)$. For the particulars on the parabolic subgroups and R-groups of SO_{2n+1}, we refer the reader to [8].

COROLLARY 3.6. *Let* $\mathbf{G} = O_{2n+1}$, *and suppose that* $\mathbf{M}^\circ \simeq \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r} \times \mathbf{G}^\circ(k)$. *Let* $\sigma_0 = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho_0 \in \mathcal{E}_2(M^\circ)$. *Let d be the number of equivalence classes of* σ_i *such that* $i_{G^\circ(m_i+k),\mathrm{GL}(m_i)\times G^\circ(k)}(\sigma_i \otimes \rho_0)$ *is reducible. Then* $R_G(\sigma_0) \simeq \mathbb{Z}_2^{d+1}$, *and* $i_{G,M^\circ}(\sigma_0)$ *is a multiplicity one representation with* 2^{d+1} *components.* ∎

4. **Relation with Arthur's conjecture.** We conclude with some remarks on the the connection between our results and the conjectural R-group $R_{\psi,\sigma}$ of Arthur. Since construction of $R_{\psi,\sigma}$ depends on the solution to the Langlands parameterization problem, we are certainly a long way from proving either Shelstad's or Arthur's conjecture. However, it is worth noting that our results, which do not rely on the conjectures of Langlands, do not contradict Arthur's conjecture.

PROPOSITION 4.1. *Suppose that* $\mathbf{G}/\mathbf{G}^\circ \simeq \mathbb{Z}/p\mathbb{Z}$, *and* $\mathbf{P} = \mathbf{MN}$ *is a parabolic subgroup of* $\mathbf{G}$. *Let* $\sigma \in \mathcal{E}_2(M)$, *and suppose that* $\sigma_0 = \sigma|_{M^\circ}$ *is irreducible. We assume that*

(1) the parameterization of discrete series L-packets of M° *by admissible homomorphisms* $\psi: L_F \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow {}^L M^\circ$ *is understood, and*

(2) if ψ *is a parameter for the L-packet of* M° *containing* σ_0, *then we have* $R_{G^\circ}(\sigma_0) \simeq R_{\psi,\sigma_0}$.

If ψ *is a parameter for the L-packet* Π_0 *of* M° *containing* σ_0, *then* $R_G(\sigma) \simeq R_{\psi,\sigma}$, *where* $R_{\psi,\sigma}$ *is the group constructed by Arthur in [3].*

PROOF. Recall that $R_{\psi,\sigma_0} = W_{\psi,\sigma_0}/W^\circ_{\psi,\sigma_0}$. We are assuming that W_{ψ,σ_0} is isomorphic to $W^\circ_G(\sigma_0)$, and and thus, $W^\circ_{\psi,\sigma_0} \simeq W'$. We have also (conjecturally) identified $W_{\psi,\sigma}$ with $W_G(\sigma_0)$, and thus,

$$R_{\psi,\sigma} = W_{\psi,\sigma}/W^\circ_{\psi,\sigma_0} \simeq W_G(\sigma_0)/W' \simeq R_G(\sigma_0),$$

the last equivalence coming from Lemma 2.7 ∎

Thus, from Theorem 2.11 and Proposition 4.1, we see that one can expect that the Arthur's conjectural R-group, $R_{\psi,\sigma}$, should determine the structure of $i_{G,M^\circ}(\sigma_0)$, at least when $\mathbf{G}/\mathbf{G}^\circ$ is of prime order. Furthermore, $R_G(\sigma_0)$ predicts the structure of $i_{G,M^\circ}(\sigma_0)$, even if $\sigma|_{M^\circ}$ is reducible. Consequently, there should be a dual side construction of such R-groups as well. In future work, we hope to extend these results to the case where $\mathbf{G}/\mathbf{G}^\circ$ is any finite cyclic group.

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