SMOOTH COMPACT INDUCTION OVER p-ADIC FIELDS

Abstract. Let F be an algebraic extension of $\mathbb{Q}_p$, and let G be a separable, locally profinite group. We consider a smooth F-representation V of G obtained by compact induction from an open subgroup that is compact modulo the center. We give a criterion for V to be supercuspidal.

In the general case, when V is not necessarily supercuspidal, we prove that there exists a finite extension E/F such that $V \otimes_F E$ decomposes as a finite direct sum of absolutely irreducible supercuspidal representations, together with a subrepresentation that contains no G-invariant supercuspidal subquotients and no absolutely irreducible admissible G-subspaces. These results are analogous to those of Bushnell [Bus90] for complex smooth representations.

1. Introduction

Let G be a separable locally profinite group. Motivated by the role of smooth representations in the theory of p-adic Banach space representations and the p-adic Langlands program, we study smooth representations of G on p-adic vector spaces. While the theory of smooth representations over $\mathbb{C}$ is well-established, we ask which results remain valid when $\mathbb{C}$ is replaced by an arbitrary field F of characteristic zero. For this, in addition to allowing F to be non-algebraically closed, we must also allow the possibility that G admits no F-valued Haar measure.

The nonexistence of a p-adic Haar measure for p-adic groups is a major obstacle to transferring results from the theory of smooth complex representations to the p-adic setting. However, for compactly supported locally constant F-valued functions on G, there exists a well-defined functional invariant under left translations, which we call a left Haar integral with values in F. In this paper, we are primarily interested in supercuspidal representations obtained by compact induction modulo the center, and the integrals we consider are taken over locally constant functions with compact support.

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This type of representation appears in the study of p-adic Banach space representations. Let F be a finite extension of $\mathbb{Q}_p$. Schneider and Teitelbaum initiated a systematic study of Banach space representations on F-vector spaces in [ST02]. Let V be an F-Banach space representation of G. If the space V^{sm} of smooth vectors of V is nonzero, then V^{sm} can provide valuable information about V.

Denote by $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ containing F. The fields $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$ are isomorphic as abstract fields, and consequently the category $\mathrm{Rep}^{\mathrm{sm}}_{\overline{\mathbb{Q}}_p}(G)$ of smooth representations of G on $\overline{\mathbb{Q}}_p$-vector spaces is equivalent to the category $\mathrm{Rep}^{\mathrm{sm}}_{\mathbb{C}}(G)$ of smooth complex representations of G. Let $V^{\mathrm{sm}}_{\overline{\mathbb{Q}}_p} = V^{\mathrm{sm}} \otimes_F \overline{\mathbb{Q}}_p$ be the scalar extension of V^{sm} to $\overline{\mathbb{Q}}_p$. Then, we can take the results for $V^{\mathrm{sm}}_{\overline{\mathbb{Q}}_p}$ and attempt to descend them to V^{sm}.

The success of this approach varies. While some properties descend, others do not, and the corresponding results for V^{sm} must be proved directly over F. In this paper, we combine both approaches. Section 2 of [Cas] is written over an arbitrary field of characteristic 0 and provides a valuable reference for basic results on smooth representations. For scalar extension, we rely on the work of Vignéras [Vig96], Henniart–Vignéras [HV19], and foundational material of Bourbaki [Bou12].

We are interested in understanding the structure of representations obtained by compact induction from open subgroups, as described in Theorems 1 and 2 of [Bus90]. The goal of this paper is to establish versions of these theorems over F, where F is an algebraic extension of $\mathbb{Q}_p$. The following theorem (Theorem 1.1) is the main result of the paper. When $F = \overline{\mathbb{Q}}_p$, it follows from [Bus90], since $\overline{\mathbb{Q}}_p \cong \mathbb{C}$.

For the notation, if V is a smooth representation of G, then V_{cusp} denotes the sum of all supercuspidal subrepresentations of V. The smooth compact induction $\text{c-Ind}_{ZH}^G W$ and the smooth induction $\mathrm{Ind}_{ZH}^G W$ are defined in the standard way (see Section 4). If V is an F-vector space and E/F is a field extension, we write $V_E = V \otimes_F E$.

Theorem 1.1 (Theorems 5.1 and 5.3)**.** *Let G be a separable, locally profinite group with center Z, and let H be an open compact subgroup of G. Let (σ, W) be an absolutely irreducible smooth F-representation of ZH, where F is an algebraic extension of $\mathbb{Q}_p$.*

(i) $\text{c-Ind}_{ZH}^{G} W$ *is admissible if and only if*

$$\text{c-Ind}_{ZH}^{G} W = \text{Ind}_{ZH}^{G} W,$$

if and only if there exists a finite extension E/F such that $\left(\text{c-Ind}_{ZH}^{G} W\right)_E$ is a finite direct sum of absolutely irreducible supercuspidal representations of G.

(ii) *Set $V = \text{c-Ind}_{ZH}^{G} W$. Then there exists a finite extension E/F such that $(V_E)_{\text{cusp}}$ is a finite direct sum of absolutely irreducible supercuspidal G-representations. Moreover,*

$$(\text{c-Ind}_{ZH}^{G} W)_E = \text{c-Ind}_{ZH}^{G} W_E = (V_E)_{\text{cusp}} \oplus X, \quad \textit{and}$$
$$(\text{Ind}_{ZH}^{G} W)_E = \text{Ind}_{ZH}^{G} W_E = (V_E)_{\text{cusp}} \oplus Y,$$

for uniquely determined G-subspaces $X \subseteq V_E$ and $Y \subseteq \text{Ind}_{ZH}^{G} W_E$ such that $X \subseteq Y$. Furthermore, Y contains no G-invariant supercuspidal subquotients, and X contains no absolutely irreducible admissible G-subspaces.

More details can be found in Theorems 5.1 and 5.3.

It would be interesting to determine whether the above theorem extends to coefficient fields F of positive characteristic, because an F-valued Haar integral on G exists whenever G contains a compact open subgroup whose pro-order is invertible in F [Vig96, I.2.4]. If G is a p-adic Lie group, this excludes the case $\text{char}(F) = p$, placing us in the classical ℓ-modular setting of Vignéras [Vig96] rather than the "natural characteristic" case. We do not pursue this question here because our arguments rely on tools developed by Casselman under the assumption that $\text{char}(F) = 0$, and Bushnell's proof likewise depends essentially on this assumption. Extending these methods to the ℓ-modular setting, in the spirit of [Vig96] and while accounting for its known subtleties, is beyond the scope of this paper.

Here is a brief overview of the paper. Let F be an arbitrary field of characteristic zero. In Section 2, we introduce notation and basic concepts and review some results on smooth F-representations.

Section 3 is on supercuspidal representations. We prove that an absolutely irreducible supercuspidal F-representation of G with central character χ is injective and projective in the category of smooth χ-representations of G on F-vector spaces (Theorem 3.3).

Section 4 is on induced representations. For a compact open subgroup H of G and a smooth irreducible F-representation W of ZH, we prove that

$$\text{c-Ind}_{ZH}^{G}(W \otimes_F E) \cong \left(\text{c-Ind}_{ZH}^{G} W\right) \otimes_F E$$

for any field extension E/F, and that

$$\left(\mathrm{Ind}_{ZH}^{G} W\right) \otimes_F E \cong \mathrm{Ind}_{ZH}^{G}(W \otimes_F E),$$

if $\mathrm{Ind}_{ZH}^{G} W$ is admissible or E/F is finite (Proposition 4.2).

In Section 5, F is an algebraic extension of $\mathbb{Q}_p$. We prove the main results, Theorems 5.1 and 5.3. The proof of Theorem 5.1 is a relatively simple application of scalar extension, while the proof of Theorem 5.3 is more involved.

In Section 6, F is a finite extension of $\mathbb{Q}_p$. We explain how the work in this paper relates to F-Banach space representations and discuss supercuspidal representations embedded in F-Banach spaces.

1.1. **Notation.** Throughout this paper, G is a separable, locally profinite group with center Z. In Sections 2-4, F is a field of characteristic zero. In Section 5, F is an algebraic extension of $\mathbb{Q}_p$, and $\overline{\mathbb{Q}}_p$ denotes an algebraic closure of $\mathbb{Q}_p$ containing F. In Section 6, F is a finite extension of $\mathbb{Q}_p$. Unless specified otherwise, all representations are on F-vector spaces.

2. Preliminaries

Let F be a field of characteristic zero. In this section, we introduce notation and review some known results on smooth representations with coefficients in F.

2.1. **Haar integrals with values in F.** Denote by $C_c^\infty(G, F)$ the space of locally constant, compactly supported functions $f\colon G \to F$. A *left Haar integral* on G with values in F is a nonzero linear functional

$$I\colon C_c^\infty(G, F) \to F$$

that is invariant under left translation by G. By [Ban23, Proposition 6.11], there exists a nontrivial left Haar integral on G with values in F, and it is unique up to multiplication by a nonzero scalar.[1]

We fix a left Haar integral $I_{\mathbb{Q}}$ on G with rational values, $I_{\mathbb{Q}}\colon C_c^\infty(G,\mathbb{Q}) \to \mathbb{Q}$. We use integral notation and write

$$\int_G f(x)\,dx$$

for $I_{\mathbb{Q}}(f)$. Moreover, for any field extension $F/\mathbb{Q}$, we choose the left Haar integral

$$I_F\colon C_c^\infty(G,F) \to F$$

such that its restriction to $C_c^\infty(G,\mathbb{Q})$ coincides with $I_{\mathbb{Q}}$.

2.2. **Modular character.** Given $g \in G$, there exists a constant $\delta_G(g) \in \mathbb{Q}^\times$ such that

$$\int_G f(xg^{-1})dx = \delta_G(g)\int_G f(x)dx$$

for all $f \in C_c^\infty(G,F)$. The function $\delta_G : G \to \mathbb{Q}^\times$ is a character, called the *modular character*.

2.3. **Smooth representations.** Let (π,V) be a representation of G. If K is a subgroup of G, let $V^K = \{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}$ be the space of K-fixed vectors. A vector v is called smooth if $v \in V^K$ for some compact open subgroup K of G. We denote by V^{sm} the subspace of all smooth vectors in V. Then

$$V^{\mathrm{sm}} = \bigcup_K V^K, \tag{1}$$

where K runs over the compact open subgroups of G. A representation (π,V) is called smooth if $V^{\mathrm{sm}} = V$. A smooth representation (π,V) of G is called admissible if V^K is finite-dimensional for every compact open subgroup K of G.

We denote by $\mathrm{Rep}_F^{\mathrm{sm}}(G)$ the category of smooth representations of G on F-vector spaces, with morphisms given by F-linear G-equivariant maps.

[1] If F has positive characteristic, a left Haar integral on G with values in F exists provided that G contains a compact open subgroup whose pro-order is invertible in F [Vig96, I.2.4].

Proposition 2.1 ([Ban23], Proposition 6.14). *If the fields E and F are isomorphic as abstract fields, then the categories $\mathrm{Rep}_E^{\mathrm{sm}}(G)$ and $\mathrm{Rep}_F^{\mathrm{sm}}(G)$ are equivalent. In particular, the category $\mathrm{Rep}_{\overline{\mathbb{Q}}_p}^{\mathrm{sm}}(G)$ of smooth representations of G on $\overline{\mathbb{Q}}_p$-vector spaces is equivalent to the category $\mathrm{Rep}_{\mathbb{C}}^{\mathrm{sm}}(G)$ of smooth complex representations of G.*

If (π, V) is a smooth representation of G, we write $(\widetilde{\pi}, \widetilde{V})$ for the contragredient or smooth dual of (π, V). Thus $\widetilde{V}$ is the space of all G-smooth elements of $\mathrm{Hom}_F(V, F)$, equipped with its natural G-action. We write

$$\langle\, ,\, \rangle : V \times \widetilde{V} \to F$$

for the canonical evaluation pairing given by $\langle\, v, \lambda\, \rangle = \lambda(v)$. For $v \in V$ and $\lambda \in \widetilde{V}$, the matrix coefficient $m_{v,\lambda}$ is the smooth function $m_{v,\lambda} : G \to F$ given by

$$m_{v,\lambda}(g) = \langle \pi(g)v, \lambda \rangle.$$

2.4. **Scalar extensions and absolute irreducibility.** A representation V is irreducible if V contains no G-invariant subspaces except zero and itself. If V is irreducible, then the intertwining algebra $\mathrm{End}_G(V)$ is a division algebra.

Let (π, V) be a representation of G on the F-vector space V and let E/F be a field extension. We denote by $\pi \otimes E$ or by π_E the representation of G on the E-vector space $V_E = V \otimes_F E$ given by

$$(\pi \otimes E)(g)(v \otimes c) = \pi(g)v \otimes c$$

and call it the scalar extension of π to E. (See [HV19] for results on scalar extensions.) We say that π is *absolutely irreducible* if $\pi \otimes E$ is irreducible for any field extension E/F.

Lemma 2.2 (Schur's lemma). *Suppose that V is an absolutely irreducible F-representation of G. Then $\mathrm{End}_G(V) \cong F$.*

Proof. This follows from Corollaire 2 in [Bou12, §3.2,p.44]. ∎

Proposition 2.3. *Let (π, V) be an F-representation of G, and let E/F be a field extension.*

(i) $(V \otimes_F E)^K = V^K \otimes_F E$ *for any subgroup K of G.*

(ii) $(V \otimes_F E)^{\mathrm{sm}} = V^{\mathrm{sm}} \otimes_F E$.

(iii) (π, V) *is smooth (respectively, admissible) if and only if* (π_E, V_E) *is smooth (respectively, admissible).*

Proof. Assertion (i) is a basic F-linear algebra. Indeed, fixing an F-basis (e_i) of E, we have $V \otimes_F E = \bigoplus_i (V \otimes e_i)$ as K-modules, and $v \otimes e_i$ is fixed by K if and only if v is.

Assertion (ii) follows from (i), using (1). Assertion (iii) follows from (i) and (ii). See also [HV19, §III.1, Lemma III.1 (ii)]. □

The following proposition can be proved using several existing results, but we include a proof for completeness.

Proposition 2.4. *Let V be a smooth F-representation of G, and let E/F be a field extension. Then the canonical injection*

$$\widetilde{V} \otimes_F E \hookrightarrow \widetilde{V \otimes_F E},$$

given by $\lambda \otimes b \mapsto (v \otimes a \mapsto ab\lambda(v))$, is an isomorphism if V is admissible or E/F is finite.

Proof. Let U be an F-vector space. By the tensor–Hom adjunction and [Bou70, II.110, §7.7, Prop. 16(i)], there is a canonical embedding

$$\mathrm{Hom}_F(U, F) \otimes_F E \hookrightarrow \mathrm{Hom}_E(U \otimes_F E, E)$$

given by $\lambda \otimes b \mapsto (v \otimes a \mapsto ab\lambda(v))$. It is an isomorphism if U is finite-dimensional over F or if E/F is finite (see [HV19, Remark II.2]).

Suppose that $U \subseteq V$ is K-invariant for some compact open subgroup K of G. Taking smooth vectors and using Proposition 2.3(ii), we obtain an embedding

$$\varphi_U : \widetilde{U} \otimes_F E \hookrightarrow \widetilde{U \otimes_F E}.$$

Again, φ_U is an isomorphism if U is finite-dimensional over F or if E/F is finite.

We now prove that φ_V is an isomorphism when V is admissible. Since injectivity has already been established, it remains to prove that φ_V is surjective.

Consider the decomposition

$$\widetilde{V \otimes_F E} = \bigcup_K \left(\widetilde{V \otimes_F E}\right)^K, \tag{2}$$

where K runs over the compact open subgroups of G. Fix such a subgroup K. Since V is admissible, V^K is finite-dimensional. Therefore,

$$\varphi_{V^K} : \widetilde{(V^K)} \otimes_F E \longrightarrow \widetilde{V^K \otimes_F} E$$

is an isomorphism. Define $V(K) = \mathrm{span}_F\{\pi(k)v - v \mid v \in V, k \in K\}$. Then

$$V = V^K \oplus V(K)$$

as representations of K ([Cas, p.20]), and any $\lambda \in (\widetilde{V})^K$ restricted to $V(K)$ is zero. Then $\lambda \mapsto \lambda|_{V^K}$ gives an isomorphism between $(\widetilde{V})^K$ and $\widetilde{(V^K)}$ (see [BH06, Proposition 2.8]). Similarly, the restriction $\lambda \otimes b \mapsto \lambda \otimes b|_{V^K \otimes_F E}$ gives an isomorphism

$$\left(\widetilde{V \otimes_F E}\right)^K \cong \widetilde{V^K \otimes_F} E.$$

Then

$$\varphi_{V^K} = (\varphi_V)|_{\widetilde{V}^K \otimes_F E} : \widetilde{V}^K \otimes_F E \to \left(\widetilde{V \otimes_F E}\right)^K$$

is an isomorphism. It follows that $\left(\widetilde{V \otimes_F E}\right)^K \subset \operatorname{im} \varphi_V$. Equation (2) implies that φ_V is surjective. $\square$

The following result follows from [Bou12, §12.2, Lemme 1] and [Pa3, Lemma 5.1].

Proposition 2.5. *Let V and W be F-representations of G, and let E/F be a field extension. If V is finitely generated over $F[G]$ or E is finite over F, then the natural injection*

$$\mathrm{Hom}_G(V, W) \otimes_F E \hookrightarrow \mathrm{Hom}_G(V \otimes_F E, W \otimes_F E)$$

is an isomorphism.

3. Supercuspidal representations

Let Z denote the center of G, and let $\chi : Z \to F^\times$ be a smooth character. A smooth representation (π, V) of G is called a χ-representation if

$$\pi(z)v = \chi(z)v \quad \text{for all } v \in V,\ z \in Z.$$

A representation (π, V) of G is called *supercuspidal* if:

(i) (π, V) is admissible;

(ii) (π, V) is a χ-representation for some χ;

(iii) for every $v \in V$ and $\lambda \in \widetilde{V}$, the matrix coefficient $m_{v,\lambda}$ is compactly supported modulo Z.

3.1. **Absolutely irreducible supercuspidal representations.** While complex supercuspidal representations satisfy many desirable properties, it turns out that, in order to prove these properties over an arbitrary field of characteristic zero, one must impose the condition that the representation be absolutely irreducible.

The next two results can be proved as in [Cas]. We remark that Section 2 of [Cas] is carried out over an arbitrary field of characteristic zero, and we may use these results as stated. The rest of Casselman's paper concerns complex representations, and we must adjust the proofs to our context.

Proposition 3.1 (Formal degree). *Let (π, V) be an absolutely irreducible supercuspidal representation of G. Then there exists a nonzero constant $d_\pi \in F$ such that*

$$\int_{G/Z} \langle \pi(g)u, \mu\rangle \langle \pi(g^{-1})v, \lambda\rangle \, d\bar{g} = d_\pi^{-1} \langle u, \lambda\rangle \langle v, \mu\rangle,$$

for all $u, v \in V$ and $\mu, \lambda \in \widetilde{V}$.

Proof. The proof uses the fact that for any G-invariant bilinear pairing $[\,,\,] : V \times \widetilde{V} \to F$, there exists $c \in F$ such that $[\,,\,] = c\langle\,,\,\rangle$. The proof of this standard result relies on Schur's lemma, and this is where we use the assumption that π is absolutely irreducible.

With this modification, the argument proceeds exactly as in the first half of the proof of Proposition 5.2.4 in [Cas]. □

Remark 3.2.

(1) The formal degree d_π of the representation π depends on the choice of Haar integral.

(2) For complex representations, d_π is a positive real number, where positivity reflects both the ordered archimedean structure of $\mathbb{R}$ and the positive-definiteness of the L^2 inner product. Neither has a canonical analogue over an arbitrary field of characteristic zero.

Theorem 3.3 (Projective-injective property of supercuspidal representations)**.** *Let V be an absolutely irreducible supercuspidal F-representation of G with central character χ. Then V is injective and projective in the category of smooth χ-representations of G on F-vector spaces.*

Proof. We can apply the first part of the proof of Theorem 5.4.1 in [Cas], which establishes projectivity for irreducible supercuspidal representations. Several ingredients are needed for the proof. The first is the existence of the formal degree, given in Proposition 3.1. Next is the fact that the matrix coefficients of V are compactly supported modulo Z, which is the defining property of supercuspidal representations in this paper and is proved in Theorem 5.2.1 of [Cas]. Finally, the proof uses the definition and properties of the Hecke algebra of G, which are covered in Section 2 of [Cas] and therefore apply to our case.

The proof that V is injective can be carried out exactly as in the final part of Theorem 5.4.1 of [Cas], using projectivity and the contragredient. □

Corollary 3.4. *Suppose $V_1, \dots, V_m$ are absolutely irreducible supercuspidal F-representations of G with central character χ. Then $V_1 \oplus \cdots \oplus V_m$ is injective and projective in the category of smooth χ-representations of G on F-vector spaces.*

Proof. By Theorem 3.3, each V_i is projective and injective in the category of smooth χ-representations. Since finite direct sums preserve both projectivity and injectivity, the direct sum $V_1 \oplus \cdots \oplus V_m$ is projective and injective. □

3.2. Descent of finite direct sums of absolutely irreducible representations.

Lemma 3.5. *Let L be an algebraic extension of F, and let (π, V) be an F-representation of G. Suppose that*

$$V_L = W_1 \oplus \cdots \oplus W_m$$

is a finite direct sum of absolutely irreducible representations.

(i) *There exists a finite extension E/F such that*

$$V_E = U_1 \oplus \cdots \oplus U_m,$$

where each U_i is an absolutely irreducible representation of G satisfying $(U_i)_L = W_i$.

(ii) *If a component W_i of V_L is supercuspidal, then its descent U_i is also supercuspidal.*

Proof. (i) Let $V_L = W_1 \oplus \cdots \oplus W_m$ be a decomposition of V_L into absolutely irreducible L-representations. For each i, choose a nonzero vector $w_i \in W_i$. Then V_L is generated as an $L[G]$-module by $w_1, \ldots, w_m$.

Each $w_j \in V_L = V \otimes_F L$ can be written as a finite sum

$$w_j = \sum_{k=1}^{n_j} v_{jk} \otimes c_{jk},$$

where $c_{jk} \in L$ and $v_{jk} \in V$. Since each c_{jk} is algebraic over F, there exists a finite extension E_{jk}/F containing c_{jk}. Let E be the composite of all such fields E_{jk}. Then E/F is a finite extension and each w_j lies in $V_E = V \otimes_F E$.

Let $U_j \subset V_E$ be the $E[G]$-subrepresentation generated by w_j. Then $V_E = U_1 + \cdots + U_m$, and $(U_j)_L = U_j \otimes_E L = W_j$. Since each W_j is absolutely irreducible, it follows that U_j is absolutely irreducible as an E-representation. Moreover, the sum above is direct because it becomes direct after extension of scalars to L. Therefore,

$$V_E = U_1 \oplus \cdots \oplus U_m$$

is a direct sum of absolutely irreducible E-representations, proving (i).

(ii) Suppose W_i is supercuspidal. Then it is admissible, and by Proposition 2.3, U_i is also admissible. Being absolutely irreducible, U_i possesses a central character χ_i, and this character must coincide with the central character of W_i.

To show that the matrix coefficients of U_i are compactly supported modulo the center, take arbitrary $v \in U_i$ and $\lambda \in \widetilde{U}_i$. By Proposition 2.4, $\widetilde{U}_i \otimes_E L \cong \widetilde{W}_i$, with an isomorphism given by $\lambda \otimes b \mapsto (v \otimes a \mapsto ab\lambda(v))$. Consider the matrix coefficient of $v \otimes 1 \in W_i$ and $\lambda \otimes 1 \in \widetilde{W}_i$. Then for every $g \in G$,

$$\begin{aligned} m_{v\otimes 1,\lambda\otimes 1}(g) &= \langle \pi(g)v \otimes 1, \lambda \otimes 1 \rangle \\ &= \lambda(\pi(g)v) \\ &= m_{v,\lambda}(g), \end{aligned}$$

so $m_{v,\lambda} = m_{v\otimes 1,\lambda\otimes 1}$. Since W_i is supercuspidal, $m_{v,\lambda}$ has compact support modulo the center, completing the proof. ∎

4. Induced representations

Let H be a closed subgroup of G and let (σ, W) be a smooth representation of H. We denote by $\mathrm{Ind}_H^G W$ the space of functions $f : G \to W$ such that

(i) $f(hg) = \sigma(h)f(g)$, for all $h \in H$, $g \in G$, and

(ii) there exists a compact open subgroup K_f of G such that $f(gk) = f(g)$, for all $g \in G$, $k \in K_f$.

We denote by $\text{c-Ind}_H^G W$ the subspace of $\mathrm{Ind}_H^G W$ of functions with compact support modulo H. The group G acts on both spaces by right translations.

Proposition 4.1. *Let H be a closed subgroup of G and let (σ, W) be a smooth representation of H. Then,*

$$\widetilde{\text{c-Ind}_H^G W} \cong \mathrm{Ind}_H^G \widetilde{W} \delta_H^{-1} \delta_G.$$

Proof. If G is unimodular, this is [Cas, Theorem 2.4.2]. For the adjustment by the modular character, see Duality Theorem in [BH06, Section 3.5] or [Ren10, Théorème III.2.7]. □

If E is an extension of F, we have a canonical embedding

$$\left(\mathrm{Ind}_{ZH}^G W\right) \otimes_F E \hookrightarrow \mathrm{Ind}_{ZH}^G (W \otimes_F E),$$

given by $f \otimes a \mapsto f \otimes a$.

Proposition 4.2. *Let H be a compact open subgroup of G and let (σ, W) be a smooth irreducible F-representation of ZH. Let E be an extension of F.*

(i) *For any extension E/F,*

$$\text{c-Ind}_{ZH}^G (W \otimes_F E) \cong \left(\text{c-Ind}_{ZH}^G W\right) \otimes_F E.$$

(ii) *If $\mathrm{Ind}_{ZH}^G W$ is admissible or E/F is finite, then*

$$\left(\mathrm{Ind}_{ZH}^G W\right) \otimes_F E \cong \mathrm{Ind}_{ZH}^G (W \otimes_F E).$$

Proof. (i) Let $\mathcal{G}$ be a set of coset representatives of G/ZH. Then

$$\text{c-Ind}_{ZH}^G (W) \cong \bigoplus_{g \in \mathcal{G}} gW \tag{3}$$

(see [BH06, Lemma 2.5])). This, together with general properties of tensor products, gives us the following:

$$\text{c-Ind}_{ZH}^{G}(W \otimes_F E) \cong \bigoplus_{g \in \mathcal{G}} g(W \otimes_F E) \cong \left(\bigoplus_{g \in \mathcal{G}} gW \right) \otimes_F E \cong \left(\text{c-Ind}_{ZH}^{G} W\right) \otimes_F E.$$

(ii) Since H is compact, every character $\chi : H \to \mathbb{R}_{>0}$ is trivial. Hence $\delta_H = 1$ and $\delta_G|_H = 1$. Moreover, $\delta_G(z) = 1$ for every $z \in Z$. Therefore, the modular characters appearing in Proposition 4.1 are trivial, and we have

$$\operatorname{Ind}_{ZH}^{G} W \cong \left(\text{c-Ind}_{ZH}^{G} \widetilde{W}\right)^{\sim}. \tag{4}$$

Then by Proposition 2.1.10 of [Cas], $\operatorname{Ind}_{ZH}^{G} W$ is admissible, if and only if $\text{c-Ind}_{ZH}^{G} \widetilde{W}$ is admissible.

Suppose that $\operatorname{Ind}_{ZH}^{G} W$ is admissible or E/F is finite. Then,

$$\begin{aligned}
\left(\operatorname{Ind}_{ZH}^{G} W\right) \otimes_F E &\cong \left(\text{c-Ind}_{ZH}^{G} \widetilde{W}\right)^{\sim} \otimes_F E && \text{(by Proposition 4.1)}\\
&\cong \left(\left(\text{c-Ind}_{ZH}^{G} \widetilde{W}\right) \otimes_F E\right)^{\sim} && \text{(by Proposition 2.4)}\\
&\cong \left(\text{c-Ind}_{ZH}^{G} \left(\widetilde{W} \otimes E\right)\right)^{\sim} && \text{(by part (i))}\\
&\cong \left(\text{c-Ind}_{ZH}^{G} (\widetilde{W \otimes E})\right)^{\sim} && \text{(by Proposition 2.4)}\\
&\cong \operatorname{Ind}_{ZH}^{G}(W \otimes_F E) && \text{(by Proposition 4.1)}
\end{aligned}$$

□

Proposition 4.3. *Let H be a compact open subgroup of G. Suppose that (σ, W) is a smooth finitely generated representation of ZH. Then $\text{c-Ind}_{ZH}^{G} W$ admits an irreducible quotient, and $\operatorname{Ind}_{ZH}^{G} W$ admits an irreducible subrepresentation.*

Proof. We use standard arguments from representation theory involving Zorn's lemma.

Set $V = \text{c-Ind}_{ZH}^{G} W$. Let R be a finite set of generators of W as an H-representation. By equation (3), the same set R generates V as a G-representation.

Let $\mathcal{S}$ be the family of all proper G-invariant subspaces of V. Then $\mathcal{S}$ is nonempty because it contains the zero space. Let $\mathcal{C}$ be a chain in $\mathcal{S}$, and define

$$Y = \bigcup_{U \in \mathcal{C}} U.$$

Then Y is a G-invariant subspace of V. Furthermore, $Y \neq V$; otherwise, we would have $R \subset U$ for some $U \in \mathcal{C}$, which would imply $U = V$, a contradiction. Hence, $Y \in \mathcal{S}$.

This shows that every chain in $\mathcal{S}$ has an upper bound. By Zorn's lemma, $\mathcal{S}$ has a maximal element M. Then the quotient V/M is irreducible, which proves the first statement.

For the second statement, we first observe that $\text{c-Ind}_{ZH}^G \widetilde{W}$ admits an irreducible quotient Q. Equation (4) implies $\left(\text{c-Ind}_{ZH}^G \widetilde{W}\right)^{\sim} \cong \text{Ind}_{ZH}^G W$. The contragredient $\widetilde{Q}$ is then an irreducible subrepresentation of $\text{Ind}_{ZH}^G W$. $\square$

5. Structure of induced representations

In this section, F is an algebraic extension of $\mathbb{Q}_p$, and $\overline{\mathbb{Q}}_p$ denotes an algebraic closure of $\mathbb{Q}_p$ containing F. We establish versions of Theorems 1 and 2 of [Bus90] that hold over F. Both theorems automatically hold over $\overline{\mathbb{Q}}_p$ via an isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ (see Proposition 2.1 and Remark 5.2).

5.1. Admissibility of induced representations.

Theorem 5.1. *Let G be a separable, locally profinite group with center Z. Let H be an open compact subgroup of G. Let (σ, W) be an absolutely irreducible smooth F-representation of ZH. The following conditions are equivalent:*

- (i) $\text{c-Ind}_{ZH}^G(\sigma)$ *is admissible;*
- (ii) $\text{Ind}_{ZH}^G(\sigma)$ *is admissible;*
- (iii) $\text{c-Ind}_{ZH}^G(\sigma) = \text{Ind}_{ZH}^G(\sigma)$ *(i.e., the canonical inclusion map* $\text{c-Ind}_{ZH}^G(\sigma) \hookrightarrow \text{Ind}_{ZH}^G(\sigma)$ *is an isomorphism);*
- (iv) *There exists a finite extension E/F such that $\left(\text{Ind}_{ZH}^G(\sigma)\right)_E$ is a finite direct sum of absolutely irreducible supercuspidal representations of G.*

Moreover, if (σ, W) satisfies these conditions, then so does $(\tilde{\sigma}, \tilde{W})$.

Remark 5.2. Theorems 1 and 2 of [Bus90] assume that G is unimodular. As stated in the paper, "This restriction is largely for convenience." This assumption can be removed with only minor modifications to the proofs, and these changes do not affect the statements of the theorems.

Consequently, Theorem 5.1 holds for complex representations (noting that the finite extension $E/\mathbb{C}$ in assertion (iv) is simply $\mathbb{C}$). By Proposition 2.1, it also holds for $\overline{\mathbb{Q}}_p$-representations.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) can be proved in the same way as in [Bus90].

It remains to prove (i) $\Leftrightarrow$ (iv).

Assume $\text{c-Ind}_{ZH}^G W$ is admissible. By Proposition 4.2,

$$\text{c-Ind}_{ZH}^G(W \otimes_F \overline{\mathbb{Q}}_p) \simeq \big(\text{c-Ind}_{ZH}^G W\big) \otimes_F \overline{\mathbb{Q}}_p.$$

Then Proposition 2.3 implies that $\text{c-Ind}_{ZH}^G(W \otimes_F \overline{\mathbb{Q}}_p)$ is also admissible. The implication (i) $\Rightarrow$ (iv) for $\text{c-Ind}_{ZH}^G(W \otimes_F \overline{\mathbb{Q}}_p)$ tells us that

$$\text{c-Ind}_{ZH}^G(W \otimes_F \overline{\mathbb{Q}}_p) = W_1 \oplus \cdots \oplus W_r,$$

a finite direct sum of irreducible supercuspidal representations of G. Now, Lemma 3.5 implies that there exists a finite extension E/F such that

$$\big(\text{c-Ind}_{ZH}^G W\big) \otimes_F E = U_1 \oplus \cdots \oplus U_r,$$

where each U_i is an absolutely irreducible supercuspidal representation, proving (iv).

The converse (iv) $\Rightarrow$ (i) is clear.

For the final statement of the theorem, assume $\text{c-Ind}_{ZH}^G W$ is admissible, so (i)-(iv) hold for (σ, W). By Proposition 2.1.10 of [Cas], $\widetilde{\text{c-Ind}_H^G W}$ is also admissible. Proposition 4.1 gives us

$$\widetilde{\text{c-Ind}_H^G W} \cong \text{Ind}_H^G \widetilde{W}.$$

Then $\text{Ind}_H^G \widetilde{W}$ is also admissible, and (i)-(iv) hold for $(\tilde{\sigma}, \tilde{W})$. $\square$

5.2. **Decomposition of induced representations.** If V is a smooth representation of G, we denote by V_{cusp} the sum of all supercuspidal subspaces of V.

Theorem 5.3. *Let G be a separable, locally profinite group with center Z. Let H be an open compact subgroup of G. Let (σ, W) be an absolutely irreducible smooth F-representation of ZH. Set*

$$V = \text{c-Ind}_{ZH}^{G} W.$$

Then, there exist a finite extension E/F and uniquely determined G-subspaces $X \subseteq V_E$ and $Y \subseteq \mathrm{Ind}_{ZH}^{G} W_E$ such that

$$(\text{c-Ind}_{ZH}^{G} W)_E = \text{c-Ind}_{ZH}^{G} W_E = (V_E)_{\mathrm{cusp}} \oplus X, \quad \text{and}$$
$$(\mathrm{Ind}_{ZH}^{G} W)_E = \mathrm{Ind}_{ZH}^{G} W_E = (V_E)_{\mathrm{cusp}} \oplus Y,$$

Moreover,

(i) *$(V_E)_{\mathrm{cusp}}$ is a finite sum of absolutely irreducible supercuspidal G-spaces;*
(ii) *$X \subseteq Y$;*
(iii) *X has no absolutely irreducible admissible G-subspaces;*
(iv) *Y has no G-invariant supercuspidal subquotients.*

Remark 5.4.

(1) For complex representations, this is Theorem 2 of [Bus90], except for (iv), which is stronger than the statement of that theorem but is established in its proof.
(2) Proposition 2.1 implies that the theorem also holds for $\overline{\mathbb{Q}}_p$-representations.
(3) If $\text{c-Ind}_{ZH}^{G} W$ is admissible, then by Theorem 5.1 we have $X = Y = 0$.

Proof. We divide the proof into several steps.

Step 1: Existence of X. By Proposition 4.2(i),

$$V_{\overline{\mathbb{Q}}_p} = \text{c-Ind}_{ZH}^{G} W_{\overline{\mathbb{Q}}_p}.$$

Then, we know that the theorem holds for $V_{\overline{\mathbb{Q}}_p}$. Hence, there exists a uniquely determined G-subspace $\mathcal{X} \subseteq V_{\overline{\mathbb{Q}}_p}$ such that

$$V_{\overline{\mathbb{Q}}_p} = (V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}} \oplus \mathcal{X}, \tag{5}$$

and $\mathcal{X}$ has no irreducible admissible G-subspaces. Moreover, $(V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}}$ decomposes as a finite direct sum of absolutely irreducible representations

$$(V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}} = W_1 \oplus \cdots \oplus W_m.$$

Using similar arguments as in the proof of Lemma 3.5, we can show that there exist a finite extension E/F and absolutely irreducible supercuspidal subrepresentations $U_1, \ldots, U_m$ of V_E such that $U_1 \oplus \cdots \oplus U_m \subseteq V_E$ and

$$(U_i)_{\overline{\mathbb{Q}}_p} = W_i. \tag{6}$$

Notice that the center Z acts by a character χ. Then all the representations in the theorem are χ-representations. In particular, the U_i are χ-representations.

By Corollary 3.4, the representation $U_1 \oplus \cdots \oplus U_m$ is injective and projective in the category of smooth χ-representations of G on E-vector spaces. Therefore, the embedding

$$U_1 \oplus \cdots \oplus U_m \hookrightarrow V_E$$

splits, and there exists a G-subspace X of V_E such that

$$V_E = (U_1 \oplus \cdots \oplus U_m) \oplus X.$$

By equations (5) and (6), using the uniqueness of $\mathcal{X}$, we get

$$\mathcal{X} = X_{\overline{\mathbb{Q}}_p}.$$

It follows that X has no absolutely irreducible admissible G-subspaces; otherwise, $\mathcal{X}$ would also have irreducible admissible G-subspaces. This proves (iii). Moreover,

$$(V_E)_{\mathrm{cusp}} = U_1 \oplus \cdots \oplus U_m,$$

proving (i). We also remark that

$$(V_E)_{\mathrm{cusp}} \otimes_E \overline{\mathbb{Q}}_p = W_1 \oplus \cdots \oplus W_m = (V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}}. \tag{7}$$

Finally,

$$V_E = (V_E)_{\mathrm{cusp}} \oplus X.$$

Step 2: Existence of Y. We consider the inclusions

$$(V_E)_{\mathrm{cusp}} \hookrightarrow V_E = \text{c-Ind}_{ZH}^G W_E \hookrightarrow \mathrm{Ind}_{ZH}^G W_E.$$

We use again the injectivity of $(V_E)_{\mathrm{cusp}}$. It follows that there exists a G-invariant subspace Y such that

$$\mathrm{Ind}_{ZH}^G W_E = (V_E)_{\mathrm{cusp}} \oplus Y.$$

Let $\mathcal{Y}$ be the unique G-invariant subspace of $\mathrm{Ind}_{ZH}^G W_{\overline{\mathbb{Q}}_p}$ such that

$$\mathrm{Ind}_{ZH}^G W_{\overline{\mathbb{Q}}_p} = (V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}} \oplus \mathcal{Y}.$$

Then, by (iv), $\mathcal{Y}$ has no G-invariant supercuspidal subquotients. By Proposition 4.2(ii),

$$(\mathrm{Ind}_{ZH}^G W) \otimes E = \mathrm{Ind}_{ZH}^G W_E \quad \text{and} \quad (\mathrm{Ind}_{ZH}^G W) \otimes_F \overline{\mathbb{Q}}_p \subset \mathrm{Ind}_{ZH}^G W_{\overline{\mathbb{Q}}_p}.$$

Tensoring with $\overline{\mathbb{Q}}_p$ and using (7), we obtain an embedding

$$(V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}} \oplus Y_{\overline{\mathbb{Q}}_p} \hookrightarrow (V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}} \oplus \mathcal{Y}.$$

Every $x \in Y_{\overline{\mathbb{Q}}_p}$ can be written uniquely as $x = x_{\mathrm{cusp}} + x_{\mathcal{Y}}$, where $x_{\mathrm{cusp}} \in (V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}}$ and $x_{\mathcal{Y}} \in \mathcal{Y}$. The map $x \mapsto x_{\mathcal{Y}}$ is injective because $(V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}} \cap Y_{\overline{\mathbb{Q}}_p} = 0$, so $\mathcal{Y}$ contains a subrepresentation isomorphic to $Y_{\overline{\mathbb{Q}}_p}$. On the other hand, $x \mapsto x_{\mathrm{cusp}}$ gives a G-equivariant map

$$\phi\colon Y_{\overline{\mathbb{Q}}_p} \longrightarrow (V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}}.$$

Since $(V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}}$ is a finite direct sum of supercuspidal representations, any nonzero element in $\mathrm{Hom}_G(Y_{\overline{\mathbb{Q}}_p}, (V_{\overline{\mathbb{Q}}_p})_{\mathrm{cusp}})$ would produce a supercuspidal quotient of $Y_{\overline{\mathbb{Q}}_p}$. However, $Y_{\overline{\mathbb{Q}}_p}$ embeds into $\mathcal{Y}$, which has no G-invariant supercuspidal subquotients. Therefore $\phi = 0$, and every $x \in Y_{\overline{\mathbb{Q}}_p}$ satisfies $x_{\mathrm{cusp}} = 0$, giving

$$Y_{\overline{\mathbb{Q}}_p} \subseteq \mathcal{Y}.$$

From this we deduce that Y has no G-invariant supercuspidal subquotients, proving (iv).

Step 3: Uniqueness of X and Y. Suppose that Y and Y' are subspaces of $\operatorname{Ind}_{ZH}^G W_E$ such that

$$\operatorname{Ind}_{ZH}^G W_E = (V_E)_{\mathrm{cusp}} \oplus Y = (V_E)_{\mathrm{cusp}} \oplus Y'.$$

By Step 2, neither Y nor Y' has G-invariant supercuspidal subquotients. The decomposition $\operatorname{Ind}_{ZH}^G W_E = (V_E)_{\mathrm{cusp}} \oplus Y$ yields two projections onto $(V_E)_{\mathrm{cusp}}$ and Y. Denote by ϕ the projection onto $(V_E)_{\mathrm{cusp}}$. Since Y' has no G-invariant supercuspidal quotients, we have $\phi|_{Y'} = 0$. This shows that $Y' \subseteq Y$. By symmetry, $Y \subseteq Y'$, and hence $Y = Y'$. This proves the uniqueness of Y.

The uniqueness of X is proved in the same way. Finally, $\phi|_X$ must also be zero, so $X \subseteq Y$, proving (ii). □

6. Smooth representations embedded in p-adic Banach spaces

The purpose of this section is illustrative. It shows how the study of p-adic Banach space representations led to the consideration of smooth F-representations described in Theorems 5.1 and 5.3.

Let $\mathbb{Q}_p \subseteq L \subseteq F$ be a sequence of finite extensions. Denote by $\mathcal{O}_F$ the ring of integers of F. In this section, G denotes the group of L-points of a reductive group defined over L. Then G is unimodular.

Suppose that U is an F-Banach space representation of G. The space U^{sm} of smooth vectors may be zero. However, if U^{sm} is nonzero and, moreover, if U is topologically irreducible, then U^{sm} is dense in U.

Among F-Banach space representations, there is an important class of admissible Banach space representations. A Banach space representation U is said to be *admissible* if its continuous dual U' is finitely generated as a module over the Iwasawa algebra $F[[H]]$[2] for some compact open subgroup H of G [ST02]. A Banach space representation is said to be *unitary* if it is norm-preserving.

[2]The Iwasawa algebra of a compact Lie group H over $\mathcal{O}_F$ is defined by

$$\mathcal{O}_F[[H]] = \varprojlim_N \mathcal{O}_F[H/N],$$

where N runs over the set of open normal subgroups of H. Then $F[[H]] = F \otimes_{\mathcal{O}_F} \mathcal{O}_F[[H]]$.

Admissible unitary Banach space representations play a fundamental role in the p-adic Langlands program. Those containing smooth (or, more generally, locally algebraic) vectors lie at the heart of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Col10], [CDP14]).

Since admissibility for a Banach space representation U is defined in terms of its continuous dual U', if we wish to study admissible Banach space representations that contain smooth vectors, it is natural to consider the smooth dual $\widetilde{U^{\mathrm{sm}}}$ and its relationship with U'. In this section, we briefly discuss this relationship.

6.1. **The completion of a compactly induced representation.** Suppose that (π, V) is an absolutely irreducible supercuspidal representation of G obtained by compact induction as in Theorem 5.1. Thus, there exist a compact open subgroup H and an absolutely irreducible smooth F-representation (σ, W) of ZH such that

$$V = \text{c-Ind}_{ZH}^{G} W.$$

Assume that the central character of W is unitary. Then we can equip V with a G-invariant norm as follows. First, we observe that there exists a norm $\| \, \|_W$ on W making it a unitary Banach representation of H (this follows from [Sch02, Proposition 4.13] and [Ban23, Lemma 4.32]). Next, for a function $f\colon G \to W$ in $\text{c-Ind}_{ZH}^{G} W$, we define

$$\|f\| = \sup_{g \in G} \|f(g)\|_W.$$

Let U denote the completion of V with respect to this norm. It can be shown (full details will appear elsewhere) that U is a unitary Banach space representation of G, but is not admissible if G/ZH is infinite. Moreover, U is the universal unitary completion of V,[3] provided that we equip V with the finest locally convex topology.

[3]Let V be a locally convex topological F-vector space equipped with a continuous G-action, and let U be an F-Banach space equipped with a unitary G-action. We say that a continuous F-linear G-equivariant map $\alpha\colon V \to U$ realizes U as a *universal unitary completion* of V if, for every continuous F-linear G-equivariant map $\beta\colon V \to X$, where X is an F-Banach space equipped with a unitary G-action, there exists a unique continuous F-linear G-equivariant map $U \to X$ such that β factors through α. [Eme05]

If X is an admissible quotient representation of U, then X' is a submodule of U' and it is finitely generated as an $F[[H]]$-module. Thus, we want to find finitely generated $F[[H]]$-submodules of U'.

6.2. **Smooth and continuous duals.** We use the decomposition of V given in (3). Let $\mathcal{G}$ be a set of representatives for G/ZH. Then

$$V = \text{c-Ind}_{ZH}^{G}(W) \cong \bigoplus_{g\in\mathcal{G}} gW \quad \text{and} \quad U = \hat{\bigoplus}_{g\in\mathcal{G}} gW, \tag{8}$$

where $\hat{\bigoplus}$ denotes the completed direct sum.

We claim that $U^{\text{sm}} = V$. To prove this, fix a set $\mathcal{R}$ of representatives for $H\backslash G/ZH$, and let V_r denote the subspace of V consisting of vectors supported on $HrZH$. Then

$$V = \bigoplus_{r\in\mathcal{R}} V_r \quad \text{and} \quad U = \hat{\bigoplus}_{r\in\mathcal{R}} V_r.$$

Every vector $u \in U$ can be written uniquely as $u = \sum_{r\in\mathcal{R}} v_r$, where $v_r \in V_r$. Let $u \in U^{\text{sm}}$, so $u \in U^{K_0}$ for some compact open subgroup K_0 of G. Since $U^{K_0} \subseteq U^{K_0\cap H}$, we may replace K_0 by $K := K_0 \cap H \subseteq H$, and assume $u \in U^K$. For $k \in H \subseteq ZH$ we have $ZHk^{-1} = ZH$, so $HrZH\,k^{-1} = Hr\,(ZHk^{-1}) = HrZH$ for every $r \in \mathcal{R}$; hence $\pi(k)$ preserves each V_r. Since $u = \sum_r v_r$ is K-invariant, comparing components in the decomposition of $\pi(k)u = u$ shows that each v_r is itself K-invariant, i.e., $v_r \in V^K$. As V is admissible, V^K is finite-dimensional, so only finitely many v_r are nonzero. Therefore $u \in V$, and hence $U^{\text{sm}} = V$.

As a supercuspidal representation, V is admissible. Then, by Theorem 5.1,

$$V = \text{c-Ind}_{ZH}^{G}(W) = \text{Ind}_{ZH}^{G}(W).$$

Let $W^* = \text{Hom}_F(W, F)$ be the algebraic dual of W. Since W is finite-dimensional, we have $\widetilde{W} = W^*$. By Proposition 4.1 and Theorem 5.1 for $\widetilde{W}$,

$$\widetilde{V} = \widetilde{\text{c-Ind}_{ZH}^{G}}\, W \cong \text{Ind}_{ZH}^{G}\,\widetilde{W} = \text{c-Ind}_{ZH}^{G}\,\widetilde{W} = \bigoplus_{g\in\mathcal{G}} (gW)^*.$$

Both $\widetilde{V}$ and U' belong to the algebraic dual

$$\text{Hom}_F\left(\bigoplus_{g\in\mathcal{G}} gW, F\right) \cong \prod_{g\in\mathcal{G}} (gW)^*.$$

A linear functional on a Banach space is continuous if and only if it is bounded with respect to the operator norm. From equation (8), it follows that

$$U' = \left\{ \lambda = (\lambda_g) \in \prod_{g \in \mathcal{G}} (gW)^* \;\middle|\; \sup_{g \in \mathcal{G}} \|\lambda_g\|_{W^*} < \infty \right\},$$

namely, the subspace of $\prod_{g \in \mathcal{G}} (gW)^*$ consisting of bounded elements. If $\lambda = (\lambda_g) \in \widetilde{V}$, then only finitely many λ_g are nonzero, so λ is bounded. Hence,

$$\widetilde{V} \subset U'.$$

In the Schneider–Teitelbaum duality (or Schikhof duality), we equip U' with the bounded-weak topology. Then $\widetilde{V}$ is dense in U' with respect to this topology (a proof will appear elsewhere). So proper submodules of U' are generated by elements outside $\widetilde{V}$.

Note that the definition of $\widetilde{V}$ involves the G-action, whereas the definition of U' involves only the Banach space structure. We may consider vectors in the algebraic dual of V that are smooth with respect to a subgroup of G. Those that are bounded belong to U'. This flexibility provides a wide variety of vectors in U' not contained in $\widetilde{V}$.

Example. Let P be a minimal parabolic subgroup of G, and let K be a maximal compact subgroup such that $G = PK$. Assume that V is a supercuspidal representation obtained by compact induction from ZK:

$$V = \text{c-Ind}_{ZK}^{G}(W).$$

Choose a set of representatives $\mathcal{P}$ for $P/Z(P \cap K)$. Then $\mathcal{P}$ is also a set of representatives for G/ZK because $G/K \cong P/(P \cap K)$. It is interesting to note that

$$V = \text{c-Ind}_{ZK}^{G}(W) \cong \bigoplus_{g \in \mathcal{P}} gW \cong \text{c-Ind}_{Z(P\cap K)}^{P}\big(W|_{Z(P\cap K)}\big),$$

as vector spaces and also as P-representations. By Proposition 4.1, together with the facts that $\delta_P(k) = 1$ for all $k \in Z(P \cap K)$ and $\widetilde{W} = W^*$, the smooth dual of $V|_P$ is isomorphic to

$$\text{Ind}_{Z(P\cap K)}^{P}\big(\widetilde{W}|_{Z(P\cap K)}\big).$$

This is significantly larger than the smooth dual of V, which is isomorphic to $\text{c-Ind}_{ZK}^{G} \widetilde{W}$. In the notation of Theorem 5.3 applied to $\text{c-Ind}_{Z(P\cap K)}^{P}\big(\widetilde{W}|_{Z(P\cap K)}\big)$, we have $X = 0$ and every nonzero vector of Y is a P-smooth vector in V^* that is not compactly supported modulo $Z(P \cap K)$. It is easy to construct both bounded and unbounded vectors in Y. Thus $Y \cap U' \neq 0$ but $Y \not\subset U'$.

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