SOME RESULTS ON REDUCIBILITY FOR UNITARY GROUPS AND LOCAL ASAI L-FUNCTIONS

David Goldberg*

Introduction. Let F be a p-adic field of characteristic zero and let $\bar{F}$ be the algebraic closure of F. In [**20**], Shahidi describes the relationship of the poles of certain Langlands L-functions attached to representations of $GL(n,F)$, to the theory of twisted endoscopy. Here we carry out this program for the generalized Asai L-function attached to a representation of $GL(n,E)$, where E/F is a quadratic extension. The problem is equivalent to certain reducibility questions for unitary groups.

More precisely, let $\mathbf{G}=\mathbf{U}(n,n)$ be the quasi-split unitary group in $2n$ variables defined with respect to E/F. Let $G=\mathbf{G}(F)$. Then there is a maximal parabolic subgroup $\mathbf{P}=\mathbf{MN}$ of $\mathbf{G}$, with $\mathbf{M}$ isomorphic to $\mathrm{Res}_{E/F}(\mathbf{GL}_n)$. Thus, the F-points are given by $M=\mathbf{M}(F)\simeq GL(n,E)$. The L-group ${}^LM$ of $\mathbf{M}$ is isomorphic to

$$(GL(n,\mathbb{C})\times GL(n,\mathbb{C}))\rtimes W_F,$$

where W_F is the Weil group of $\bar{F}/F$. Let Ψ be the adjoint representation of ${}^LM$ acting on the Lie algebra ${}^L\mathfrak{n}$ of ${}^LN$ [**17**, §4]. This is the generalization to rank n of the situation studied by Asai [**1**]. Let w be the non-trivial element of the Weyl group of G with respect to M. Suppose π is an irreducible unitary supercuspidal representation of M. Then $\pi^w\simeq\pi$ if and only if π is invariant under the automorphism ε of $GL(n,E)$ which takes g to ${}^t\bar{g}^{-1}$ (here $\bar{g}$ is the Galois conjugate of g). If $\pi^w\simeq\pi$, then the unitarily induced representation $\mathrm{Ind}_P^G(\pi\otimes 1_N)$ is irreducible if and only if $L(s,\pi,\Psi)$ has a pole at $s=0$ [**18**].

We compute the poles of $L(s,\pi,\Psi)$ by computing the residue of the standard intertwining operator that determines the reducibility of $I(\pi)=\mathrm{Ind}_P^G(\pi\otimes 1_N)$. In

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Section 2, we prove that if $\pi^w \simeq \pi$, then $I(\pi)$ is irreducible if and only if there is some matrix coefficient φ of π for which a certain sum of ε-twisted orbital integrals is non-zero (cf. Theorems 2.7 and 2.8). More precisely, if $\pi^w \simeq \pi$, then $I(\pi)$ is irreducible if and only if there is some matrix coefficient φ of π such that

$$\begin{cases} \Phi_\varepsilon^\kappa(\delta,\varphi) \neq 0 & \text{if } n \text{ is even} \\ \Phi_\varepsilon^{st}(\delta,\varphi) \neq 0 & \text{if } n \text{ is odd,} \end{cases}$$

where δ is a hermitian form in $GL(n,E)$, $\Phi_\varepsilon^\kappa(\delta,\varphi)$ is the non-stable sum of twisted orbital integrals, and $\Phi_\varepsilon^{st}(\delta,\varphi)$ is the stable sum (cf. Section 1). The theory of twisted endoscopy, as developed by Kottwitz and Shelstad [**12**,**13**], says that the non-vanishing of such sums points to the representation π being a lift from the group $U(n)$ (cf. Section 4). In Section 4 we show that, for $n=3$, the pole of $L(s,\pi,\Psi)$ at $s=0$ determines whether or not π is a standard base change lift from $U(3)$ [**14**]. This is similar to our results for $n=2$ [**6**].

In Section 5, we explicitly determine the generalized Asai L-function $L(s,\sigma,\Psi)$ for any irreducible admissible representation σ of $GL(n,E)$. We use the results of Section 2 to compute $L(s,\pi,\Psi)$ for any irreducible supercuspidal representation π of $GL(n,E)$. We then use Theorem 3.5 of [**18**] to compute $L(s,\sigma,\Psi)$ for σ in the discrete series. This determines $L(s,\sigma,\Psi)$ in general. One consequence of these computations is the following identity (cf. Corollary 5.5);

$$L(s,\sigma\times\bar\sigma) = L(s,\sigma,\Psi)\; L(s,\sigma\otimes\mu\circ\det,\Psi), \tag{1}$$

where $\bar\sigma(g)=\sigma(\bar g)$, μ is an extension to $E^\times$ of the local class field theory character attached to E/F, and $L(s,\sigma\times\bar\sigma)$ is the Rankin-Selberg product L-function attached to σ and $\bar\sigma$ [**10**].

In Section 6 we consider a second group where reducibility of induced representations is also determined by Asai L-functions (by means of (1)). Let $\mathbf{G}'=\mathbf{U}(n,n+1)$ be the quasi-split unitary group in $2n+1$ variables defined with respect to E/F. Then there is a maximal parabolic $\mathbf{P}'=\mathbf{M}'\mathbf{N}'$, with $\mathbf{M}'\simeq \mathrm{Res}_{E/F}(\mathbf{GL}_n)\times\mathbf{U}(1)$. Thus, $M'=\mathbf{M}'(F)\simeq GL(n,E)\times U(1)$. If π is an irreducible unitary representation of $GL(n,E)$, and ν is a character of $U(1)$, then we denote by (π,ν) the representation of M' given by $(\pi,\nu)(g,y)=\pi(g)\nu(y\det(g\varepsilon(g)))$. Suppose that

π is an irreducible unitary supercuspidal representation of $GL(n,E)$. Then, unless $n=1$ and $\pi=1$, the reducibility of $I(\pi,\nu)'=\mathrm{Ind}_{P'}^{G'}(\pi,\nu)$ is determined by whether or not $L(s,\pi\otimes\mu\circ\det,\Psi)$ has a pole at $s=0$. If $\pi^w\simeq\pi$, then $L(s,\pi\times\bar{\pi})$ has a simple pole at $s=0$ [**10**], and therefore, by (1), exactly one of $L(s,\pi,\Psi)$ and $L(s,\pi\otimes\mu\circ\det,\Psi)$ has a pole at $s=0$. Thus, exactly one of $I(\pi)$ and $I(\pi,\nu)'$ is reducible (cf. Proposition 6.2). The results of Section 2 allow us to explicitly describe the reducibility criteria for $I(\pi,\nu)'$ in terms of twisted endoscopy (cf. Theorem 6.3). It is interesting to note how these results parallel those of Shahidi [**20**], Theorem 6.3.

In Section 7, we use the explicit formulas for $L(s,\sigma,\Psi)$, with σ a discrete series representation to determine the reducibility for $\mathrm{Ind}_P^G(\sigma)$ and $\mathrm{Ind}_{P'}^{G'}(\sigma,\nu)$ (cf. Theorem 7.1). These results are similar to the reducibility criteria for $Sp(2n,F)$ and $SO(2n+1,F)$ determined by Shahidi [**20**].

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§1 Twisted Orbital Integrals for $GL(n)$.

Let F be a nonarchimedean local field of characteristic 0. Let $\mathcal{R}_F$ be its ring of integers and $\mathfrak{p}_F$ the unique maximal ideal in $\mathcal{R}_F$. Let ϖ_F be a uniformizer in F, that is, $\mathfrak{p}_F=\varpi_F\mathcal{R}_F$. Let $q_F=|\mathcal{R}_F/\mathfrak{p}_F|$ be the residual characteristic of F. Let $\bar{F}$ be a (separable) algebraic closure of F.

Let E be a quadratic extension of F. Suppose $E=F(\beta)$, with $\beta^2\in F^\times\setminus(F^\times)^2$. Let $\mathcal{R}_E$, $\mathfrak{p}_E$, ϖ_E, and q_E be the appropriate objects in E. Let $\tau: E\longrightarrow E$ be the non-trivial Galois automorphism of E/F. We also denote the action of τ by $\tau(x)=\overline{x}$. Let $N: E^\times\longrightarrow F^\times$ be the norm map, $N(x)=x\overline{x}$. We assume that $\bar{\beta}=-\beta$.

Let $\mathbf{H} = \mathbf{U}(n)$ as an algebraic group over F. $\mathbf{H}$ is defined as follows. Let

$$\Phi_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \cdot & & & \\ \cdot & & & \end{pmatrix}, \text{ and } \delta_n = \begin{cases} \Phi_n & n \text{ odd} \\ \beta\Phi_n & n \text{ even.} \end{cases}$$

Then we let $\mathbf{H} = \{g \in \mathbf{GL}(n) | g\delta_n {}^t\bar{g} = \delta_n\}$. Let $\widetilde{\mathbf{H}} = \mathrm{Res}_{E/F}(\mathbf{H})$. Then $\widetilde{\mathbf{H}}$ is an algebraic group such that $\widetilde{\mathbf{H}}(F) = \mathbf{H}(E) = GL(n, E)$ [**3**]. Over E, $\widetilde{\mathbf{H}} \simeq \mathbf{H} \times \mathbf{H}$. Let $H = \mathbf{H}(F)$ be the F-rational points of $\mathbf{H}$. Let $\widetilde{H} = \widetilde{\mathbf{H}}(F)$ be the F-rational points of $\widetilde{\mathbf{H}}$. We define the automorphism $\varepsilon : \widetilde{H} \to \widetilde{H}$ by $g \mapsto {}^t\bar{g}^{-1}$.

Definition 1.1. An element δ of $\widetilde{H}$ is said to be ε-**semisimple** if (δ, ε) is semisimple in the non-connected group $\widetilde{H} \rtimes < \varepsilon >$.

Definition 1.2. Two elements δ and δ' of $\widetilde{H}$ are said to be ε-**conjugate** if there is a $g \in \widetilde{H}$ such that $\delta' = g^{-1}\delta\varepsilon(g)$.

Let δ be an ε-semisimple element of $\widetilde{H}$. Let $\widetilde{H}_{\delta\varepsilon} = \left\{g \in \widetilde{H} \mid g^{-1}\delta\varepsilon(g) = \delta\right\}$, and let $\widetilde{H}'_{\delta\varepsilon} = \left\{g \in \widetilde{H} \mid g^{-1}\delta\varepsilon(g)\delta^{-1} \in Z_F\right\}$, where Z_F is the set of F scalar matrices. Similarly, for γ a semisimple element of H, we define $H_\gamma = \left\{g \in H \mid g^{-1}\gamma g = \gamma\right\}$.

definition 1.3. An element $\delta \in \widetilde{H}$ is **stably** ε-**conjugate** to δ' if there is a $g \in \widetilde{\mathbf{H}}(\bar{F})$ so that $\delta = g^{-1}\delta'\varepsilon(g)$. In this case $\tau(g)g^{-1} \in \widetilde{\mathbf{H}}_{\delta\varepsilon}$ [**14**]. Two elements of H are **stably conjugate** if they are conjugate by some g in $\mathbf{H}(\bar{F})$. This implies that $\tau(g)g^{-1} \in \mathbf{H}_\gamma$.

Lemma 1.4.

(1) *A stable conjugacy class in H is a union of conjugacy classes.*

(2) *A stable ε-conjugacy class in $\widetilde{H}$ is the union of ε-conjugacy classes.* □

For $\gamma \in H$ we let $\mathcal{O}(\gamma)$ be the conjugacy class of γ and $\mathcal{O}_{st}(\gamma)$ the stable conjugacy class of γ in H. For $\delta \in \widetilde{H}$ we let $\mathcal{O}_\varepsilon(\delta)$ be the ε-conjugacy class of δ and $\mathcal{O}_{\varepsilon-st}(\delta)$ the stable ε-conjugacy class of δ in $\widetilde{H}$.

Definition 1.5. We define the norm map for $\delta \in \widetilde{H}$ by $N(\delta) = \delta\varepsilon(\delta)$. Note that $N(g^{-1}\delta\varepsilon(g)) = g^{-1}N(\delta)g$. Thus, N defines an injection $\mathcal{N} : [\delta] \mapsto N([\delta])$ from ε-stable conjugacy classes of $\widetilde{H}$, to the set of stable conjugacy classes of H.

Proposition 1.6 (Rogawski [14, proposition 3.11.1(c)]). *The norm map defines a bijection between stable ε-conjugacy classes in $\widetilde{H}$ and stable conjugacy classes in H.* □

Corollary 1.7. *For any n, $\mathcal{O}_{\varepsilon-st}(\delta_n)$ consists of all hermitian matrices in $\widetilde{H}$.*

Proof. δ_n is hermitian, i.e. $\delta_n = {}^t\bar{\delta}_n$. Therefore, $N(\delta_n) = I_n$. Since $\{I_n\}$ is a stable conjugacy class in H, proposition 1.6 implies δ is stably ε-conjugate to δ_n if and only if $N(\delta) = I_n$. This holds if and only if δ is hermitian. □

Lemma 1.8. *For any n,*

$$\mathcal{O}_{\varepsilon-st}(\delta_n) = \mathcal{O}_{\varepsilon}(\delta_n) \cup \mathcal{O}_{\varepsilon}(\delta_n'),$$

where δ_n' is any nondegenerate hermitian form which is inequivalent with δ_n.

Proof. Note that if δ is hermitian, then δ' is ε-conjuagte to δ if and only if δ and δ' define equivalent hermitian forms. For each n there are two classes of hermitian forms in $GL(n,E)$, indexed by $F^\times/NE^\times$. That is, if δ and δ' are hermitian, then δ is equivalent to δ' if and only if $\det(\delta'\delta^{-1}) \in NE^\times$ [**9**]. The result then follows from Corollary 1.7. □

Remark. Suppose $z \in F^\times$. Then δ and $z\delta$ define the same unitary group. Thus, the isomorphism class of $\widetilde{H}_{\delta\varepsilon}$ only depends on $F^\times/NE^\times(F^\times)^n$. Therefore, when n is odd, $\widetilde{H}_{\delta_n\varepsilon} \simeq \widetilde{H}_{\delta_n'\varepsilon}$.

Let E^1 be the norm 1 elements in E, i.e. $E^1 = \{z \in E^\times | z\bar{z} = 1\}$. Note that $Z = Z(H) \simeq E^1$, and $\widetilde{Z} = Z(\widetilde{H}) \simeq E^\times$. Let ω be a character of E^1. Let $C(H,\omega)$ be the space of locally constant functions, compactly supported modulo Z, such that $f(zg) = \omega^{-1}(z)f(g)$ for all $z \in Z$, and $g \in H$. Let $\widetilde{\omega}$ be the character of $E^\times$ given by $\widetilde{\omega}(z) = \omega(z/\bar{z})$. Then we let $C(\widetilde{H},\widetilde{\omega})$ be the space of locally constant functions, compactly supported modulo $\widetilde{Z}$, such that $\varphi(zg) = \widetilde{\omega}^{-1}(z)\varphi(g)$ for all $z \in \widetilde{Z}$, and $g \in \widetilde{H}$.

DEFINITION 1.9.

(a) For γ a semisimple element of H and $f \in C(H,\omega)$, we define

$$\Phi(\gamma, f) = \int_{H_\gamma \backslash H} f(g^{-1}\gamma g)\, dg^\times,$$

where $dg^\times$ is the right invariant measure on the quotient coming from Haar measure dg on H. This is referred to as the **orbital integral** of f at γ.

(b) Similarly, for δ an ε-semisimple element of $\widetilde{H}$ and $\varphi \in C(\widetilde{H}, \widetilde{\omega})$, we define

$$\Phi_\varepsilon(\delta, \varphi) = \int_{\widetilde{H}'_{\delta\varepsilon} \backslash \widetilde{H}} \varphi(g^{-1}\delta\varepsilon(g))\, dg^\times,$$

where again the measure $dg^\times$ is the right invariant one coming from Haar measure. This is called the ε-**twisted orbital integral** of φ at δ.

DEFINITION 1.10.

(a) Let G be a reductive F group. Let $q(G)$ be the F rank of the derived group of G. Let $e(G) = (-1)^{q(G)-q(G')}$, where G' is the quasi split form of G.

(b) Let γ be a semisimple element of H. Then we let $e(\gamma) = e(H_\gamma)$.

(c) Let δ be an ε-semisimple element of $\widetilde{H}$. Then we let $e(\delta) = e(\widetilde{H}_{\delta\varepsilon})$.

DEFINITION 1.11. Let $\gamma \in H$ be a semisimple element and let $\{\gamma'\}$ be a collection of representatives of the conjugacy classes in $\mathcal{O}_{st}(\gamma)$. Let ω be a character of E^1 and let $f \in C(H,\omega)$. Then we let

$$\Phi^{st}(\gamma, f) = \sum_{\{\gamma'\}} e(\gamma')\Phi(\gamma', f).$$

DEFINITION 1.12. Let $\Delta_n = \{\delta_n, \delta'_n\}$ be the chosen representatives of the ε-classes in $\mathcal{O}_{\varepsilon-st}(\delta_n)$. Consider the bijection $\Delta_n \leftrightarrow F^\times/NE^\times$ given by $\delta \longmapsto \det(\delta) \pmod{NE^\times}$. Let $\kappa : F^\times/NE^\times \longrightarrow \{\pm 1\}$ be a character. Let $\varphi \in C(\widetilde{H}, \widetilde{\omega})$. Define

$$\Phi_\varepsilon^\kappa(\delta_n, \varphi) = \sum_{\Delta_n} \kappa(\delta)e(\delta)\Phi_\varepsilon(\delta, \varphi).$$

When $\kappa = 1$ we write $\Phi_\varepsilon^{st}(\delta_n, \varphi)$. By the remark following Lemma 1.8 we know that, when n is odd, $e(\delta_n) = e(\delta_n') = 1$. Therefore, in this case $\Phi_\varepsilon^{st}(\delta_n, \varphi) = \Phi_\varepsilon(\delta_n, \varphi) + \Phi_\varepsilon(\delta_n', \varphi)$. However, when n is even, $e(\delta_n) = 1$ and $e(\delta_n') = -1$. Therefore, in the even case, the sum of the two twisted orbital integrals is $\Phi_\varepsilon^\kappa(\delta_n, \varphi)$, where κ is the non-trivial character.

§2 **Reducibility Criteria.** In this Section we repeat the argument of Section 2 of [**6**] in the wider context of the group $U(n,n)$. Let E/F be as in §1. Recall that $E = F(\beta)$. In this section we use the hermitian form $J' = \begin{pmatrix} 0 & \beta I_n \\ -\beta I_n & 0 \end{pmatrix}$. Let $\mathbf{G} = \mathbf{U}(2n)$ be defined with respect to J', and $G = U(n,n) = \mathbf{G}(F)$. Let $\mathbf{T}$ be the maximal torus of diagonal elements in $\mathbf{G}$. Then

$$T = \mathbf{T}(F) = \left\{ \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & \bar{x}_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \bar{x}_n^{-1} \end{pmatrix} \middle| \; x_i \in E^\times \right\}.$$

Let $\mathbf{T}_d$ be the maximal F-split sub-torus of $\mathbf{T}$. Then

$$T_d = \mathbf{T}_d(F) = \left\{ \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_1^{-1} & & \\ & & & & \ddots & \\ & & & & & x_n^{-1} \end{pmatrix} \middle| \; x_i \in F^\times \right\}.$$

The restricted root system $\Phi(\mathbf{G}, \mathbf{T}_d)$ is of type C_n. Let $\mathbf{A}$ be the subtorus of $\mathbf{T}_d$ given by $\theta = \{e_i - e_{i+1}\}_{i=1}^{n-1}$. Then

$$A = \mathbf{A}(F) = \left\{ \begin{pmatrix} x I_n & 0 \\ 0 & x^{-1} I_n \end{pmatrix} \middle| \; x \in F^\times \right\}.$$

Let $\mathbf{M}$ be the centralizer of $\mathbf{A}$ in $\mathbf{G}$. Then

$$\mathbf{M} = \left\{ \begin{pmatrix} g & 0 \\ 0 & {}^t\bar{g}^{-1} \end{pmatrix} \middle| g \in \operatorname*{Res}_{E/F}(\mathbf{GL}_n) \right\} \simeq \widetilde{\mathbf{H}}.$$

The Weyl group $W(A)$ is of order two, with the non-trivial element w represented by $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Let $\mathbf{P} = \mathbf{MN}$ with

$$\mathbf{N} = \left\{ \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \middle| {}^t\overline{X} = X \right\}.$$

Then $\mathbf{P}$ is a maximal parabolic subgroup of $\mathbf{G}$. Let $P = \mathbf{P}(F) = MN$, with $M = \mathbf{M}(F) \simeq GL(n, E)$, and

$$N = \mathbf{N}(F) = \left\{ \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \middle| X \in M(n, E);\ {}^t\overline{X} = X \right\}.$$

Let $X(\mathbf{M})_F$ denote the F-rational characters of $\mathbf{M}$. Let $\mathfrak{a}$ be the real Lie algebra of $\mathbf{A}$. Then $\mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R})$ [**7**]. Let $\mathfrak{a}^* = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R}$ be its dual, and let $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$. There is a homomorphism [**7**] $H_P : M \to \mathfrak{a}$ defined by

$$q_F^{<\chi, H_P(m)>} = |\chi(m)|_F, \forall\, \chi \in X(\mathbf{M})_F.$$

Let ρ be half the sum of the positive roots in N. Let $\alpha = 2e_n$ be the unique simple root in N. Let $\widetilde{\alpha} = <\rho, \alpha>^{-1} \rho$, where $<\gamma_1, \gamma_2>$ is defined as follows. Choose non-restricted roots γ_1' and γ_2' restricting to γ_1 and γ_2, respectively. Then

$$<\gamma_1, \gamma_2> = \frac{2(\gamma_1', \gamma_2')}{(\gamma_2', \gamma_2')},$$

where $(\cdot, \cdot)$ is the standard Euclidean inner product on $\Phi(\mathbf{G}, \mathbf{T})$ [**17**]. Clearly $<\rho, \alpha>$ is well defined. Let $\mathfrak{z}$ be the complexified lie algebra of the split component of $\mathbf{G}$. Then we identify $\mathfrak{a}^*_{\mathbb{C}}/\mathfrak{z}$ with $\mathbb{C}$ via the map $s \mapsto s\widetilde{\alpha}$. Note that the non-restricted roots $\Phi(\mathbf{G}, \mathbf{T})$ are of type A_{2n-1}. Let $\widetilde{\Delta} = \{\beta_1, \ldots, \beta_{2n-1}\}$ be the set of simple roots of $\Phi(\mathbf{G}, \mathbf{T})$, with $\beta_i = e_i - e_{i+1}$. Then $\widetilde{\theta} = \widetilde{\Delta} \setminus \{\beta_n\}$. Thus,

$$\rho_{\tilde{\theta}} = \frac{n}{2} \sum_{j=1}^{n} j(\beta_j + \beta_{2n-j}).$$

Therefore, $<\rho, \alpha> = (\rho_{\tilde{\theta}}, \beta_n) = n$. This implies that, in terms of non-restricted roots,

$$\widetilde{\alpha} = \frac{1}{2} \sum_{j=1}^{n} j(\beta_j + \beta_{2n-j}).$$

Thus, if $m \in GL(n,E)$, we have

$$q_{F}^{<\tilde{\alpha},H_P(m)>} = |\det m|_{E}^{1/2}.$$

Let $^{\circ}\mathcal{E}(M)$ be the set of equivalence classes of irreducible unitary supercuspidal representations of M. Let $(\pi, V) \in {}^{\circ}\mathcal{E}(M)$. Let $\widetilde{\omega}$ be the central character of π. Let $s \in \mathbb{C}$ and let $V(s,\pi) =$

$$\left\{ f \in C^{\infty}(G,V) \big| f(mng) = \pi(m) q_{F}^{<s\tilde{\alpha},H_P(m)>} \delta_P^{1/2}(m) f(g) \forall\ g \in G, m \in M, n \in N \right\}.$$

Then G acts on $V(s,\pi)$ by right translations. We denote this action by

$$I(s,\pi) = \mathrm{Ind}_P^G \left(\pi \otimes q_{F}^{<s\tilde{\alpha},H_P()>} \right) = \mathrm{Ind}_P^G \left(\pi \otimes |\det()|_{E}^{s/2} \right).$$

We write $I(\pi)$, or $\mathrm{Ind}_P^G(\pi)$ for $I(0,\pi)$. Note that by Bruhat theory [**7**] $\mathrm{Ind}_P^G(\pi)$ is irreducible if $\pi^w \not\simeq \pi$. If $m = \begin{pmatrix} g & 0 \\ 0 & {}^t\overline{g}^{-1} \end{pmatrix}$ with $g \in GL(n,E)$, then $wmw^{-1} = \begin{pmatrix} {}^t\overline{g}^{-1} & 0 \\ 0 & g \end{pmatrix}$. Therefore, $\pi^w = \pi^{\varepsilon}$, where $\pi^{\varepsilon}(g) = \pi(\varepsilon(g))$.

We formally define an operator $A(s,\pi)$ on $V(s,\pi)$ by

$$[A(s,\pi) f](g) = \int_N f(w^{-1}ng)\ dn$$

for $f \in V(s,\pi)$, $g \in G$. If $A(s,\pi)$ converges, then it defines an intertwining operator between $I(s,\pi)$ and $I(-s,\pi^w)$. It is a Theorem of Harish-Chandra [**15**] that, for π supercuspidal, $A(s,\pi)$ converges for $\mathrm{Re}\, s > 0$. Moreover, $s \mapsto A(s,\pi)$ is meromorphic as an operator valued function, and has a meromorphic continuation to the whole plane. This means that there is some fixed polynomial $P(t)$ so that $s \mapsto P(q_{F}^{-s}) < A(s,\pi) f(g), \tilde{v} >$ is holomorphic for each $g \in G, \tilde{v} \in \widetilde{V}$, $f \in V(s,\pi)$, and the operator $P(q_{F}^{-s}) A(s,\pi)$ is non-zero.

Harish-Chandra's completeness Theorem, [**21**], implies $\mathrm{Ind}_P^G(\pi)$ is reducible if and only if $\pi \simeq \pi^w$ and 0 is not a pole of $s \mapsto A(s,\pi)$.

Lemma 2.1 (Rallis, Shahidi [20]). *Let*

$$V(s,\pi)_0 = \left\{ f \in V(s,\pi) \middle| \operatorname{supp} f \subset P\overline{N} \text{ and is compact mod } P \right\}.$$

Then every pole of $s \mapsto A(s,\pi)$ *is a pole of* $s \mapsto A(s,\pi)f(e)$ *for some* $f \in V(s,\pi)_0$. □

Thus, we study poles of $s \mapsto A(s,\pi)f(e)$ for $f \in V(s,\pi)_0$ and $\pi \simeq \pi^{\varepsilon}$. Let $L = M(n, \mathfrak{p}_E^m)$ for some $m \in \mathbb{Z}^+$. Let $L' \subset \overline{N}$ be given by

$$L' = \left\{ \begin{pmatrix} I & 0 \\ x & I \end{pmatrix} \middle| x \in L \right\}.$$

Let $f \in V(s,\pi)_0$. We assume that there is a $v \in V$ so that, for $y \in \overline{N}$,

$$f(y) = \begin{cases} v & y \in L' \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2. *If* $y \in N$, *then* $w^{-1}y \in P\overline{N}$ *if and only if* $y = \begin{pmatrix} I & a \\ 0 & I \end{pmatrix}$ *with* $a \in GL(n,E)$ *and* $a = {}^t\bar{a}$.

Proof. Suppose $a, b, c \in M(n,E)$, and $g \in GL(n,E)$. If

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & a \\ 0 & I \end{pmatrix} = \begin{pmatrix} g & b \\ 0 & {}^t\overline{g}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ c & I \end{pmatrix},$$

then

$$\begin{pmatrix} 0 & -I \\ I & a \end{pmatrix} = \begin{pmatrix} g + bc & b \\ {}^t\overline{g}^{-1}c & {}^t\overline{g}^{-1} \end{pmatrix}.$$

Therefore, $a = {}^t\overline{g}^{-1}$, which implies $a \in GL(n,E)$. Moreover, if $y \in N$ then $a = {}^t\bar{a}$. Conversely, if $a \in GL(n,E)$ with ${}^t\bar{a} = a$, then the above calculation shows

$$\begin{pmatrix} 0 & -I \\ I & a \end{pmatrix} = \begin{pmatrix} a^{-1} & -I \\ 0 & a \end{pmatrix} \begin{pmatrix} I & 0 \\ a^{-1} & I \end{pmatrix}. \quad \square$$

Let $\tilde{v} \in \widetilde{V}$. By Lemma 2.2

$$\begin{aligned}
\langle A(s,\pi)f(e),\tilde{v}\rangle &= \int_N < f(w^{-1}y),\tilde{v} > \ dy \\
&= \int_{\substack{\det a\neq 0\\ \overline{a}=\ {}^t a}} \left\langle f\left(\begin{pmatrix} \varepsilon(a) & -I \\ 0 & a\end{pmatrix}\begin{pmatrix} I & 0 \\ a^{-1} & I\end{pmatrix}\right),\tilde{v}\right\rangle \ da \\
&= \int_{\substack{\det a\neq 0\\ \overline{a}=\ {}^t a}} \left\langle \pi(\varepsilon(a))f\left(\begin{pmatrix} I & 0 \\ a^{-1} & I\end{pmatrix}\right),\tilde{v}\right\rangle |\det a|_E^{-s/2-<\rho,\tilde{\alpha}>} da \\
&= \int_{\substack{\det a\neq 0\\ \overline{a}={}^t a\\ a^{-1}\in L}} \langle \pi(\varepsilon(a))v,\tilde{v}\rangle\, |\det a|_E^{-s/2-<\rho,\tilde{\alpha}>} \ da. \qquad (2.1)
\end{aligned}$$

Lemma 2.3. *If* $d^\times a = |\det a|_E^{-<\rho,\tilde{\alpha}>}\ da$ *then* $d^\times a^{-1} = d^\times a$.

Proof. Let M act on N by conjugation. If ${}^t\overline{X} = X \in M(n,E)$ and $g \in GL(n,E)$ then

$$\begin{pmatrix} g^{-1} & 0 \\ 0 & {}^t\overline{g}\end{pmatrix}\begin{pmatrix} I & X \\ 0 & I\end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & {}^t\overline{g}^{-1}\end{pmatrix} = \begin{pmatrix} I & g^{-1}X{}^t\overline{g}^{-1} \\ 0 & I\end{pmatrix}.$$

Therefore, on N, $d(g^{-1}X{}^t\overline{g}^{-1})/dX = |\det X|_E^{-2<\rho,\tilde{\alpha}>}$.

Now suppose $g = X$. Then

$$|\det X^{-1}|_E^{-<\rho,\tilde{\alpha}>} dX^{-1} = |\det X|_E^{-<\rho,\tilde{\alpha}>} dX,$$

and therefore, $d^\times a^{-1} = d^\times a$. $\square$

Let $\varphi(g) = <\pi(g)v,\tilde{v}>$. For $x \in M(n,E)$ let

$$\xi_L(x) = \begin{cases} 1 & x \in L \\ 0 & x \notin L. \end{cases}$$

Then using the relation $\bar{a} = {}^t a$ and Lemma 2.3, we rewrite (2.1) as

$$\begin{aligned}
< A(s,\pi)f(e),\tilde{v} > &= \int_{\substack{\det a\neq 0\\ \overline{a}=\ {}^t a}} \varphi(a^{-1})|\det a|_E^{-s/2}\xi_L(a^{-1})\ d^\times a \\
&= \int_{\substack{\det a\neq 0\\ \overline{a}=\ {}^t a}} \varphi(a)|\det a|_E^{s/2}\xi_L(a)\ d^\times a. \qquad (2.2)
\end{aligned}$$

Remark. Since v and $\tilde{v}$ were arbitrary, any matrix coefficient φ of π can appear in (2.2).

Recall that $\Delta_n = \{\delta_n, \delta_n'\}$ are the representatives of the ε-classes in $\mathcal{O}_{\varepsilon-st}(\delta_n)$. If $a \in GL(n,E)$ is hermitian, then $a = g^{-1}\delta\varepsilon(g)$ for some g and a unique $\delta \in \Delta_n$. Thus, using the notation in Section 1, we can rewrite (2.2) as

(2.3)

$$< A(s,\pi)f(e), \tilde{v} > = \sum_{\Delta_n} \int_{\widetilde{H}_{\delta\varepsilon}\backslash \widetilde{H}} \varphi(g^{-1}\delta\varepsilon(g))|\det(g^{-1}\delta\varepsilon(g))|_E^{s/2}\xi_L(g^{-1}\delta\varepsilon(g))\; d^\times g,$$

where $d^\times g$ is the invariant measure on the quotient coming from $d^\times a$.

Notice that the integrals in (2.3) are not of the form given in Section 1. The next lemma allows us to decompose the integrals in (2.3) into an iterated integral involving twisted orbital integrals, as defined in Section 1.

Lemma 2.4.

(a) *If n is even then $\widetilde{H}_{\delta\varepsilon}\backslash\widetilde{H}'_{\delta\varepsilon} \simeq F^\times$, for any $\delta \in \Delta_n$.*

(b) [**14**, proposition 3.11.2(c)] *If n is odd then $\widetilde{H}_{\delta\varepsilon}\backslash\widetilde{H}'_{\delta\varepsilon} \simeq NE^\times$, for any $\delta \in \Delta_n$.*

Proof. (a) Note that $\psi : \widetilde{H}'_{\delta\varepsilon} \longrightarrow F^\times$ given by $\psi(g) = g^{-1}\delta\varepsilon(g)\delta^{-1}$ is a homomorphism. By its definition $\widetilde{H}_{\delta\varepsilon}$ is the kernel of ψ. Let $z \in F^\times$. Note that $\widetilde{H}_{\delta\varepsilon} = \widetilde{H}_{(z\delta)\varepsilon}$, and therefore, since n is even, the two hermitian forms δ and $z\delta$ are equivalent. Since ε-conjugacy is equivalence of hermitian forms, there is some $g \in \widetilde{H}$ so that $g^{-1}\delta\varepsilon(g) = z\delta$. Therefore, g is in $\widetilde{H}'_{\delta\varepsilon}$. Since $\psi(g) = z$, we see ψ is surjective.

The proof of (b) is similar, once one notes that if δ and $z\delta$ are equivalent, then $z \in NE^\times$ [**14**]. □

Suppose $g \in \widetilde{H}$ is a representative of a coset of $\widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H}$. If n is even and $z_0 \in F^\times$, then there is some $g_0 \in \widetilde{H}'_{\delta\varepsilon}$ so that

$$(g_0g)^{-1}\delta\varepsilon(g_0g) = (z_0I_n)g^{-1}\delta\varepsilon(g).$$

If n is odd, and $z_0 \in E^\times$, then there is some $g_0 \in \widetilde{H}'_{\delta\varepsilon}$ so that

$$(g_0g)^{-1}\delta\varepsilon(g_0g) = (Nz_0I_n)g^{-1}\delta\varepsilon(g).$$

Therefore, we choose representatives for $\widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H}$ so that, if $z \in \begin{cases} F^\times & n \text{ odd} \\ NE^\times & n \text{ even,} \end{cases}$

then

$$g^{-1}z\delta\varepsilon(g) \in L \text{ if and only if } |z|_E \leq 1. \tag{2.4}$$

If n is even, we now can rewrite (2.3) as $< A(s,\pi)f(e), \tilde{v} >=$

$$\sum_{\Delta_n}\int_{F^\times}\int_{\widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H}} \varphi(g^{-1}(z\delta)\varepsilon(g))|\det(g^{-1}(z\delta)\varepsilon(g))|_E^{s/2}\xi_L(g^{-1}(z\delta)\varepsilon(g))d^\times g\ d^\times z$$

$$(2.5a) = \int_{\mathcal{R}_F}\widetilde{\omega}(z)|z|_E^{ns/2}\ d^\times z \sum_{\Delta_n}\int_{\widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H}}\varphi(g^{-1}\delta\varepsilon(g))|\det(g^{-1}\delta\varepsilon(g))|_E^{s/2}\ d^\times g.$$

If n is odd, we rewrite (2.3) as $< A(s,\pi)f(e), \tilde{v} >=$

$$\sum_{\Delta_n}\int_{NE^\times}\int_{\widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H}} \varphi(g^{-1}(z\delta)\varepsilon(g))|\det(g^{-1}(z\delta)\varepsilon(g))|_E^{s/2}\xi_L(g^{-1}(z\delta)\varepsilon(g))d^\times g\ d^\times z$$

(2.5b)

$$= \int_{\mathcal{R}_F\cap NE^\times}\widetilde{\omega}(z)|z|_E^{ns/2}\ d^\times z \sum_{\Delta_n}\int_{\widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H}}\varphi(g^{-1}\delta\varepsilon(g))|\det(g^{-1}\delta\varepsilon(g))|_E^{s/2}\ d^\times g.$$

Lemma 2.5. $\{g \in \widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H} \mid g^{-1}\delta\varepsilon(g) \in \operatorname{supp}\varphi\}$ *is compact.*

Proof. Let $||\ ||_\infty$ denote the supremum norm on $M(n,E)$. Let $X = \{q_E^{-m}, q_E^{-m+1}\}$. Let $L_0 = \{x|\ ||x||_\infty \in X\} \subset M(n,E)$. We have already chosen representatives g with $g^{-1}\delta\varepsilon(g) \in L_0$. Let $S \subset M(n,E)$ be the set of hermitian matrices. Since transposition and Galois conjugation are continuous, S is closed in $M(n,E)$. Define $\psi : \widetilde{H} \longrightarrow S$ by $\psi(g) = g^{-1}\delta\varepsilon(g)$. Then

$$\operatorname{Im}(\psi) = \{s \in S \big| \det s \equiv \det\delta\ mod\ NE^\times\}.$$

Since both $NE^\times$ and its complement are closed in $F^\times$, and the determinant is continuous, $\operatorname{Im}(\psi)$ is closed in S, thus is closed in $\widetilde{H}$.

If $y \in S$ and $\psi(g) = y$, then $\psi^{-1}(\{y\}) = \widetilde{H}_{\delta\varepsilon}g$. Note that $\psi^{-1}(Z_F y) = \widetilde{H}'_{\delta\varepsilon}g$. Let C be a compact subset of $\widetilde{H}$ such that $\operatorname{supp}\varphi \subset C\widetilde{Z}$. Since $x \mapsto ||x||_\infty$

and $x \mapsto |\det x|_E$ are continuous we can choose integers j,k,l,t so that $q_E^k \leq ||c||_\infty \leq q_E^j$ and $q_E^l \leq |\det c|_E \leq q_E^t$ for all $c \in C$. Therefore, if $c \in C$ and $cz \in C\widetilde{Z} \cap L_0$ then $q_E^{-m-j} \leq |z|_E \leq q_E^{-m+1-k}$. We let

$$\Omega = \{z |\ q_E^{-m-j} \leq |z|_E \leq q_E^{-m+1-k}\}.$$

Then $C\widetilde{Z} \cap L_0 \subset C\Omega$, which is compact. Therefore, $\operatorname{Im}\psi \cap \operatorname{supp}\varphi \cap L_0$ is compact in $\widetilde{H}$. Hence the lemma holds. □

By Lemma 2.5,

$$\lim_{s\to 0} \sum_{\Delta_n} \int_{\widetilde{H}'_{\delta\varepsilon}\backslash \widetilde{H}} \varphi(g^{-1}\delta\varepsilon(g)) |\det(g^{-1}\delta\varepsilon(g))|_E^{s/2}\, dg^\times =$$

$$\sum_{\Delta_n} \int_{\widetilde{H}'_{\delta\varepsilon}\backslash \widetilde{H}} \varphi(g^{-1}\delta\varepsilon(g))\, d^\times g = \begin{cases} \Phi_\varepsilon^\kappa(\delta_n,\varphi) & n \text{ even} \\ \Phi_\varepsilon^{st}(\delta_n,\varphi) & n \text{ odd.} \end{cases} \tag{2.6}$$

Consequently, if n is even, the residue at 0 of $A(s,\pi)f(e)$ is given by

$$\lim_{s\to 0} s < A(s,\pi)f(e), \tilde{v} > = \left(\lim_{s\to 0} s \int_{\mathcal{R}_F} \widetilde{\omega}(z)|z|_E^{ns/2}\, d^\times z\right) \Phi_\varepsilon^\kappa(\delta_n,\varphi). \tag{2.7a}$$

Similarly, when n is odd, the residue is given by

$$\lim_{s\to 0} s < A(s,\pi)f(e), \tilde{v} > = \left(\lim_{s\to 0} s \int_{\mathcal{R}_F \cap NE^\times} \widetilde{\omega}(z)|z|_E^{ns/2}\, d^\times z\right) \Phi_\varepsilon^{st}(\delta_n,\varphi). \tag{2.7b}$$

Consider the case where n is even. If $A(s,\pi)f(e)$ has a pole at $s=0$ then

$$s \mapsto \int_{\mathcal{R}_F} \widetilde{\omega}(z)|z|_E^{ns/2} d^\times z$$

has a pole at $s = 0$.

$$\int_{\mathcal{R}_F} \widetilde{\omega}(z)|z|_E^{ns/2} d^\times z = \int_{\mathcal{R}_F} \widetilde{\omega}(z)|z|_F^{ns}\, d^\times z = L(ns, \widetilde{\omega}_{|_{F^\times}}), \tag{2.8}$$

where $L(s,\chi)$ is the local Hecke-Tate L-function attached to a character χ of $F^\times$. Thus, (2.7a) is zero, unless $\widetilde{\omega}_{|_{F^\times}}$ is unramified, in which case it is proportional to

$$(2.9)\qquad \left(\sum_{m=0}^{\infty}\left(\widetilde{\omega}(\varpi_F)q_F^{-ns}\right)^m\right)\Phi_\varepsilon^\kappa(\delta_n,\varphi)=\left(\frac{1}{1-\widetilde{\omega}(\varpi_F)q_F^{-ns}}\right)\Phi_\varepsilon^\kappa(\delta_n,\varphi).$$

Therefore, there is no pole at $s=0$ unless $\widetilde{\omega}(\varpi_F)=1$. Since $\widetilde{\omega}_{|_{F^\times}}$ is unramified and $\omega(\varpi_F)=1$, $\widetilde{\omega}_{|_{F^\times}}\equiv 1$.

Lemma 2.6. *Let $\eta: E^\times \longrightarrow \mathbb{C}^\times$ be a character.*

(a) *Suppose n is even and, for some $\psi\in C(\widetilde{H},\eta)$, $\Phi_\varepsilon^\kappa(\delta_n,\psi)\neq 0$. Then $\eta(z)=1$ for all $z\in F^\times$.*

(b) *Suppose n is odd and ,for some $\psi\in C(\widetilde{H},\eta)$, $\Phi_\varepsilon^{st}(\delta_n,\psi)\neq 0$. Then $\eta(z)=1$ for any $z\in NE^\times$.*

Proof. (a) Since $\Phi_\varepsilon^{st}(\delta_n,\psi)\neq 0$, we know that $\Phi_\varepsilon(\delta,\psi)\neq 0$, for some $\delta\in\Delta_n$. Let $z\in F^\times$. By Lemma 2.4 we can choose some $g_0\in\widetilde{H}'_{\delta\varepsilon}$ with $g_0^{-1}\delta\varepsilon(g_0)=z\delta$. Therefore, changing g to g_0g in $\Phi_\varepsilon(\delta,\psi)$, we have

$$0\neq\Phi_\varepsilon(\delta,\psi)=\int_{\widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H}}\psi(g^{-1}\delta\varepsilon(g))\,dg^\times$$
$$=\int_{\widetilde{H}'_{\delta\varepsilon}\backslash\widetilde{H}}\psi((g_0g)^{-1}\delta\varepsilon(g_0g))\,dg^\times=\eta(z^{-1})\Phi_\varepsilon(\delta,\psi).$$

The proof of (b) is similar. □

Therefore, the condition $\widetilde{\omega}_{|_{F^\times}}\equiv 1$ is guaranteed by the non-vanishing of $\Phi_\varepsilon^\kappa(\delta_n,\psi)$. We can now state the result precisely.

Theorem 2.7. *Let n be even. Let $\pi\in{}^\circ\mathcal{E}(M)$. The intertwining operator $A(s,\pi)$ has a pole at $s=0$ if and only if $\pi\simeq\pi^\varepsilon$, and there is a matrix coefficient φ of π such that $\Phi_\varepsilon^\kappa(\delta_n,\varphi)\neq 0$.* □

Now consider the case where n is odd. Since we are assuming that $\pi\simeq\pi^\varepsilon$, we know that $\widetilde{\omega}(z\bar{z})=1$. Therefore, the integral appearing in (2.7b) can be rewritten as

$$\int_{\mathcal{R}_E} |Nz|_E^{ns/2}\, d^\times z = \int_{\mathcal{R}_E} |z|_E^{ns}\, d^\times z = (1 - 1/q_E)\frac{1}{1 - q_E^{-ns}}. \tag{2.10}$$

Therefore, we have the following Theorem.

Theorem 2.8. *Let n be odd and $\pi \in {}^\circ\mathcal{E}(M)$. Then the intertwining operator $A(s,\pi)$ has a pole at $s = 0$ if and only if $\pi \simeq \pi^\varepsilon$, and there is a matrix coefficient φ of π such that $\Phi_\varepsilon^{st}(\delta_n, \varphi) \neq 0$.* $\square$

Theorem 2.9. *Let $G = U(n,n)$ and $M = GL(n,E)$ as above. Let $\pi \in {}^\circ\mathcal{E}(M)$. Let κ denote the trivial character of $F^\times/NE^\times$ if n is odd, and the non-trivial character if n is even. Then $I(\pi)$ is reducible if and only if $\pi \simeq \pi^\varepsilon$ and $\Phi_\varepsilon^\kappa(\delta_n, \varphi) = 0$ for every matrix coefficient φ of π.* $\square$

§**3 Complementary series.** In this Section we use the results of Shahidi [**18**] to determine when the representation $I(s,\pi)$ is reducible, for $s \notin i\mathbb{R}$. Let $\bar{F}$ be a separable algebraic closure of F, and let $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$. We denote by W_F the Weil group of $\bar{F}$ over F [**23**].

Recall that, for a connected, reductive, algebraic group $\mathbf{G}$, defined over F, the L-group is given by ${}^LG = {}^LG^0 \rtimes W_F$, where ${}^LG^0$ is the complex group whose canonical root datum is dual to that of $\mathbf{G}$, and W_F acts on ${}^LG^0$ through the action of Γ_F on root data [**3**].

For $\mathbf{H} = \mathbf{U}(n)$ we know that $\mathbf{H}(E) \simeq GL(n,E)$, and therefore ${}^LH^0 = GL(n,\mathbb{C})$. Since $\widetilde{\mathbf{H}}(E) \simeq \mathbf{H}(E) \times \mathbf{H}(E)$, we have ${}^L\widetilde{H}^0 \simeq GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$. Let τ be the generator of $\mathrm{Gal}(E/F)$. Then the action of τ on ${}^LH^0$ is given by $\tau(x) = \Phi_n({}^tx^{-1})\Phi_n^{-1}$. The Galois group $\mathrm{Gal}(E/F)$ acts on ${}^L\widetilde{H}^0$ by $\tau(x,y) = (\tau(y), \tau(x))$. The action of τ determines the action of W_F in each case [**14**, pg. 47].

We compute the constituents of the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}$, where ${}^LM$ is the L-group of $\mathbf{M}$, ${}^L\mathfrak{n}$ is the Lie algebra of ${}^LN$, and ${}^LP = {}^LM{}^LN$ [**3**]. Recall that $\mathbf{G} = \mathbf{U}(2n)$ arises from an action of $\mathrm{Gal}(E/F)$ on the root system A_{2n-1}. Let $\check{\Delta} = \{\check{\alpha}_i\}_{i=1}^{2n-1}$ be the simple roots in A_{2n-1}, with $\check{\alpha}_i = e_i - e_{i+1}$. The action is given by $\tau(\check{\alpha}_i) = \check{\alpha}_{2n-i}$. This is case ${}^2A_{2n-1} - 2$ of [**17**].

By the above discussion, ${}^LG^0 = GL(2n,\mathbb{C})$ is a group of type A_{2n-1}, while the restricted root system $\Phi(\mathbf{G},\mathbf{T}_d)$ is of type C_n. The Levi subgroup $\mathbf{M}$ is generated by the subset $\theta = \{\alpha_i\}_{i=1}^{n-1}$. As described above, each α_i is the restriction of two roots from A_{2n-1}, namely $\check{\alpha}_i$ and $\check{\alpha}_{2n-i}$. Thus, $\check{\theta} = \{\check{\alpha}_i\}_{i\neq n}$.

Therefore, ${}^LM^0 = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \middle| g,h \in GL(n,\mathbb{C}) \right\} \simeq GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$. Note that the action of $\mathrm{Gal}(E/F)$ on ${}^LM^0$ is given by

$$\tau(g,h) = \Phi_{2n} \begin{pmatrix} {}^tg & 0 \\ 0 & {}^th \end{pmatrix}^{-1} \Phi_{2n}^{-1} = \begin{pmatrix} \Phi_n {}^th^{-1}\Phi_n^{-1} & 0 \\ 0 & \Phi_n {}^tg^{-1}\Phi_n^{-1} \end{pmatrix} = (\tau(h),\tau(g)),$$

which is consistent with the description given above. ${}^LM = {}^L M^0 \rtimes W_F$ with this action. The unipotent radical ${}^LN = {}^LN^0$ is given by ${}^LN = \left\{ \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \middle| X \in M(n,E) \right\}$. Thus, ${}^L\mathfrak{n} = \left\{ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \middle| X \in M(n,E) \right\}$. Let ${}^LM$ act on ${}^L\mathfrak{n}$ by the adjoint representation. We denote this representation by Ψ. Then $\Psi(m)Y = mYm^{-1}$. Let $(g,h) \in {}^LM^0$. Then

$$\Psi((g,h,1)) \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & gXh^{-1} \\ 0 & 0 \end{pmatrix}.$$

Therefore, $r_{|_{{}^LM^0}} \simeq \rho_n \otimes \widetilde{\rho}_n$, where ρ_n is the standard representation of $GL(n,\mathbb{C})$. Thus, $\Psi_{|_{{}^LM^0}}$ is irreducible, so Ψ must also be irreducible. Note that the action of τ on Δ shows that

$$\Psi((1,1,\tau)) \begin{pmatrix} 0 & (x_{ij}) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (y_{ij}) \\ 0 & 0 \end{pmatrix}$$

where $(x_{ij}) \in M(n,\mathbb{C})$ and $y_{ij} = x_{(n+1-j)(n+1-i)}$. We will use this in Section 5 in computing $L(s,\sigma,\Psi)$ for σ in the discrete series of M.

Theorem 3.1. *Let $G = U(n,n)$ and suppose $P = MN$, with $M \simeq GL(n,E)$. Let $\pi \in {}^\circ\mathcal{E}(M)$. Suppose that $\pi \simeq \pi^\varepsilon$ and $I(\pi)$ is irreducible (see Theorem 2.9).*

(a) *For $0 < s < 1$ the representation $I(s,\pi) = \mathrm{Ind}_P^G\left(\pi \otimes |\det()|_E^{s/2}\right)$ is irreducible and unitarizable (i.e. in the complementary series).*

(b) *The representation* $I(1,\pi)$ *is reducible. It has a unique generic non-supercuspidal discrete series subrepresentation. Its Langlands quotient is degenerate (non-generic) pre-unitary, and nontempered.*

(c) *If* $s>1$ *then* $I(s,\pi)$ *is irreducible and never unitarizable.*

PROOF. Fix a non-trivial additive character ψ_F of F. Theorem 3.5 of [**18**] describes a function $\gamma(s,\pi,\Psi,\psi_F)$, which is rational in q_F^{-s}. If $P_\pi(t)$ is the polynomial such that $P_\pi(0)=1$ and $P_\pi(q_F^{-s})$ is the numerator of $\gamma(s,\pi,\Psi,\psi_F)$, then $L(s,\pi,\Psi)=P(q_F^{-s})^{-1}$ [**18**, §7]. Since Ψ is irreducible, $P_\pi=P_{\pi,1}$ in the language of [**18**]. Therefore, Corollary 7.6 of [**18**] implies that the polynomial $P_\pi(t)$ has a zero at $t=1$. Since $\pi\simeq\pi^\varepsilon$ and $I(\pi)$ is irreducible, (a), (b) and (c) follow immediately from Theorem 8.1 of [**18**]. □

Theorem 3.2. *Suppose* $\pi\simeq\pi^\varepsilon$ *and* $I(\pi)$ *is reducible (see Theorem 2.9). Then for all* $s>0$*, the representation* $I(s,\pi)$ *is irreducible, but not unitarizable.*

PROOF. This follows from Theorem 2.9 and Theorem 8.1 (d) of [**18**].

§4 Reducibility and Base Change.

In this Section we discuss the relation of the reducibility criteria derived in Section 2 to the theory of base change from $U(n)$ to $GL(n,E)$. We will obtain specific results when $n=3$. In [**6**] we obtained similar results when $n=2$.

Let $\omega_{E/F}$ be the local class field theory character attached to E/F. That is

$$\omega_{E/F}(x)=\begin{cases} 1 & x\in NE^\times \\ -1 & x\notin NE^\times. \end{cases}$$

Let $\mu:E^\times\longrightarrow\mathbb{C}$ be any character whose restriction to $F^\times$ is $\omega_{E/F}$.

There are two base change maps $\psi_H,\psi_H':{}^LH\longrightarrow{}^L\widetilde{H}$. These homomorphisms of L-groups are described by Rogawski [**14**, pg. 50]. The maps ψ_H and ψ_H' are related by the cocycle $\alpha:W_F\longrightarrow{}^L\widetilde{H}$ which defines $\chi_\mu=\mu\circ\det$ over F. We will describe some properties of these base change maps.

DEFINITION 4.1. A distribution on $C(H,\omega)$ is called **stable** if it vanishes on every f such that $\Phi^{st}(\gamma,f)=0$ for every regular semisimple $\gamma\in H$.

Let $\mathcal{E}(H)$ be the set of equivalence classes irreducible admissible representations of H, $\mathcal{E}_2(H) \subset \mathcal{E}(H)$ the collection of square integrable equivalence classes, and ${}^\circ\mathcal{E}(H)$ the unitary supercuspidal equivalence classes. We make no notational distinction between a class $[\pi]$ and its representative π. For any $\pi \in \mathcal{E}(H)$ with central character ω, we define the distribution character of π on $C(H,\omega)$ by $\chi_\pi(f) = Tr(\pi(f))$, where

$$\pi(f) = \int_{Z(H)\backslash H} f(g)\pi(g)\, dg \quad .$$

We assume that, for every reductive group $G = \mathbf{G}(F)$, $\mathcal{E}_2(G)$ can be partitioned into finite subsets, called L-**packets**, with the following property: for every L-packet Π we can choose non-zero integers, $m(\pi)$ for each $\pi \in \Pi$, so that $\chi_\Pi = \sum_{\pi\in\Pi} m(\pi)\chi_\pi$ is a stable distribution. This distribution is referred to as the **stable character** of the L-packet Π. We let $\bar{\mathcal{E}}_2(G)$ be the collection of discrete series L-packets. One can then define tempered L-packets by parabolic induction [**18**, §9]. We let $\bar{\mathcal{E}}(G)$ be the collection of tempered L-packets.

For each $\Pi \in \bar{\mathcal{E}}(H)$ there should be two base change lifts, $\psi_H(\Pi)$ and $\psi'_H(\Pi)$, given by the Langlands correspondence. Namely, if $\xi : W_F \longrightarrow {}^LH$ is an admissible homomorphism defining Π then $\psi_H \circ \xi : W_F \longrightarrow {}^L\widetilde{H}$ should define $\psi_H(\Pi)$. Similarly $\psi'_H \circ \xi : W_F \longrightarrow {}^L\widetilde{H}$ should define $\psi'_H(\Pi)$. Since L-packets of $GL(n)$ are singletons, these lifts will actually define representations. Since ψ_H and ψ'_H differ by the cocycle α, Theorem 10.3(2) of [**3**] implies the two lifts are related by

$$\psi_H(\Pi) = \psi'_H(\Pi) \otimes \chi_\mu. \tag{4.1}$$

Let $\pi = \psi_H(\Pi)$. One of the properties of base change lifting is that the central characters of Π and $\psi_H(\Pi)$ should be related by $\omega_\pi = \omega_\Pi \circ N$, where N is the norm map. Recall that $Z(\widetilde{H}) \simeq E^\times$ and $Z(H) \simeq E^1$. If $z \in F^\times$ then $\omega_\pi(z) = \omega_\Pi(zz^{-1}) = 1$. Therefore, $\psi_H(\Pi)$ always has central character whose restriction to $F^\times$ is trivial. Note that if n is even, then χ_μ is trivial on $F^\times$ as well, while if n is odd, then $\chi_{\mu|_{F^\times}} = \omega_{E/F}$. Therefore, if $\pi' = \psi'_H(\Pi)$, then $\omega_{\pi'|_{F^\times}} = \omega_{E/F}$ if n is odd, and $\omega_{\pi'|_{F^\times}} = 1$ if n is even.

In [**6**] we discussed the relation of the reducibility criteria given in Theorem 2.7 to the theory of base change when $n = 2$. For the remainder of this section we consider the case where $n = 3$. Recall that we must determine when $\Phi_{\varepsilon}^{st}(\delta_3, \varphi) \neq 0$ for some matrix coefficient φ of π. Let ω be a character of E^1, and let $\widetilde{\omega}$ be the character of E defined by $\widetilde{\omega}(z) = \omega(z/\bar{z})$.

Theorem 4.2 (Rogawski). *Let* $n = 3$ *and* $\varphi \in C(\widetilde{H}, \widetilde{\omega})$.

(a) *There exists an* $f \in C(H, \omega)$ *so that* $\Phi_{\varepsilon}^{st}(\delta, \varphi) = \Phi^{st}(\gamma, f)$ *whenever* δ *is a regular* ε*-semisimple element of* $\widetilde{H}$ *and* $\gamma = N(\delta) \in H$ [**14**, Proposition 4.10.1].

(b) *If* $\gamma = N(\delta)$ *is central then* $\Phi_{\varepsilon}^{st}(\delta, \varphi) = f(\gamma)$, *where* f *is given by (a)* [**14**, Proposition 8.4.1]. □

Let $\mathcal{E}'(\widetilde{H})$ be the collection of irreducible admissible representations of $\widetilde{H}$ such that $\pi^{\varepsilon} \simeq \pi$, and the central character of π has trivial restriction to $F^{\times}$. Let $\pi(\varepsilon)$ be an equivalence between π and π^{ε}. Let $\widetilde{\omega}$ be the central character of π. We define a distribution on $C(\widetilde{H}, \widetilde{\omega})$ by $\chi_{\pi\varepsilon}(\varphi) = Tr(\pi(\varphi)\pi(\varepsilon))$.

DEFINITION 4.3. If $\pi \in \mathcal{E}'(\widetilde{H})$ is tempered, then $\pi = \psi_H(\Pi)$ if and only if there is a choice of $\pi(\varepsilon)$ so that the character identity $\chi_{\pi\varepsilon}(\varphi) = \chi_{\Pi}(f)$ holds whenever $\varphi \to f$ as in Theorem 4.2 [**14**, pg. 200].

Let $H_0 = U(2) \times U(1)$. Then $H_0 \hookrightarrow H$, and we consider H_0 as a subgroup of H. Moreover, H_0 is an endoscopic group for H [**14**]. The L-packets for H_0 are understood, and some L-packets for H should arise from the map ${}^L H_0 \longrightarrow {}^L H$ [**14**]. Such L-packets are said to come from *endoscopic transfer* in the sense of character identities. Suppose ρ_0 is an L-packet of H_0. Denote by $\xi_{H_0}(\rho_0)$ the corresponding L-packet of H.

Theorem 4.4 (Rogawski [14, Theorem 13.1.1,proposition 13.1.3]).

(1) *An* L*-packet* Π *of* H *has more than one element if and only if* $\Pi = \xi_{H_0}(\rho_0)$ *for some* L*-packet* ρ_0 *of* H_0.

(2) *If* Π *is a tempered* L*-packet of* H *with more than one element, then* Π *is either a discrete series* L*-packet, or* Π *consists of the constituents of* $\mathrm{Ind}_B^H(\theta)$, *for a character* θ *of the torus* $U(1) \times U(1) \times U(1)$. *Here* B *is the standard Borel subgroup of* H.

(3) *If* Π *is a tempered* L*-packet of* H *with more than one element, then*

$$\chi_\Pi = \sum_{\rho\in\Pi} \chi_\rho,$$

i.e. $m(\rho) = 1,\ \forall \rho \in \Pi.$

(4) *If* $\Pi \in \bar{\mathcal{E}}_2(H)$ *and* $\Pi = \xi_{H_0}(\rho_0)$ *then every element of* Π *has the same formal degree.* □

Theorem 4.5 (Rogawski [14, propositions 13.2.1, 13.2.2]).

(1) *For every tempered* L*-packet* Π *of* H*, there is a unique standard base change lift* $\pi = \psi_H(\Pi) \in \mathcal{E}'(\widetilde{H})$*. The map* $\psi_H : \bar{\mathcal{E}}(H) \to \mathcal{E}'(\widetilde{H})$ *is injective.*

(2) *The representation* $\psi_H(\Pi)$ *is square integrable if and only if* Π *is a square integrable* L*-packet which consists of one element. Moreover, if* $\bar{\mathcal{E}}_s(H)$ *is the collection of singleton square integrable* L*-packets, then*

$$\psi_H : \bar{\mathcal{E}}_s(H) \longrightarrow \mathcal{E}'(\widetilde{H}) \cap \mathcal{E}_2(\widetilde{H})$$

is bijective. Furthermore, $\psi_H(\Pi)$ *is supercuspidal if and only if* $\Pi \in \bar{\mathcal{E}}_s(H)$ *is supercuspidal.* □

Therefore, if π is a supercuspidal representation of $\widetilde{H}$ such that $\pi^\varepsilon \simeq \pi$, then $\pi = \psi_H(\Pi)$, or $\pi = \psi'_H(\Pi)$ for a unique L-packet Π of H. Which map π lifts through is determined by the restriction of the central character of π to $F^\times$.

Corollary 4.6. *Suppose* $G = U(3,3)$ *and* $M = GL(3,E)$*. If* $\pi \in {}^\circ\mathcal{E}(M)$ *and* $A(s,\pi)$ *has a pole at* $s = 0$*, then* π *is a base change lift from* $U(3)$*.*

Proof. This follows immediately from Theorem 2.8. □

We will now proceed to show that we can find a matrix coefficient φ of π such that $\Phi^{st}_\varepsilon(\delta_3,\varphi) \neq 0$ if and only if π is a standard base change lift. Since $N(\delta_3) = I_3$, Theorem 4.2(b) implies $\Phi^{st}_\varepsilon(\delta_3,\varphi) = f(I_3)$, where $\varphi \to f$ is as in Theorem 4.2(a).

Let $e = I_3$. We use the Plancherel formula [**8**] to expand $f(e)$. The description of the tempered L-packets, given in Theorem 4.4, allows us to rewrite the

Plancherel formula in terms of stable tempered characters. Using the character relation which defines base change, and orthogonality relations of twisted characters, we show that $f(e) \neq 0$ for some choice of φ if and only if $\pi = \psi_H(\Pi)$ for some Π. Note that if π has central character $\widetilde{\omega}$ then $\varphi \in C(\widetilde{H}, \widetilde{\omega}^{-1})$.

Let $\mathcal{M}$ be the collection of standard Levi components of H. Suppose ρ is a discrete series representation of L for some $L \in \mathcal{M}$. Then we write $\hat{\rho}$ for $\mathrm{Ind}_{LN}^{H}(\rho)$, where N is the standard unipotent radical associated to L. For each $L \in \mathcal{M}$, let $\mathcal{E}_2(L)_{\omega^{-1}}$ be the collection of discrete series representations of L with central character ω^{-1}. We denote by $\bar{\mathcal{E}}_2(L)_{\omega^{-1}}$ the collection of discrete series L-packets of L with central character ω^{-1}. Then, by the Plancherel formula

$$f(e) = \sum_{L \in \mathcal{M}} C(L) \int_{\rho \in \mathcal{E}_2(L)_{\omega^{-1}}} d(\rho)\mu(\rho)\chi_{\hat{\rho}}(f)\, d\rho, \tag{4.2}$$

where $C(L) > 0$ is a constant, $d(\rho)$ is the formal degree of ρ, $\mu(\rho)$ is its Plancherel measure, and $d\rho$ is the Euclidean measure given in [**8**].

Suppose $\Pi \in \bar{\mathcal{E}}_2(L)_{\omega^{-1}}$. Let $\hat{\Pi}$ be the tempered L-packet of H obtained by induction. If $\hat{\Pi}$ is not a singleton L-packet, then $\hat{\Pi}$ is either a discrete series L-packet, or the collection of constituents of $\mathrm{Ind}_B^H(\theta)$. In the first case the representations in $\hat{\Pi}$ appear in (4.2) with $L = H$. So $\rho = \hat{\rho}$ and $\mu(\rho) = 1$ for all $\rho \in \hat{\Pi} = \Pi$. By Theorem 4.4(4), $d(\rho) = d(\rho')$ for all $\rho, \rho' \in \Pi$. We let $\lambda_H(\Pi)$ be this common formal degree. Then, collecting terms for this L-packet, and applying Theorem 4.4(3), the stable character χ_Π appears with coefficient $\lambda_H(\Pi)$ in (4.2). In the second case $d(\theta) = 1$, and part (3) of Theorem 4.4 implies $\chi_{\hat{\Pi}}$ appears in (4.2) with coefficient $\lambda_H(\Pi) = \mu(\theta)$. If $\hat{\Pi} = \{\hat{\rho}\}$ then we let $\lambda_H(\Pi) = d(\rho)\mu(\rho)$. Collecting terms according to L-packets we rewrite (4.2) as

$$f(e) = \sum_{L \in \mathcal{M}} C(L) \int_{\Pi \in \bar{\mathcal{E}}_2(L)_{\omega^{-1}}} \lambda_H(\Pi)\ \chi_{\hat{\Pi}}(f)\ d\Pi. \tag{4.3}$$

The character identity defining base change (Definition 4.3) shows that we can replace $\chi_{\hat{\Pi}}(f)$ by $\chi_{\pi'\varepsilon}(\varphi)$, where $\pi' = \psi_H(\hat{\Pi})$. We need the following orthogonality relation whose proof is identical to that of Lemma 2.7 of [**6**].

Lemma 4.7. *Let π, π' be irreducible admissible representations of M. Suppose their central characters are related by $\widetilde{\omega}' \simeq \widetilde{\omega}^{-1}$. Further suppose that π is supercuspidal and $\pi' \not\simeq \widetilde{\pi}$. Then, for any matrix coefficient φ of π, $\pi'(\varphi) = 0$. In particular, if π and π' are both ε-invariant, then $\chi_{\pi'\varepsilon}(\varphi) = 0$.* □

Theorem 4.8. *Let $\pi \in {}^\circ\mathcal{E}(M)$. Then $A(s,\pi)$ has a pole at $s=0$ if and only if π is a standard base change lift from $U(3)$, i.e. $\pi = \psi_H(\Pi)$ for some $\Pi \in \bar{\mathcal{E}}(H)$.*

Proof. We already noted, in Corollary 4.6, that if $A(s,\pi)$ has a pole at $s=0$, then π is a base change lift from $U(3)$. Now suppose that π is a base change lift. By Theorem 2.8, $A(s,\pi)$ will have a pole at $s=0$ if and only if $\Phi_\varepsilon^{st}(\delta_3,\varphi) \neq 0$, for some matrix coefficient φ of π. Let $f \in C(H,\omega^{-1})$ be the function associated to φ by Theorem 4.2(a). Then $f(e)$ is given by (4.3). By Theorem 4.5(2), $\widetilde{\pi}$ is also a base change lift from H. Moreover, by considering central characters, π is a standard base change lift if and only if $\widetilde{\pi}$ is a standard base change lift.

Then Lemma 4.7 and Definition 4.3 show that every term of (4.3) vanishes, unless $\widetilde{\pi} = \psi_H(\Pi)$ for some Π. Thus, if $\pi = \psi'_H(\Pi)$, then Theorem 4.5(2) implies $f(e) = 0$. Therefore, if π is a non-standard base change lift, then there is no pole of $A(s,\pi)$ at $s=0$. On the other hand, if $\pi = \psi_H(\Pi)$, then

$$\Phi_\varepsilon^{st}(\delta_3,\varphi) = f(e) = c\chi_{\tilde{\pi}\varepsilon}(\varphi),$$

for some $c > 0$. Thus, it would be enough to know that, for some matrix coefficient φ of π, $\chi_{\tilde{\pi}\varepsilon}(\varphi) \neq 0$. Such a matrix coefficient is called an ε- pseudo coefficient, and their existence is guaranteed by [**14**, pg. 188]. Therefore, we can find a matrix coefficient φ of π for which $\Phi_\varepsilon^{st}(\delta_3,\varphi) \neq 0$ and hence $A(s,\pi)$ has a pole at $s = \pi$. □

§5 Computation of Local Asai L-functions.

We wish to compute the local L-function $L(s,\pi,\Psi)$ referred to in Section 3. Once we have computed this L-function when π is supercuspidal, we will compute $L(s,\sigma,\Psi)$ in general.

Recall that $\mathbf{G}$ is of type A_{2n-1}. Let $\Delta = \{\beta_j\}$ be the set of simple roots, where

$\beta_j = e_j - e_{j+1}$. Then, as described in Section 3, the Galois action identifies β_j and β_{2n-j} for $j = 1, 2, \ldots, n-1$. Therefore, the L-function in question is the generalization of the Asai L-function [**1**] as described in Section 4 of [**17**].

Let π be a supercuspidal representation of M. Then $L(s, \pi, \Psi) = P_\pi(q_F^{-s})^{-1}$, where P_π is the polynomial defined in Section 7 of [**18**]. Let $\pi_s = \pi \otimes |\det()|_E^{s/2}$. Then, for any $s_1 \in \mathbb{C}$, Theorem 3.5(2) of [**18**] implies,

$$P_{\pi_{s_1}}(q_F^{-s}) = P_\pi(q_F^{-s-s_1}).$$

Suppose that, for every $s \in \mathbb{C}$, we have $\pi_s \not\simeq (\pi_s)^\varepsilon$. Then $1 = P_{\pi_s}(1) = P_\pi(q_F^{-s})$. Thus, under this assumption, $L(s, \pi, \Psi) = 1$. Therefore, we may suppose that $\pi \simeq \pi^\varepsilon$. We need to find the normalized polynomial $P_\pi(q_F^{-s})$ such that $P_\pi(q_F^{-s})A(s, \pi)$ is holomorphic and non-zero.

Let $f \in V(s, \pi)_0$ be as in Section 2, and let $\widetilde{v} \in \widetilde{V}$. Let φ be the matrix coefficient associated to f and $\widetilde{v}$. For $s \in \mathbb{C}$ we let $\varphi_s(g) = \varphi(g)|\det g|_E^{s/2}$. Then φ_s is a matrix coefficient of π_s.

Suppose n is odd. The poles of $A(s, \pi)$ are among those of $\left(1 - q_E^{-ns}\right)^{-1}$. Let $s\ \mathbb{C}$. Suppose $q_E^{-ns_0} = 1$. Then (2.7b) shows that

$$\operatorname*{Res}_{s=s_0} < A(s, \pi) f(e), \widetilde{v} >= \left(\operatorname*{Res}_{s=s_0} \left(\frac{1}{1 - q_E^{-ns}} \right) \right) \Phi_\varepsilon^{st}(\delta_n, \varphi_{s_0}).$$

Therefore, $A(s, \pi)$ has a pole at $s = s_0$ if and only if $\Phi_\varepsilon^{st}(\delta_n, \varphi_{s_0}) \neq 0$, for some matrix coefficient φ of π.

Since $\pi^\varepsilon \simeq \pi$, the central character $\widetilde{\omega}$ of π satisfies $\widetilde{\omega}(z) = 1$ for all $z \in NE^\times$. By Lemma 2.6(b), the non-vanishing of $\Phi_\varepsilon^{st}(\delta_n, \varphi_{s_0})$ implies that $\widetilde{\omega}|\ |_E^{-ns_0/2}$ is trivial on $NE^\times$. Therefore, the unramified character $|\ |_E^{-ns_0/2}$ is trivial on $NE^\times$. Thus, evaluating at $N\varpi_E$, we see that $s = s_0$ is a root of $1 - q_E^{-ns} = 0$. So, if $\Phi_\varepsilon^{st}(\delta_n, \varphi_{s_0}) \neq 0$ for some φ, then $(1 - q_E^{s_0 - s})$ divides $P_\pi(q_F^{-s})$. Note that

$$1 - q_E^{s_0 - s} = \begin{cases} 1 - q_F^{s_0 - s} & E/F \text{ ramified} \\ 1 - q_F^{2s_0 - 2s} & E/F \text{ unramified.} \end{cases}$$

Finally notice that if $\eta = |\ |_E^{s_0/2}$, then $(\pi \otimes \eta \circ \det)^\varepsilon \simeq \pi^\varepsilon \otimes \eta^{-1} \circ \det$. Therefore, the above discussion shows that $\pi \otimes \eta \circ \det$ is ε-invariant at the points in question. We summarize our results below.

Theorem 5.1. *Let n be odd. Suppose that π is an irreducible supercuspidal representation of M such that $\pi \simeq \pi^{\varepsilon}$. Let Λ be the set of all unramified characters $\eta \in \hat{E}^{\times}$, no two of which have equal squares, such that $\Phi_{\varepsilon}^{st}(\delta_n, \psi) \neq 0$ for some matrix coefficient ψ of $\pi \otimes \eta \circ \det$.*

(a) *Suppose E/F is ramified. Then*

$$L(s, \pi, \Psi) = \prod_{\eta \in \Lambda} (1 - \eta^2(\varpi_E) q_E^{-s})^{-1} = \prod_{\eta \in \Lambda} (1 - \eta(\varpi_F) q_F^{-s})^{-1}.$$

(b) *If E/F is unramified then*

$$L(s, \pi, \Psi) = \prod_{\eta \in \Lambda} (1 - \eta^2(\varpi_E) q_E^{-s})^{-1} = \prod_{\eta \in \Lambda} (1 - \eta^2(\varpi_F) q_F^{-2s})^{-1}. \quad \square$$

Note than when $n = 3$, then Lemma 4.7 and the argument of Theorem 4.8 show that $\eta \in \Lambda$ if and only if $\pi \otimes \eta \circ \det = \psi_H(\Pi)$ for some supercuspidal L-packet Π of $H = U(3)$.

Now suppose that n is even, and $\pi \simeq \pi^{\varepsilon}$. Then (2.9) shows that the poles of $A(s, \pi)$ are among those of $\left(1 - \widetilde{\omega}(\varpi_F) q_F^{-ns}\right)^{-1}$. Moreover, if s_0 is a zero of $(1 - \widetilde{\omega}(\varpi_F) q_F^{-ns})$, then s_0 is a pole of $A(s, \pi)$ if and only if $\Phi_{\varepsilon}^{\kappa}(\delta_n, \psi) \neq 0$ for some matrix coefficient ψ of $\pi \otimes |\det()|_E^{s_0/2}$. Since $\widetilde{\omega}$ is trivial on $NE^{\times}$, Lemma 2.6(a) implies $|\ |_E^{s_0/2} = \widetilde{\omega}$ on $F^{\times}$. Therefore, if $\eta = |\ |_E^{s_0/2}$, then $\pi \otimes \eta \circ \det$ is ε-invariant.

Theorem 5.2. *Let n be even. Suppose that π is an irreducible supercuspidal representation of M such that $\pi \simeq \pi^{\varepsilon}$. Let Λ be the collection of unramified characters $\eta \in \hat{E}^{\times}$, no two of which have equal value at ϖ_F, such that $\Phi_{\varepsilon}^{\kappa}(\delta_n, \psi) \neq 0$ for some matrix coefficient ψ of $\pi \otimes \eta$. Then*

$$L(s, \pi, \Psi) = \prod_{\eta \in \Lambda} (1 - \eta(\varpi_F) q_F^{-s})^{-1}. \quad \square$$

If $n = 2$, then the results of [**6**] show that Λ is the set of η such that $\pi \otimes \eta \circ \det = \psi'_H(\Pi)$ for some supercuspidal L-packet Π of $U(2)$.

We now compute the L-function $L(s, \sigma, \Psi)$ for any irreducible admissible representation σ of M. By the discussion in Section 7 of [**18**], it is enough to compute

$L(s,\sigma,\Psi)$ where σ is in the discrete series of M. From now on we write Ψ_n for Ψ, and denote by α the character $|\det()|_E$.

Let σ be an irreducible admissible discrete series representation of M. Let ψ_F be a non-trivial additive character of F. Let $\gamma(s,\sigma,\Psi_n,\psi_F)$ be the rational function of q_F^{-s} attached to σ and Ψ_n by Theorem 3.5 of [**18**]. Then $L(s,\sigma,\Psi_n)$ is the inverse of the normalized numerator of $\gamma(s,\sigma,\Psi_n,\psi_F)$. That is, there is a monomial $\varepsilon(s,\sigma,\Psi_n,\psi_F)$ in q_F^{-s}, such that

$$\gamma(s,\sigma,\Psi_n,\psi_F)=\varepsilon(s,\sigma,\Psi_n,\psi_F)L(1-s,\widetilde{\sigma},\Psi_n)/L(s,\sigma,\Psi_n). \tag{5.1}$$

Let Δ be the set of simple roots of M. Suppose $\sigma\in\mathcal{E}_2(M)$. By Jacquet's theorem [**7**], we can find a parabolic $M'N'$ of M, and an irreducible supercuspidal representation σ_1 of M' so that σ is a subquotient of $\mathrm{Ind}_{M'N'}^{M}(\sigma_1\otimes 1_{N'})$. We can assume that there is a subset θ of Δ so that $M'N'=M_\theta N_\theta$. By [**2**,**24**] there are integers a and b, with $ab=n$, so that

$$M_\theta\simeq GL(a,E)\times\cdots\times GL(a,E).$$

Moreover, we can assume that there is an irreducible unitary supercuspidal representation π_0 of $GL(a,E)$ so that $\sigma_1=\pi_1\otimes\cdots\otimes\pi_b$, where

$$\pi_i=\pi_0\otimes\alpha^{(b+1-2i)/2}.$$

Note that

$${}^LM_\theta^0=GL(a,\mathbb{C})\times\cdots\times GL(a,\mathbb{C})\subset GL(2n,\mathbb{C}).$$

Suppose that

$$g=(g_1,\ldots,g_b,h_1,\ldots,h_b)\in {}^LM_\theta^0. \tag{5.2}$$

Then, computing directly, we see that

$$\tau(g)=\left(\tau(h_b),\ldots,\tau(h_1),\tau(g_b),\ldots,\tau(g_1)\right),$$

where $\tau(g_i)$ is, up to an inner automorphism, the element described in Section 3.

Consider the restriction Ψ_θ of Ψ_n to ${}^LM_\theta$. If $X \in M(n, \mathbb{C})$, then we write

$$X = \begin{pmatrix} X_{11} & \dots & X_{1b} \\ \vdots & \ddots & \vdots \\ X_{b1} & \dots & X_{bb} \end{pmatrix}$$

with each $X_{ij} \in M(a, \mathbb{C})$. Then if g is given by (5.2), we have

$$\Psi_\theta((g,1)) \cdot (Xij) = (g_i X_{ij} h_j^{-1}) \text{ and} \tag{5.3}$$

$$\Psi_\theta((1,\tau)) \cdot (X_{ij}) = (X_{(b+1-j)(b+1-i)}). \tag{5.4}$$

We look at the irreducible constituents of Ψ_θ. Let V_{kl} be the subspace of ${}^L\mathfrak{n}$ given by

$$\left\{(X_{ij}) \,\middle|\, X_{ij} = 0, (i,j) \neq (k,l)\right\}.$$

Let $G_1 = U(a,a)$, $M_1 \simeq GL(a,E)$, and $P_1 = M_1N_1$. Then, we see that ${}^LM_1 = (GL(a,\mathbb{C}) \times GL(a,\mathbb{C})) \ltimes W_F$. Let ${}^L\mathfrak{n}_1$ be the lie algebra of ${}^LN_1$. Then, for $1 \le i \le b$, $V_{i(b+1-i)}$ is irreducible, and Ψ_θ restricted to $V_{i(b+1-i)}$ is isomorphic to the representation Ψ_a of ${}^LM_1$ on ${}^L\mathfrak{n}_1$.

If $j \neq b+1-i$, then $W_{ij} = V_{ij} \cup V_{(b+1-j)(b+1-i)}$ is irreducible, and the action of Ψ_θ restricted to W_{ij} is given by $\mathrm{Ind}_{{}^LM_1^0}^{{}^LM_1}(\rho_a \otimes \widetilde{\rho}_a)$, where ρ_a is the standard representation of $GL(a,\mathbb{C})$.

The following two lemmas and their corollary were pointed out to me by Freydoon Shahidi. I would like to thank him again for his time and effort in this matter.

Lemma 5.3. *Consider* $\rho = \rho_a \otimes \widetilde{\rho}_a$ *as a representation of* ${}^LM_1^0$. *Let* η *be the non-trivial character of* $\mathrm{Gal}(E/F)$. *Let* $I(\rho) = \mathrm{Ind}_{{}^LM_1^0}^{{}^LM_1}(\rho)$. *Then* $I(\rho) = \Psi_a \oplus (\Psi_a \otimes \eta)$.

Proof. Let $\{e_i\}$ be a basis for $\mathbb{C}^a$, and let $\{e_i^*\}$ be the dual basis for $(\mathbb{C}^a)^*$. Then

$$\rho(g_1, g_2, 1)(e_i \otimes e_j^*) = g_1 e_i \otimes e_j^* g_2^{-1}.$$

Let $\tau(\rho)$ be the representation of ${}^LM_1^0$ given by

$$\tau(\rho)(g_1, g_2, 1) = \rho\left((1,1,\tau)\,(g_1,g_2,1)\,(1,1,\tau)^{-1}\right) = \rho({}^tg_2^{-1},\ {}^tg_1^{-1}, 1).$$

Note that $\{I(\rho)((1,1,\tau))(e_i \otimes e_j^*)\}$ is a basis for $\tau(\rho)$ as a subspace of $I(\rho)_{|_{L_{M_1^0}}}$.

Let $v_{ij} = e_i \otimes e_j^* + I(\rho)((1,1,\tau))(e_j \otimes e_i^*)$. Let V_0 be the subspace generated by $\{v_{ij} \mid 1 \leq i,j \leq a\}$. Then we will show that V_0 is invariant and isomorphic to Ψ_a. First note that

$$\begin{aligned} I(\rho)((g_1,g_2,1))v_{ij} &= I(\rho)((g_1,g_2,1))e_i \otimes e_j^* + I(\rho)((g_1,g_2,1)I(\rho)((1,1,\tau))e_j \otimes e_i^* \\ &= g_1 e_i \otimes e_j^* g_2^{-1} + I(\rho)((1,1,\tau))\tau(\rho)((g_1,g_2,1))e_j \otimes e_i * \\ &= g_1 e_i \otimes e_j^* g_2^{-1} + I(\rho)((1,1,\tau))\left({}^t g_2^{-1} e_j \otimes e_i^* \, {}^t g_1\right). \end{aligned}$$

Let

$$g e_i = \sum_{l=1}^{a} c_l e_l \text{ and } e_j^* g_2^{-1} = \sum_{k=1}^{a} d_k e_k^*.$$

Then

$$e_i^* \, {}^t g_1 = \sum_{l=1}^{a} c_l e_l^* \text{ and } {}^t g_2^{-1} e_j = \sum_{k=1}^{a} d_k e_k.$$

Thus,

$$g_1 e_i \otimes e_j^* g_2^{-1} = \sum_{l,k=1}^{a} c_l d_k e_l \otimes e_k^*,$$

while

$${}^t g_2^{-1} e_j \otimes e_i^* \, {}^t g_1 = \sum_{l,k=1}^{a} c_l d_k e_k \otimes e_l^*.$$

Therefore,

$$I(\rho)((g_1,g_2,1))v_{ij} = \sum_{l,k} c_l d_k v_{lk} \in V_0.$$

Furthermore,

$$I(\rho)((1,1,\tau))v_{ij} = I(\rho)((1,1,\tau))e_i \otimes e_j^* + e_j \otimes e_i^* = v_{ji}.$$

Thus, V_0 is invariant, and the description of $I(\rho)$ acting on V_0 given above clearly shows that V_0 is isomorphic to Ψ_a (see Section 3).

Note that the complement of V_0 in $I(\rho)$ is generated by

$$\{w_{ij} = e_i \otimes e_j^* - I(\rho)((1,1,\tau))e_j \otimes e_i^*\}.$$

Moreover, $I(\rho)((1,1,\tau))w_{ij} = -w_{ji}$. Therefore, $I(\rho)$ acts on this subspace as $\Psi_a \otimes \eta$ acts on ${}^L\mathfrak{n}_1$. This proves the lemma. □

REMARK. For the following lemma, we allow the possibility that $F = \mathbb{R}$ (and $E = \mathbb{C}$ or $\mathbb{R} \oplus \mathbb{R}$).

Lemma 5.4. *Let ρ and $I(\rho)$ be as in Lemma 5.3. Suppose that π is any irreducible admissible representation of $GL(a,E)$ which can be parameterized, i.e. $F = \mathbb{R}$, or π is unramified in the sense of* [**3**]. *Then $L(s,\sigma,I(\rho)) = L(s,\sigma \times \bar{\sigma})$. Here $\bar{\sigma}(g) = \sigma(\bar{g})$, and the L-function on the right is the Rankin-Selberg product L-function attached to σ and $\bar{\sigma}$.* [**10**].

PROOF.

Since ρ_a is the standard representation, $\widetilde{\rho}_a$ is isomorphic to the representation ρ^τ, given by $\rho^\tau(g) = \rho(\tau(g))$. The action of $\mathrm{Gal}(E/F)$ on $\mathrm{Res}_{E/F}(GL_n)$ sends σ to $\bar{\sigma}$. Taking the viewpoint of E-groups, we have $L(s,\sigma,I(\rho)) = L(s,\sigma,\rho)$. However, by definition $L(s,\sigma,\rho) = L(s,\sigma \times \bar{\sigma})$. □

Corollary 5.5. *Let π be an irreducible admissible supercuspidal representation of $GL(a,E)$. Then*

$$L(s,\pi \times \bar{\pi}) = L(s,\pi,\Psi_a)\ L(s,\pi \otimes \chi_\mu,\Psi_a).$$

PROOF. By proposition 5.1 of [**18**] we can choose a number field K, with $K_{v_0} = F$ for some place v_0 of K, a quadratic extension K'/K, a place w_0 lying over v_0 so that $K'_{w_o} = E$, and a cusp form $\Pi = \bigotimes_w \Pi_w$ of $GL(a,\mathbf{A}_{K'})$, such that Π_w is unramified for every finite place $w \neq w_0$, and $\Pi_{w_0} \simeq \pi$. For each non-split place w of K' we let η_w be the non-trivial character of $\mathrm{Gal}(K'_w/K_v)$, where w lies over v. By local class field theory, η_w corresponds to the character χ_{μ_w} of $GL(a,K'_w)$. Let $\eta_{K'}$ be the non-trivial character of $\mathrm{Gal}(K'/K)$. By Lemmas 5.3 and 5.4 we have

$$L(s,\Pi_w \times \bar{\Pi}_w) = L(s,\Pi_w,\Psi_a)\ L(s,\Pi_w \otimes \chi_{\mu_w},\Psi_a), \tag{5.5}$$

for every $w \neq w_0$. We have a global functional equation

$$L(s, \Pi \times \bar{\Pi}) = \varepsilon(s, \Pi \times \bar{\Pi}) L(1-s, \widetilde{\Pi} \times \widetilde{\bar{\Pi}}),$$

as well as one for $L(s, \Pi, \Psi_a)$ and $L(s, \Pi, \Psi_a \otimes \eta'_K)$. At each place of K', we have

$$L(s, \widetilde{\Pi}_w \times \widetilde{\bar{\Pi}_w}) = L(s, \Pi_w, \widetilde{\rho}) \text{ and}$$
$$L(s, \widetilde{\Pi}_w, \Psi_a) = L(s, \Pi_w, \widetilde{\Psi}_a).$$

Using global functional equations for each of these L-functions and equation (5.5) for every $w \neq w_0$, we conclude

$$L(s, \pi \times \bar{\pi}) = L(s, \pi, \Psi_a) L(s, \pi \otimes \chi_\mu, \Psi_a),$$

as desired. □

We return to the situation in which σ is a discrete series subquotient of $\mathrm{Ind}_{M_\theta N_\theta}^{M}(\sigma_1)$, with $\sigma_1 = \bigotimes_i \pi_i$. In what follows we use the symbol $\equiv$ to denote two rational functions which differ by a monomial. Applying part 3 of Theorem 3.5 of [**18**] along with Lemmas 5.3 and 5.4 to (5.3), and (5.4), we find

$$\gamma(s, \sigma, \Psi_n, \psi_F) = \prod_{i=1}^{b} \gamma(s, \pi_i, \Psi_a, \psi_F) \prod_{1 \leq i < j \leq b} \gamma(s, \pi_i \times \bar{\pi}_j, \psi_F),$$

where $\gamma(s, \pi_i \times \bar{\pi}_j, \psi_F)$ is the Rankin-Selberg factor attached to $(\pi_i, \bar{\pi}_j)$ [**10**,**16**].

By part (2) of Theorem 3.5 of [**18**] and page 409 of [**10**], we have

$$\gamma(s, \pi_i, \Psi_a, \psi_F) = \gamma(s + b + 1 - 2i, \pi_0, \Psi_a, \psi_F), \tag{5.6}$$

and

$$\gamma(s, \pi_i \times \bar{\pi}_j, \psi_F) = \gamma(s + b + 1 - (i+j), \pi_0 \times \bar{\pi}_0, \psi_F). \tag{5.7}$$

Using this, and (5.1) we have

$$\gamma(s, \sigma, \Psi_n, \psi_F) \equiv$$
$$\prod_{i=1}^{b} \frac{L(1 - (s + b + 1 - 2i), \widetilde{\pi}_0, \Psi_a)}{L(s + b + 1 - 2i, \pi_0, \Psi_a)} \prod_{i<j} \frac{L(1 - (s + b + 1 - (i+j)), \widetilde{\pi}_0 \times \widetilde{\bar{\pi}}_0)}{L(s + b + 1 - (i+j), \pi_0 \times \bar{\pi}_0)}.$$

Since π_0 is unitary and supercuspidal, [**19**] implies that, up to a monomial, the right hand side is equal to

$$\prod_{i=1}^{b} \frac{L(s+b-2i,\pi_0,\Psi_a)}{L(s+b+1-2i,\pi_0,\Psi_a)} \prod_{i<j} \frac{L(s+b-(i+j),\pi_0\times\bar{\pi}_0)}{L(s+b+1-(i+j),\pi_0\times\bar{\pi}_0)}$$

$$= \prod_{\nu=-(b-1)/2}^{(b-1)/2} \frac{L(s+2\nu-1,\pi_0,\Psi_a)}{L(s+2\nu,\pi_0,\Psi_a)} \prod_{i=1}^{b-1} \frac{L(s-i,\pi_0\times\bar{\pi}_0)}{L(s+b-2i,\pi_0\times\bar{\pi}_0)}.$$

By [**10**], $L(s,\pi_0\times\bar{\pi}_0)$ is identically one unless some unramified twist $\pi_0\otimes\eta$ satisfies $\overline{\pi_0\otimes\eta}\simeq(\pi_0\otimes\eta)^{\sim}$. By [**2**, §7], the contragredient of an irreducible admissible representation of $GL(a,E)$ is given by composition with the automorphism $g\mapsto {}^tg^{-1}$. Therefore, $L(s,\pi_0\times\bar{\pi}_0)$ is identically one unless there is an unramified character η such that $\pi_0\otimes\eta\simeq(\pi_0\otimes\eta)^{\varepsilon}$. By Theorems 5.1 and 5.2, this is the same as the condition for $L(s,\pi_0,\Psi_a)$ to be non-trivial. Note that

$$\mathrm{Ind}_{M_\theta N_\theta}^{M}(\sigma_1^{\varepsilon}\otimes 1_{N_\theta}) = \mathrm{Ind}_{M_\theta N_\theta}^{M}(\tilde{\bar{\sigma}}_1\otimes 1_{N_\theta})$$

$$\simeq \left(\mathrm{Ind}_{M_\theta N_\theta}^{M}(\bar{\sigma}_1\otimes 1_{N_\theta})\right)^{\sim} \simeq \left(\mathrm{Ind}_{M_\theta N_\theta}^{M}(\sigma_1\otimes 1_{N_\theta})\right)^{\varepsilon}$$

(see [**4**]). Therefore, if $L(s,\sigma,\Psi_n)\not\equiv 1$, then $\sigma_1\simeq\sigma_1^{\varepsilon}$, which implies that $\pi_0\simeq\pi_0^{\varepsilon}$.

Suppose that b is even. Then

$$\prod_{i=1}^{b-1} \frac{L(s-i,\pi_0\times\bar{\pi}_0)}{L(s+b-2i,\pi_0\times\bar{\pi}_0)} = \prod_{k=0}^{(b-2)/2} \frac{L(s-(2k+1),\pi_0\times\bar{\pi}_0)}{L(s+2k,\pi_0\times\bar{\pi}_0)},$$

and

$$\prod_{\nu=-(b-1)/2}^{(b-1)/2} \frac{L(s+2\nu-1,\pi_0,\Psi_a)}{L(s+2\nu,\pi_0,\Psi_a)} = \prod_{k=0}^{(b-2)/2} \frac{L(s+2k,\pi_0,\Psi_a)L(s-2(k+1),\pi_0,\Psi_a)}{L(s-(2k+1),\pi_0,\Psi_a)L(s+(2k+1),\pi_0,\Psi_a)}.$$

Therefore, using Corollary 5.5, we have

$$\gamma(s,\sigma,\Psi_n,\psi_F) = \prod_{k=0}^{(b-2)/2} \frac{L(s+(2k+1),\pi_0,\Psi_a)^{-1}L(s+2k,\pi_0\otimes\chi_\mu,\Psi_a)^{-1}}{L(s-2(k+1),\pi_0,\Psi_a)^{-1}L(s-(2k+1),\pi_0\otimes\chi_\mu,\Psi_a)^{-1}}.$$

Note that both the numerator and denominator of the above expression are polynomials in q_F^{-s}, and there are no further cancellations. For each $0 \leq k \leq (b-2)/2$, we let $i = (b/2) - k$. Then $1 \leq i \leq b/2$. Moreover,

$$L(s+(2k+1), \pi_0, \Psi_a) = L(s, \pi_i, \Psi_a), \text{ and}$$
$$L(s+2k, \pi_0 \otimes \chi_\mu, \Psi_a) = L(s, \pi_i \otimes \chi_\mu \otimes \alpha^{-1/2}, \Psi_a).$$

Let $\check{\pi}_i = \widetilde{\pi}_0 \otimes \alpha^{b+1-2i/2}$. Then

$$L(s-2(k+1), \pi_0, \Psi_a) \equiv L(1-s, \check{\pi}_i, \Psi_a) \text{ and}$$
$$L(s-(2k+1), \pi_0 \otimes \chi_\mu, \Psi_a) \equiv L(1-s, \check{\pi}_i \otimes \chi_\mu^{-1} \otimes \alpha^{-1/2}, \Psi_a).$$

Now suppose that b is odd. Then

$$\prod_{i=1}^{b-1} \frac{L(s-i, \pi_0 \times \bar{\pi}_0)}{L(s+b-2i, \pi_0 \times \bar{\pi}_0)} = \prod_{k=1}^{(b-1)/2} \frac{L(s-2k, \pi_0 \times \bar{\pi}_0)}{L(s+(2k-1), \pi_0 \times \bar{\pi}_0)},$$

and

$$\prod_{\nu=-(b-1)/2}^{(b-1)/2} \frac{L(s+2\nu-1, \pi_0, \Psi_a)}{L(s+2\nu, \pi_0, \Psi_a)} =$$
$$\prod_{k=1}^{(b-1)/2} \frac{L(s+(2k-1), \pi_0, \Psi_a)}{L(s-2k, \pi_0, \Psi_a)} \prod_{k=0}^{(b-1)/2} \frac{L(s-(2k+1), \pi_0, \Psi_a)}{L(s+2k, \pi_0, \Psi_a)}.$$

Thus,

$\gamma(s, \sigma, \Psi_n, \psi_F) \equiv$

$$\prod_{k=0}^{(b-1)/2} \frac{L(s+2k, \pi_0, \Psi_a)^{-1}}{L(s-(2k+1), \pi_0, \Psi_a)^{-1}} \prod_{k=1}^{(b-1)/2} \frac{L(s+(2k-1), \pi_0 \otimes \chi_\mu, \Psi_a)^{-1}}{L(s-2k, \pi_0 \otimes \chi_\mu, \Psi_a)^{-1}}.$$

Rewriting the product above in terms of the π_i we get part (b) of the following theorem.

Theorem 5.6. *Let σ be a discrete series representation of $GL(n,E)$. Choose an irreducible unitary supercuspidal representation π_0 of $GL(a,E)$, $n=ab$, such that σ is the unique discrete series component of the representation of $GL(n,E)$ induced from $\pi_1\otimes\cdots\otimes\pi_b$, $\quad \pi_i=\pi_0\otimes\alpha^{(b+1-2i)/2}$. Let $\breve{\pi}_i=\widetilde{\pi}_0\otimes\alpha^{(b+1-2i)/2}$.*

(a) *Suppose b is even. Then*

$$L(s,\sigma,\Psi_n)=\prod_{i=1}^{b/2}L(s,\pi_i,\Psi_a)L(s,\pi_i\otimes\chi_\mu\otimes\alpha^{-1/2},\Psi_a),$$

and

$$L(s,\widetilde{\sigma},\Psi_n)=\prod_{i=1}^{b/2}L(s,\breve{\pi}_i,\Psi_a)L(s,\breve{\pi}_i\otimes\chi_\mu^{-1}\otimes\alpha^{-1/2},\Psi_a).$$

(b) *Suppose b is odd. Then*

$$L(s,\sigma,\Psi_n)=\prod_{i=1}^{(b+1)/2}L(s,\pi_i,\Psi_a)\prod_{i=1}^{(b-1)/2}L(s,\pi_i\otimes\chi_\mu\otimes\alpha^{-1/2},\Psi_a),$$

and

$$L(s,\widetilde{\sigma},\Psi_n)=\prod_{i=1}^{(b+1)/2}L(s,\breve{\pi}_i,\Psi_a)\prod_{i=1}^{(b-1)/2}L(s,\breve{\pi}_i\otimes\chi_\mu^{-1}\otimes\alpha^{-1/2},\Psi_a).\qquad\square$$

corollary 5.7. *Let σ be an irreducible admissible representation of $GL(n,E)$. Then*

$$L(s,\sigma\times\bar{\sigma})=L(s,\sigma,\Psi_n)L(s,\sigma\otimes\chi_\mu,\Psi_n).$$

PROOF. It is enough to prove the claim for the case when σ is in the discrete series. Suppose that σ is of the form described in Theorem 5.6. Then, by Theorem 8.2 of [**10**],

$$L(s,\sigma\times\bar{\sigma})=\prod_{i=1}^{b}L(s,\pi_1\times\bar{\pi}_i).$$

We rewrite this as

$$\prod_{i=1}^{b} L(s+b-i,\pi_0\times\bar{\pi}_0)=\prod_{i=1}^{b} L(s+b-i,\pi_0,\Psi_a)L(s,\pi_0\otimes\chi_\mu,\Psi_a). \tag{5.8}$$

Examining the formulas in Theorem 5.6 we see that $L(s,\sigma,\Psi_a)L(s,\sigma\otimes\chi_\mu,\Psi_a)$ also has the form (5.8). $\square$

§6 Reducibility for $U(n,n+1)$.

We now use the results of Section 5 to determine reducibility criteria for the group $U(n,n+1)$. Let E/F be a quadratic extension of local nonarchimedean fields of characteristic 0. Let $n\geq 1$. Let $J=\begin{pmatrix} & & \beta I_n\\ & 1 & \\ -\beta I_n & & \end{pmatrix}$, where $\beta\in E\setminus F$ satisfies $\bar{\beta}=-\beta$. Let $\mathbf{G}=\mathbf{U}(2n+1)$, defined with respect to J. Let $G=\mathbf{G}(F)=U(n,n+1)$. Then $G=\{g\in GL(2n+1,E)\mid gJ\,{}^t\bar{g}=J\}$. Let $T=\mathbf{T}(F)$ be the maximal torus of diagonal elements. Then

$$T=\left\{\begin{pmatrix} x_1 & & & & & & \\ & \ddots & & & & & \\ & & x_n & & & & \\ & & & y & & & \\ & & & & \bar{x}_1^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \bar{x}_n^{-1}\end{pmatrix} \;\middle|\; \begin{matrix} x_i\in E^\times\\ y\in E^1\end{matrix}\right\}.$$

Let $\mathbf{T}_d$ be the maximal F-split subtorus. Then

$$T_d=\mathbf{T}_d(F)=\{(x_1,\ldots,x_n,y)\in T\ |x_i\in F^\times, y=1\}.$$

The restricted root system $\Phi(\mathbf{G},\mathbf{T}_d)$ is of type BC_n. We choose the set of simple roots $\Delta=\{e_i-e_{i+1}\}_{i=1}^{n-1}\cup\{e_n\}$. Let $\theta=\Delta\setminus\{e_n\}$. Let $\mathbf{P}=\mathbf{P}_\theta$, be the maximal parabolic subgroup of $\mathbf{G}$ associated to θ. Let $\mathbf{P}=\mathbf{M}\mathbf{N}$ be the Levi decomposition of $\mathbf{P}$, and $P=\mathbf{P}(F)=MN$. Then

$$M=\mathbf{M}(F)=\left\{\begin{pmatrix} g & & \\ & y & \\ & & {}^t\bar{g}^{-1}\end{pmatrix} \;\middle|\; \begin{matrix} g\in GL(n,E)\\ y\in E^1\end{matrix}\right\}\simeq GL(n,E)\times U(1).$$

Let Π be an irreducible admissible representation of M. Then we write $\Pi = (\pi,\nu)$, in the following way. $\Pi(g,y) = \pi(g)\nu(\det(g\varepsilon(g))y)$, i.e., $(\pi,\nu) = (\pi \otimes \widetilde{\nu} \circ \det) \otimes \nu$, with π an irreducible admissible representation of $GL(n,E)$, $\nu \in \widehat{E^1}$, and $\widetilde{\nu}(z) = \nu(z/\bar{z})$. We choose this normalization so that our notation is consistent with that of Keys [**11**] and Rogawski [**14**].

Let A be the split component of M. Since $\begin{pmatrix} 0 & 0 & I_n \\ 0 & 1 & 0 \\ -I_n & 0 & 0 \end{pmatrix}$ represents the non-trivial element w of $W(A)$, we see that $\Pi \simeq \Pi^w$ if and only if $\pi \simeq \pi^\varepsilon$.

We now consider the L-groups. Note that ${}^LG^0 = GL(2n+1,\mathbb{C})$. If $g \in GL(2n+1,\mathbb{C})$, then $\tau(g) = J({}^tg^{-1})J^{-1}$. We also note that

$${}^LM^0 = \left\{ \begin{pmatrix} g & & \\ & a & \\ & & h \end{pmatrix} \;\middle|\; \begin{matrix} g,h \in GL(n,\mathbb{C}) \\ a \in \mathbb{C}^\times \end{matrix} \right\} \simeq GL(n,\mathbb{C}) \times GL(1,\mathbb{C}) \times GL(n,\mathbb{C}).$$

The action of $\mathrm{Gal}(E/F)$ restricted to ${}^LM^0$ is given by $\tau((g,a,h)) = ({}^th^{-1}, a^{-1}, {}^tg^{-1})$.

The Lie algebra ${}^L\mathfrak{n}$ of ${}^LN$ is given by

$$\left\{ \begin{pmatrix} 0 & y & X \\ 0 & 0 & {}^tz \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{matrix} z,y \in \mathbb{C}^n \\ X \in M(n,\mathbb{C}) \end{matrix} \right\}.$$

The adjoint action r of ${}^LM$ on ${}^L\mathfrak{n}$ has two constituents. We order these constituents as in [**18**] and write $r = r_1 \oplus r_2$. Thus,

$$r_{1|\,{}^LM^0} \simeq \rho_n \otimes \widetilde{\rho}_1 \oplus \rho_1 \otimes \widetilde{\rho}_n.$$

Moreover, [**18**, pp. 297-298], shows that $L(s,\Pi,r_2) = L(s,\pi,\Psi_n \otimes \eta)$, where η is the nontrivial character of $\mathrm{Gal}(E/F)$. Thus, $L(s,\Pi,r_2) = L(s,\pi \otimes \chi_\mu,\Psi_n)$.

Lemma 6.1. *Let π be an irreducible admissible representation of $GL(n,E)$, and $\nu \in \widehat{E^1}$. Suppose $\Pi = (\pi,\nu)$. Then $L(s,\Pi,r_1) = L(s,\pi)$, where $L(s,\pi)$ is the Godement-Jacquet L-function attached to π* [**5**].

Proof. Let $\varphi : W_F \longrightarrow GL(1,\mathbb{C}) \rtimes W_F$ be an admissible homomorphism attached to ν, and $\varphi_0 : W_F \longrightarrow GL(1,\mathbb{C})$, the associated 1-cocycle. Then $\widetilde{\varphi}_0 :$

$W_F \longrightarrow GL(1,\mathbb{C}) \times GL(1,\mathbb{C})$, given by $\widetilde{\varphi}_0(w) = (\varphi_0(w), \varphi_0(w))$, is attached to the character $\widetilde{\nu}$ of $E^\times$ [**14**, pg. 50]. Let $\mathbf{M}_1 = \mathrm{Res}_{E/F}(\mathbf{GL}_n)$. Then ${}^L M_1 = (GL(n,\mathbb{C}) \times GL(n,C)) \rtimes W_F$. We consider a representation r_1' of ${}^L M_1$ on $\mathbb{C}^n \oplus \mathbb{C}^n$. Namely,

$$r_1'(g,h,1) \cdot (y,\ {}^t z) = (gy,\ {}^t z h^{-1}),$$

and

$$r_1(1,1,\tau) \cdot (y,\ {}^t z) = (z',\ {}^t y'),$$

where $(y_1, \ldots, y_n)' = (y_n, \ldots, y_1)$. Suppose π is a class one representation, and $\xi : W_E \longrightarrow GL(n,\mathbb{C})$ parameterizes π. Then there is an admissible homomorphism $\xi' : W_F \longrightarrow {}^L M_1$ which parameterizes π as a representation of an F-group [**14**, pg. 48]. For simplicity, we write $\xi_0'(w) = (\xi_1(w), \xi_2(w))$. Then, by our choice of normalization, Π is parameterized by the map $\psi : W_F \longrightarrow {}^L M$, given by

$$\psi(w) = \left(\begin{pmatrix} \xi_1(w)\varphi_0(w)I_n & & \\ & \varphi_0(w) & \\ & & \xi_2(w)\varphi_0(w)I_n \end{pmatrix}, w \right).$$

Let ψ_0 be the map from W_F to ${}^L M^0$. Thus, if $(y,\ {}^t z) \in \mathbb{C}^n \oplus \mathbb{C}^n$, then

$$r_1 \circ \psi_0(w) \cdot (y, {}^t z) = (\xi_1(w) y,\ {}^t z \xi_2(w)^{-1}) = r_1' \circ \xi_0'(w) \cdot (y,\ {}^t z).$$

Similarly, $r_1(1,1,1,\tau) = r_1'(1,1,\tau)$. Therefore, $L(s,\Pi,r_1) = L(s,\pi,r_1') = L(s,\pi)$. The lemma now follows from global considerations, as in the proof of Corollary 5.5. $\square$

By Lemma 6.1 and Proposition 5.11 of [**5**], $L(s,\Pi,r_1) = 1$, for $n \neq 1$. If $n = 1$, then, by [**22**], $L(s,\pi)$ is holomorphic and non-zero at $s = 0$, unless $\pi = 1$. Assume that if $n = 1$, then $\pi \neq 1$. By Corollary 7.6 of [**18**], we know that $I(\Pi) = \mathrm{Ind}_P^G(\Pi)$ is reducible if and only if $\Pi^w \simeq \Pi$ and $L(s,\Pi,r_i)$ has no pole at $s = 0$ for $i = 1$ or 2. Since $L(s,\Pi,r_1)$ is holomorphic and non-zero at $s = 0$, we know that $I(\Pi)$ is reducible if and only if $\pi^\varepsilon \simeq \pi$, and $L(s,\pi\otimes\chi_\mu,\Psi_n)$ has no pole at $s = 0$.

Proposition 6.2. *Let* π *be an irreducible unitary supercuspidal representation of* $GL(n,E)$, *with* $\pi^{\varepsilon} \simeq \pi$, *and let* ν *be a unitary character of* E^1. *If* $n=1$, *then we assume that* $\pi \neq 1$. *Let* $G = U(n,n)$, $P = MN$, *with* $M \simeq GL(n,E)$. *Let* $G' = U(n,n+1)$, $P' = M'N'$, *with* $M' \simeq GL(n,E) \times U(1)$. *Let* $\Pi = (\pi,\nu)$ *as above. We denote by* $I(\pi)$ *the representation of* G *induced from* π, *and* $I(\Pi)'$ *the representation of* G' *induced from* Π. *Then* $I(\pi)$ *is reducible if and only if* $I(\Pi)'$ *is irreducible.*

PROOF. Since $\pi \simeq \pi^{\varepsilon}$, [**10**] implies $L(s,\pi\times\bar{\pi})$ has a simple pole at $s=0$. By Corollary 5.5, exactly one of $L(s,\pi,\Psi_n)$ and $L(s,\pi\otimes\chi_\mu,\Psi_n)$ has a pole at $s=0$. The first L-function determines reducibility of $I(\pi)$, while the second determines the reducibility of $I(\Pi)'$. Therefore, the proposition holds. □

For the remainder of this section we let G, G', P, and P' be as in Proposition 6.2. Let $\rho = \rho_\theta$ be the half sum of the positive roots in N'. Let $\alpha = e_n$ be the unique simple root in N', and let $\widetilde{\alpha} =< \rho, \alpha >^{-1} \rho$, where $< \rho, \alpha >$ is defined as in Section 2. Then $\widetilde{\Delta} = \{\beta_1, \ldots, \beta_{2n}\}$, and $\widetilde{\theta} = \widetilde{\Delta} \setminus \{\beta_n, \beta_{n+1}\}$. Here, as is Section 2, $\beta_i = e_i - e_{i+1}$ is the i-th simple, non-restricted root. Therefore,

$$\rho_{\tilde{\theta}} = \frac{n+1}{2}\sum_{j=1}^{n} j(\beta_j + \beta_{2n+1-j}),$$

and thus

$$< \rho, \alpha >= (\rho_{\tilde{\theta}}, \beta_n) = \frac{n+1}{2}.$$

So, we have

$$q_F^{<\tilde{\alpha}, H_{P'}(m)>} = |\det(m)|_E.$$

Therefore, $I(s,\Pi)' = \mathrm{Ind}_{P'}^{G'}(\Pi \otimes |\det()|_E^s)$.

Theorem 6.3. *Let* G', *and* P' *be as in Proposition 6.2. Let* π *be an irreducible unitary supercuspidal representation of* $GL(n,E)$, *and let* $\Pi = (\pi,\nu) \in {}^{\circ}\mathcal{E}(M')$. *Let* κ *be the trivial character of* $F^{\times}/NE^{\times}$ *if* n *is odd, and the non-trivial character if* n *is even.*

(1) $I(\Pi)'$ *is reducible if and only if* $\pi \simeq \pi^{\varepsilon}$ *and there is some matrix coefficient* φ *of* π *so that* $\Phi_{\varepsilon}^{\kappa}(\delta_n,\varphi) \neq 0$.

(2) *Suppose that if* $n = 1$, *then* $\pi \neq 1$. *If* $\pi \simeq \pi^{\varepsilon}$ *and* $I(\Pi)'$ *is irreducible, then the following hold.*

(a) *For* $0 < s < 1/2$, $I(s, \Pi)'$ *is irreducible and unitarizable.*

(b) *The representation* $I(1/2, \Pi)'$ *is reducible. It has a unique generic non-supercuspidal discrete series subrepresentation. It's Langlands quotient is degenerate, pre-unitary and non-tempered.*

(c) *If* $s > 1/2$ *then* $I(s, \Pi)'$ *is irreducible and never unitarizable.*

(3) *If* $n = 1$ *and* $\pi = 1$, *then* $I(\Pi)'$ *is irreducible, and the complementary series is of length 1. That is, we replace "1/2" by "1" in (a)-(c) above.*

(4) *If* $I(\Pi)'$ *is reducible, then* $I(s, \Pi)'$ *is irreducible and never unitarizable for* $s > 0$.

Proof. (1) follows from Theorem 2.9 and Proposition 6.2. Parts (2) and (4) follow from Theorem 8.1 of [**18**]. Part (3) follows from Corollary 5.5, part (2), and Theorem 8.1 of [**18**]. Part (3) also follows from [**11**]. □

DEFINITION 6.4. Let π be an irreducible unitary supercuspidal representation of $GL(n, E)$ satisfying $\pi \simeq \pi^{\varepsilon}$. Let κ be the non-trivial character of $F^{\times}/NE^{\times}$. Then we say that π is a **standard lift** from $U(n)$ if there is a matrix coefficient φ of π satisfying $\Phi_{\varepsilon}^{st}(\delta_n, \varphi) \neq 0$. We say that π is a κ-**lift** from $U(n)$ if there is a matrix coefficient φ of π so that $\Phi_{\varepsilon}^{\kappa}(\delta_n, \varphi) \neq 0$.

In this language, Theorems 2.7, 2.8, 6.3(1), and Proposition 6.2 can be restated as follows.

Theorem 6.5. *Let* $G = U(n, n)$ *and* $M = GL(n, E)$. *Let* $G' = U(n, n+1)$ *and* $M' = GL(n, E) \times U(1)$. *Let* $\pi \in {}^{\circ}\mathcal{E}(M)$ *with* $\pi \simeq \pi^{\varepsilon}$. *If* $n = 1$, *then we suppose that* $\pi \neq 1$. *Suppose* n *is odd. Then the following are equivalent:*

(1) π *is a standard lift from* $U(n)$,

(2) $L(s, \pi, \Psi_n)$ *has a pole at* $s = 0$.

(3) $\pi \otimes \chi_{\mu}$ *is not a standard lift from* $U(n)$.

(4) $I(\pi)$ *is irreducible and* $I(\pi, \nu)'$ *is reducible for any character* ν *of* $U(1)$.

Suppose that n *is even. Then (1)-(4) are valid if we replace "standard lift" by "*κ*-lift" in (1) and (3).* □

§7 Reducibility from discrete series.

We finally compute the reducibility of $I(\sigma)$ for discrete series representations of M. We first consider the group $G = U(n,n)$ and $M = GL(n,E)$. Suppose that $\sigma \in \mathcal{E}_2(M)$. If $\sigma \not\simeq \sigma^\varepsilon$, then $I(\sigma)$ is irreducible [**21**]. So we assume that $\sigma \simeq \sigma^\varepsilon$. Let a and b be integers with $ab = n$, such that σ is the unique discrete series constituent of the representation induced from $\pi_1 \otimes \cdots \otimes \pi_b$. Then $\sigma \simeq \sigma^\varepsilon$ implies $\pi_0 \simeq \pi_0^\varepsilon$.

By [**21**], Corollary 5.4.2.3, we have to check whether $\mu(s,\sigma)$ has a zero at $s = 0$. Here $\mu(s,\sigma)$ is the Plancherel measure attached to σ and s. By Corollary 3.6 of [**18**], we need to check whether $L(1+s,\sigma,\Psi_n)L(s,\sigma,\Psi_n)^{-1}$ has a zero at $s = 0$. Suppose that b is even. Then Theorem 5.6 and equations (5.6) and (5.7) imply $L(1,\sigma,\Psi_n)L(0,\sigma,\Psi_n)^{-1} =$

$$\prod_{i=1}^{b/2} \frac{L(b+2-2i,\pi_0,\Psi_a)L(b+1-2i,\pi_0\otimes\chi_\mu,\Psi_a)}{L(b+1-2i,\pi_0,\Psi_a)L(b-2i,\pi_0\otimes\chi_\mu,\Psi_a)}. \tag{7.1}$$

Since both $L(s,\pi_0,\Psi_a)$ and $L(s,\pi_0\otimes\chi_\mu,\Psi_a)$ are holomorphic for $Re\ s > 1$, we see that (7.1) is zero if and only if $L(s,\pi_0\otimes\chi_\mu,\Psi_a)$ has a pole at $s = 0$. By Theorem 6.5 this holds if and only if $\pi_0 \otimes \chi_\mu$ is a standard lift from $U(a)$ if a is odd, and a κ-lift from $U(a)$ if a is even. Now suppose b is odd. Then using Theorem 5.6, and making a computation similar to (7.1) we find that $\mu(s,\sigma)$ has a zero at $s = 0$ if and only if $L(s,\pi_0,\Psi_a)$ has a pole at $s = 0$. Therefore $I(\sigma)$ is irreducible if and only if π_0 is a standard lift from $U(a)$ if a is odd, and a κ-lift from $U(a)$, otherwise.

Now suppose that $G' = U(n,n+1)$ and $M' = GL(n,E) \times U(1)$. Let $\sigma_1 \in \mathcal{E}_2(M')$, and suppose that $\sigma_1 \simeq (\sigma,\nu) = (\sigma \otimes \widetilde{\nu} \circ \det) \otimes \nu$. If $\sigma \not\simeq \sigma^\varepsilon$ then $I(\sigma_1)'$ is irreducible. Let $\sigma \in \mathcal{E}_2(GL(n,E))$ be the discrete series constituent of an induced representation as above. We again wish to determine whether $\mu(0,\sigma) = 0$. By Corollary 3.6 of [**18**] we need to examine the behavior of

$$L(1+s,\sigma_1,r_1)L(1+2s,\sigma_1\otimes\chi_\mu,\Psi_n)L(s,\sigma_1,r_1)^{-1}L(2s,\sigma\otimes\chi_\mu,\Psi_n)^{-1} \tag{7.2}$$

at $s = 0$. Here r_1, is as in Section 6. If $a > 1$, then $L(s,\sigma_1,r_1) \equiv 1$. In this case, our computations for $U(n,n)$ above determine the zeros of (7.2). If b is even then (7.2) is zero if and only if π_0 is a standard lift from $U(a)$ if a is odd, and a κ-lift from $U(a)$ if a is even. If b is odd, then (7.2) is zero if and only if $\pi_0 \otimes \chi_\mu$ is a standard lift if a is odd, and a κ-lift if a is even.

Now suppose $a = 1$. If $n = 1$, then $\pi_0 = \chi$ is a character of $E^\times$. Let $H = U(1,1)$ and B be it's Borel subgroup. Then $\mathrm{Ind}_B^H(\chi)$ is reducible if and only if $\chi_{|_{F^\times}} = \omega_{E/F}$. Therefore $L(s,1,\Psi_1)$ has a pole at s=0, and $L(s,\chi_\mu,\Psi_1)$ does not. By [**11**] $\mathrm{Ind}_P^G(\chi,\nu)$ is reducible if and only if $\chi \neq 1$ and $\chi_{|_{F^\times}} = 1$. Therefore, if $\chi = 1$, then $L(s,(\chi,\nu),r_1)$ must have a pole at $s = 0$. Notice that this is consistent with $L(s,(\chi,\nu),r_1) = L(s,\chi)$, which has a pole at $s = 0$ if and only if $\chi = 1$ [**22**]. If $n > 1$, then Proposition 7.11 of [**5**] shows that $L(s,\sigma_1,r_1) = L(s,\pi_1)$, which cannot have a pole at $s = 0$.

Theorem 7.1. *Let $G = U(n,n)$, $M = GL(n,E)$, $G' = U(n,n+1)$, and $M' = GL(n,E) \times U(1)$. Let σ, σ_1, a, b, ν, and π_0 be as above. Suppose that $\sigma \simeq \sigma^\varepsilon$. We first assume that if $n = 1$, then $\sigma \neq 1$. If both a and b are odd, then $I(\sigma)$ is irreducible if and only if π_0 is a standard lift from $U(a)$. If a is even and b is odd then $I(\sigma)$ is irreducible if and only if π_0 is a κ-lift from $U(a)$. If a is odd and b is even, then $I(\sigma)$ is irreducible if and only if $\pi_0 \otimes \chi_\mu$ is a standard lift from $U(a)$. If a and b are both even then $I(\sigma)$ is irreducible if and only if $\pi_0 \otimes \chi_\mu$ is a $\kappa-lift$ from $U(a)$. Furthermore, $I(\sigma)$ is irreducible if and only if $I(\sigma_1)'$ is reducible. Finally, if $n = 1$ and $\sigma = 1$, then both $I(\sigma)$ and $I(\sigma_1)'$ are irreducible.*

Proof. Only the next to last statement remains to be proved. But this follows immediately from the above computations and Theorem 6.5. □

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Department of Mathematics
Purdue University
West Lafayette, IN 47907