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Applications of the Evaluation Map and Transfer Map Theorems

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§ 1. Introduction

Let $M \xrightarrow{i} E \xrightarrow{p} B$ be a fibre bundle furnished with a fibre preserving map $f: E \to E$ which covers the identity. Let $f|M = g$. The first goal of this paper is to establish a *Lefschetz number transfer* for the bundle. By this we mean a homomorphism $\tau_f: H^*(E) \to H^*(B)$ such that $\tau_f \circ p^* = \Lambda_g$, where Λ_g denotes the Lefschetz number of g and its appearance in the equation is to be interpreted as the homomorphism $H^*(B) \to H^*(B)$ given by multiplying by Λ_g. We can show such a transfer exists when M is a compact manifold, possibly with boundary.

This Lefschetz number transfer generalizes the "Euler-Poincaré transfer" of [13]. Its existence heavily depends on the arguments found in [13]. We prove formulas involving Λ_g and the fibre inclusion $i: M \to E$ and the evaluation map $\omega: \Omega B \to M$. For ω we have $\Lambda_g \omega^* = 0$, where Λ_g again represents multiplication by Λ_g and $\omega^*: \tilde{H}^*(M) \to \tilde{H}^*(\Omega B)$.

We will give some diverse applications of these formulas.

We show that the orbit map of an action of G on M induces the trivial homomorphism on integral homology and to some extent on homotopy groups when M satisfies certain conditions [for example $\chi(M) \neq 0$].

We study the question: when does a closed manifold M admit a submersion $s: M \to B$ onto a closed manifold B? We find that RP^n and CP^n admits a submersion if and only if n is odd.

We shall show that $G_1(BO_{2n})$, the first evaluation subgroup, is trivial.

Finally, we show that various spaces of equivariant maps to free S^1-actions must split as a cartesian product of simpler factors.

We shall always let M denote a compact topological manifold with or without boundary ∂M. Given a space X of homeomorphism S of M, we shall always denote the action by $\hat{\omega}: X \times M \to M$. Given a base point $*$, we shall write $\omega(x) = \hat{\omega}(x, *)$. We always assume that the topology chosen on X is such that $\hat{\omega}$ is continuous, we shall call $\hat{\omega}$ *the* action of X *on* M. We shall call ω *the evaluation map* at $* \in M$.

We shall always let $M \xrightarrow{i} E \xrightarrow{p} B$ denote a fibre bundle over a CW complex B. By fibre bundle, we mean there is a covering of B by open sets

$\{U\}$ such that $p^{-1}(U)$ is homeomorphic to $U \times M$. We shall let $\dot{E} \subset E$ be the subspace consisting of all points of E contained in the boundary of some fibre.

By $H_*(X)$ we mean singular homology with any coefficients, unless it is clear from the context we are using specific coefficients.

§ 2. The Lefschetz Number Transfer

Let $M \xrightarrow{i} E \xrightarrow{p} B$ be a fibre bundle with fibre a compact manifold M with or without boundary. Let $f: E \to E$ be a fiber preserving map covering the identity and let $g: M \to M$ denote the induced map. Let L be a subcomplex of B.

Theorem 1. a) *There exists a transfer homomorphism* $\tau_f: H^*(E, p^{-1}(L)) \to H^*(B, L)$ *such that* $\tau_f \circ p^* = \Lambda_g$ *for all coefficients.*

b) *There exists a transfer homomorphism* $\tau_f: H_*(B, L) \to H_*(E, p^{-1}(L))$ *such that* $p_* \circ \tau_f = \Lambda_g$.

The proof of Theorem 1 follows closely the proofs of Theorems B and C of [13]. The first step is to prove the Lefschetz number version of Theorem D of [13]. Let f and g be as above.

Theorem 2. *Let M be oriented and let $\pi_1(B)$ operate trivially on $H^n(M^n, \partial M; Z) \cong Z$. Then there exists an element $\Lambda \in H^n(E, \dot{E}; Z)$ such that $i^*(\Lambda) = \Lambda_g \mu \in H^n(M^n, \partial M; Z)$.*

Here μ is a generator of $H^n(M^n, \partial M; Z) \cong Z$.

We shall outline the proofs of Theorems 1 and 2 below. First we need the following lemma which is a special case of Theorem 2.

Lemma. *Let $M \xrightarrow{i} E \xrightarrow{p} B$ be a fibre bundle with M a closed connected oriented manifold. Suppose $\pi_1(B)$ operates trivially on $H^n(M^n; Z)$. Then there exists a $\Lambda \in H^n(E; Z)$ such that $i^*(\Lambda) = \Lambda_g \mu$.*

Proof. Consider the pullback D of p originating from the diagram

$$\begin{array}{ccc} D & \longrightarrow & E \\ \downarrow & & \downarrow p \\ E & \xrightarrow{p} & B \end{array}$$

There is a canonical cross-section $\Delta: E \to D$, which we shall call the *diagonal.* This gives rise to a fibred pair $(D, D - \Delta(E)) \to E$ with fibre $(M, M - *)$. Thus there is a natural "Thom isomorphism" $\phi: H^i(E; Z) \cong H^{i+n}((D, D - \Delta(D)); Z)$ for all i. Define $U = \phi(1) \in H^n(D, D - \Delta(D))$. Let $j: D \to (D, D - \Delta(D))$. We define $\Lambda = \phi^{-1}(U \cup f^* j^*(U)) \in H^n(E; Z)$.

We make the same constructions for the trivial bundle $M \times M \to M$. We denote the similar cohomology classes with a bar. Thus

$\bar{\Lambda} = \bar{\phi}^{-1}(\bar{U} \cup g^* \bar{j}^*(\bar{U})) \in H^n(M; Z)$. By the naturality of ϕ we see that $i^*(\Lambda) = \bar{\Lambda}$.

Now we must show that $\bar{\Lambda} = \Lambda_g \mu$. This follows by a computation similar to that of Brown ([6], Theorem 2.2). Also see Spanier ([16], p. 348).

Proof of Theorem 2. Let $M \xrightarrow{i} E \xrightarrow{p} B$ be the fibre bundle. Let $D(M)$ be the double of M. The "double" $D(E)$ of E can also be defined and this leads to a fibre bundle

$$D(M) \xrightarrow{\bar{i}} D(E) \xrightarrow{\bar{p}} B$$

which retracts onto the fibre bundle $M \to E \to B$.

In fact, we have the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\alpha'} & D(M) & \xrightarrow{\varrho'} & M \\ \downarrow i & & \downarrow i & & \downarrow \\ E & \xrightarrow{\alpha} & D(E) & \xrightarrow{\varrho} & E \\ \downarrow & & \downarrow p & & \downarrow p \\ B & \xrightarrow{1} & B & \xrightarrow{1} & B \end{array} \qquad (*)$$

where α and α' are the canonical inclusions and ϱ and ϱ' are the canonical retractions.

Now we define a map $\bar{f} = \alpha f \varrho: D(E) \to D(E)$.

Now $\bar{f}$ clearly induces the identity on B. Also $\bar{f}$ restricts on a fibre $D(M)$, to $\bar{g} = \alpha' g \varrho'$. We have $H_*(D(M)) \cong H_*(M) \oplus H_*(D(M), M)$ and $\bar{g}_*$ is the homomorphism

$$g_* \oplus 0: H_*(M) \oplus H_*(D(M), M) \to H_*(M) \oplus H_*(D(M), M).$$

Thus $\Lambda_{\bar{g}} = \Lambda_g$.

The lemma now tells us that there is an element $\bar{\Lambda} \in H^n(D(E))$ such that $\bar{i}^*(\bar{\Lambda}) = \Lambda_g \bar{\mu}$ where $\bar{\mu} \in H^n(D(M))$ is the generator. Now

$$\bar{i}^* = i^* \oplus \bar{i}^*: H^*(E) \oplus H^*(D(E), E) \to H^*(M) \oplus H^*(D(M), M)$$

and $\bar{\mu} = (0, \mu')$, for μ' a generator of $H^n(D(M), M)$. So there is an element Λ' such that $\bar{i}^*(\Lambda') = \Lambda_g \mu'$. Now the diagram

$$\begin{array}{ccc} (D(M), M) & \longleftarrow & (M, \partial M) \\ \downarrow \bar{i} & & \downarrow i \\ (D(E), E) & \longleftarrow & (E, \dot{E}) \end{array}$$

where the horizontal inclusions are excisions establishes the lemma.

Proof of Theorem 1. This is strictly analogous to the proof of Theorem B of [13], given Theorem D of [13]. The transfer $\tau: H^*(E) \to H^*(B)$ is given by $\tau(\alpha) = p_!(\alpha \cup \Lambda)$ for $M \xrightarrow{i} E \xrightarrow{p} B$ as in Theorem 1. Then we extend the theorem step by step, first to the case of an unoriented fibre, then to the case of an oriented fibre and unoriented fibre bundle, and then to the case of disconnected fibres. In each case the trick is the same as in [13], with the added complication that an appropriate map f on the total space must be constructed. In each case, there is no difficulty in constructing f.

The proof of the relative case and the homology case follow exactly as in [13] with modifications similar to those above.

Remarks. 1) Theorem 2 is also true for Z_2 coefficients and no orientability conditions on M of the fibre bundle. The proof is the same.

2) In the case when $f = 1_E$, the identity, then $g = 1_M$ and $\Lambda_g = \chi(M)$. Thus Theorems 1 and 2 imply Theorems B–D of [13], and the Euler-Poincaré transfer is a special case of the Lefschetz transfer.

3) If G is a topological group acting as a group on M, and if $g: M \to M$ is G-equivariant, then every fibre bundle with fibre M and group G will admit a map f on the total space which restricts to g on M and induces the identity on B. Thus Theorems 1 and 2 apply. In this case we have the following formula: $\Lambda_g \omega^* = 0$, where $\omega^*: \tilde{H}^*(M) \to \tilde{H}^*(G)$. If $g = 1_M$, we have $\chi(M)\omega^* = 0$. This is Theorem A of [13]. The proof that $\Lambda_g \omega^* = 0$ follows from Theorem 2 exactly in the same manner as Theorem A follows from Theorem D in [13].

4) If there exists an $f: E \to E$ as above such that g_* is trivial on $H_*(M)$, then $\Lambda_g = 1$ and we have a "homology cross-section" $\tau_f: H_*(B) \to H_*(E)$. Compare this to the fact: If f is a constant map on every fibre, then there is a cross-section.

5) Theorem 1 can be proved without using the full force of Theorem 2. In fact we need only the lemma. Then we prove Theorem 1 in the case of "oriented fibre bundles" and connected oriented M without boundary. Then we extend to the case with boundary by using diagram (*) and defining the transfer $\tau = \bar{\tau}\varrho^*$ where $\bar{\tau}$ is the transfer defined for the middle fibre bundle. We may prove an analogue of Theorem 1 under restrictive conditions, based on Pontryagin numbers. Suppose M is a smooth, oriented, connected, closed, manifold and G is the structural group of a fibre bundle $M \xrightarrow{i} E \xrightarrow{p} B$ acting smoothly on M and preserving orientation of M.

Theorem 3. *Let N be a Pontryagin number of M. Then $N\mu \in H^n(M; Z)$ is in the image of $i^*: H^n(E; Z) \to H^n(M; Z)$.*

Proof. Let $T(M)$ be the total space of the tangent bundle of M. Then G acts on $T(M)$ by letting $g \in G$ act as the differential dg. Let $G \to \bar{E} \to B$ be the principal bundle associated with $M \xrightarrow{i} E \xrightarrow{p} B$. It is also associated with $T(M) \to \bar{E} \times_G T(M) \to B$. Now $\bar{E} \times_G T(M) \to E$ is an n-vector bundle, called the bundle of tangents along the fibre and we have a map of n-vector bundles

$$\begin{array}{ccc} T(M) & \longrightarrow & \bar{E} \times_G T(M) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ i\ } & E \end{array}$$

Thus the Pontryagin classes $T(M) \to M$ are in the image of i^*. This fact and approach may be found in [3], p. 480.

This leads to the following.

Theorem 4. (a) $N\omega^* = 0: \tilde{H}^*(M) \to \tilde{H}^*(G)$. (b) *There exists a homomorphism $\tau: H^*(E) \to H^*(B)$ such that $\tau \circ p^* =$ multiplication by N.*

§ 3. Transformation Groups

In this section G will be a compact transformation group acting as a group on a space X. The evaluation map $\omega: G \to X$ is the orbit map evaluated at the base point. The induced homomorphism ω_* is clearly useful in considering the following basic question: Given a subspace A of X, when is A an orbit of an action of G on X? We shall first obtain some results about ω_* for integral homology and for homotopy.

3.1.

Theorem 5. *Let G be a Lie group acting on a compact connected CW complex X with nonzero Euler Poincare number. Then ω induces the trivial homomorphism on homology with integer coefficients.*

Proof. By a well known fact, a torus acting on X has a fixed point. Let T be the maximal torus of G and let $x \in M$ be a fixed point of the action of T on X. Then the isotropy subgroup of G_x must contain T. Thus $\omega_x: G \xrightarrow{\varrho} G/T \to G/G_x \subset X$ is the orbit map. Now by a theorem of Bott and Borel [4], $H^*(G/T; Z)$ has no torsion. By a theorem of Hopf and Samelson [14], $\chi(G/T) \neq 0$. Now, with rational coefficients, Theorem A of [13] states that $\chi(G/T)\varrho^*: \tilde{H}^*(G/T; Q) \to \tilde{H}^*(G; Q)$ is zero, hence ϱ_* is trivial for rational coefficients. Since $H_*(G/T; Z)$ has no torsion, ϱ_* is zero on integral homology. Hence ω_* is zero.

Remarks. 1) It is reasonable to conjecture that $\omega_* = 0$ on integral homology for a much wider class of groups than Lie groups. It can be shown that $\omega_* = 0$ on rational homology for G the space of homotopy equivalences of X, [11]. This tends to support the conjecture as does Theorem 4.1 of [9].

2) Integral homology is essential. For example, $\omega_*: H_*(SO(n); Z_2) \to H_*(S^{n-1}; Z_2)$ is not trivial, [18].

3) Under the appropriate conditions, the same theorem holds for Pontryagin numbers. To see this, first note that a torus acting differentiably on a smooth closed manifold with a nonzero Pontryagin number must

have a fixed point ([7], Corollary 43.8). Then the same argument as in Theorem 5 establishes the remark.

4) A similar theorem holds for the Lefschetz number.

In fact we have

Proposition. *Suppose that a compact Lie group G acts on a compact manifold M and suppose $g: M \to M$ is an equivariant map. If $\Lambda_g \neq 0$, then $\omega_*: H_*(G; Z) \to H_*(M; Z)$ is the zero homomorphism.*

Proof. As before, we must show that for some x, the isotropy subgroup contains a maximal torus. Thus we must show that any torus in G must leave a point fixed.

If $H = S^1$ has no fixed point, then $(H^*(M/H); Q) \cong H^*(E_H \times_H M; Q)$, [2] or [5, p. 374]. Here $S^1 \to E_H \to K(Z, 2)$ is the universal fibration. The fact that g is equivariant gives rise to a fibre preserving map $f: E_H \times_H M \to E_H \times_H M$ as in Theorem 1. Now $H^*(M/H; Q)$ is finite dimensional. The transfer makes $H^*(E_H \times_H M; Q)$ infinite dimensional. Thus S^1 must have a fixed point. It can easily be seen, now, by standard techniques that the maximal torus must have a fixed point. Thus the Proposition is proved.

3.2. In this section we investigate the induced homomorphism of the orbit map on homotopy groups. To do this we need the main result of [1]. Let $M \xrightarrow{i} E \xrightarrow{p} B$ be a fibre bundle with M a smooth manifold and B a finite complex and with structural group G where G is a Lie group acting smoothly on M. Then we have the following Proposition.

Proposition [1] *There is an S-map $\tau: B \to E$ which induces the Euler-Poincaré transfer on reduced singular homology and cohomology.*

Remark. In fact, what is shown in [1] is that there exists an S-map $\tau: B^+ \to E^+$ (where X^+ denotes the disjoint union of X and a base point) which induces the Euler-Poincaré transfer. However, it is an easy matter to see that this implies the above proposition.

The advantage of realizing transfers by S-maps is great. We know, for example, that τ induces transfers in generalized homology theories. The next result follows from the realizability by S-maps in a different way.

Theorem 6. *Let G be a Lie group acting smoothly on a smooth compact manifold M. Suppose M is k-connected. Then $\chi(M)\omega_* = 0: \pi_i(G) \to \pi_i(M)$ for $i \leqq 2k$.*

Lemma. *Let $M \to E \xrightarrow{p} S^r$ be a fibre bundle with group G acting smoothly on M. Let M be k-connected. Then there is an $\alpha \in \pi_r(E)$ such that $p_*(\alpha) = \chi(M)\mu$ where $\mu \in \pi_r(S^r)$ is a generator and $r \leqq 2k+1$.*

Proof of Lemma: If $r > k+1$, the homotopy exact sequence implies that E must be k-connected. The proposition [1] states that there is an S-map $\tau: S^r \to E$. Thus we have $\tau: \Sigma^l S^r \to \Sigma^l E$ representing an element

$\bar{\alpha} \in \pi_{r+l}(\Sigma^l E)$. Since $r \leqq 2k+1$, the suspension homomorphism $S^l: \pi_n(E) \to \pi_{n+l}(\Sigma^l E)$ is surjective, therefore there exists an $\alpha \in \pi_r(E)$ such that $S^l(\alpha) = \bar{\alpha}$. This α has the property that $p_*(\alpha) = \chi(M)\mu$. To see this, first note that $\Sigma^l S^n \xrightarrow{\tau} \Sigma^l E \xrightarrow{\Sigma^l p} \Sigma^l S^n$ induces multiplication by $\chi(M)$ on integral homology. Thus $(\Sigma^l p)_*(\bar{\alpha}) = \chi(M)\Sigma^l \mu$ as can be seen by using the Hurewicz homomorphism. Now $p_*(\alpha) = r\mu$, hence $(\Sigma^l p)_*(S^l(a)) = r(S^l \mu)$, so $r = \chi(M)$.

If $r \leqq k+1$, the lemma is obvious.

Proof of Theorem. Consider the universal fibre bundle $M \to B_{G_0} \to B_G$. Let $\alpha \in \pi_i(B_G)$. Then α gives rise to a diagram

$$\begin{array}{ccc} M & \longrightarrow & M \\ \downarrow & & \downarrow \\ E & \longrightarrow & B_{G_0} \\ \downarrow & & \downarrow \\ S^r & \xrightarrow{\alpha} & B_G \end{array}$$

which implies, with the aid of the lemma, that $0 = \chi(M)d: \pi_i(B_G) \to \pi_{i-1}(M)$. But d can be regarded as the composition $d: \pi_i(B_G) \xrightarrow{\cong} \pi_{i-1}(G) \xrightarrow{\omega_*} \pi_{i-1}(M)$. Hence $\chi(M)\omega_* = 0$ for $i \leqq 2k$.

Corollary. *If $M \to E \to B$ is a fibre bundle with structural group G as above acting smoothly on M, then $i_*: \pi_i(M) \to \pi_i(E)$ has a kernel consisting of elements of order $\chi(M)$ for $i \leqq 2k$.*

Remark. We would like to thank Stephen Weingram for pointing out the lemma to us.

Corollary. *Suppose M is a closed differentiable manifold and G is a compact Lie group acting on M such that for some point $x \in M$, the isotropy subgroup G_x is trivial. Then if $\chi(M) \neq 0$:*

a) *Every cycle in G is homologous to zero in M.*

b) *Every map $S^r \to G$ is homotopic to a constant in M for $r \leqq 2k$, where k is the connectivity of M.*

Proof. Theorems 5 and 6.

3.3. Another example of the use of results about orbit maps in transformation groups is the answer to the following question. There is a canonical embedding of $S^2 = CP^1$ into CP^{2n}. Now $SO(3)$ acts on S^2 in the usual way. Is there an action of $SO(3)$ on CP^{2n} which restricts to the action of $SO(3)$ on CP^1? The answer is *no*.

Proof. Assume such an action exists. Then we obtain the commutative diagram

$$\begin{array}{ccc} SO(3) & & \\ \downarrow{\scriptstyle \omega} & \searrow^{\omega} & \\ S^2 & \xrightarrow{i} & CP^{2n} \end{array}$$

Now $i^*: H^2(CP^{2n}; Z_2) \to H^2(S^2; Z_2)$ is an isomorphism, $\omega^*: H^2(S^2; Z_2) \to H^2(SO(3); Z_2)$ is nonzero [18], and by Theorem A, $0 = \chi(CP^{2n})\omega^* = (2n+1)\omega^* = \omega^*: H^2(CP^{2n}; Z_2) \to H^2(SO(3); Z_2)$. This is a contradiction.

§ 4. Fibre Bundles

We shall use our techniques to investigate certain questions arising in the theory of fibre bundles. First we shall show how transfer theorems may be used to decide if projective spaces admit submersions onto other manifolds. Then we shall show how the evaluation map can be used to compute the homology of certain total spaces. Next we shall show how the Lefschetz transfer implies that $G_1(BO_n)$ is trivial if and only if n is even. We shall conclude with some results on spaces of equivariant maps.

4.1. A smooth map $f: M \to N$ between smooth manifolds is a *submersion* if it is onto and if its Jacobian has maximal rank at every point of M. If M is compact, then it is well known that $f: M \to N$ is a fibre bundle with fibre a closed submanifold, [8]. The question we ask here is; are there any manifolds which admit no submersion aside from the trivial ones (i.e. $M \to$ point and identity: $M \to M$)?

Let KP^n stand for either real projective space RP^n, complex projective space CP^n or quaternionic projective space QP^n.

Theorem 7. *If KP^n admits a nontrivial submersion then n is odd.*

We need the following well known consequence of the Serre spectral sequence.

Lemma. *Suppose we have a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ such that F, E, and B are homotopy equivalent to finite complexes. Then $\chi(E) = \chi(B) \cdot \chi(F)$.*

Proof of Theorem 7. Consider RP^{2n}. Suppose there exists a submersion $p: RP^{2n} \to B$. Then there is a fibre bundle

$$M \to RP^{2n} \xrightarrow{p} B$$

with M and B closed manifolds. Now $1 = \chi(RP^{2n}) = \chi(M) \cdot \chi(B)$. Hence $\chi(M) = \pm 1$. Thus M must be connected. By the transfer theorem, $p^*: H^i(B; Z) \to H^i(RP^{2n}; Z)$ is a monomorphism. Since $\tilde{H}^i(RP^{2n}; Z)$ is either 0 or Z_2, B must be an unoriented manifold. From this and the homotopy exact sequence of the fibration, we see that $\pi_1(B) \cong Z_2$. Consider the pullback of the fibre bundle over the universal cover $\tilde{B}$. We have

$$\begin{array}{ccc} M & & M \\ \downarrow & & \downarrow \\ S^{2n} & \longrightarrow & RP^{2n} \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ \tilde{B} & \xrightarrow{p} & B \end{array}$$

Now p^* is a monomorphism by the transfer theorem. Hence $\tilde{H}^*(\tilde{B}) = 0$ or $\tilde{B}$ has dimension $2n$. In the first case we have $\tilde{B}$ a point which yields a trivial submersion; in the second case M is 0-dimensional and is connected, so M is a point and we have a trivial submersion.

For the case of CP^{2n}, assume we have a submersion p which is a fibre bundle $M \xrightarrow{i} CP^{2n} \xrightarrow{p} B$. First note that M must be positive dimensional: If not, then CP^{2n} is a covering space of B and the deck transformations are fixed point free maps on CP^{2n}. Since $H^*(CP^{2n}; Z)$ is a truncated polynomial algebra of height $2n+1$ generated by $\alpha \in H^2(CP^{2n}, Z)$, we see by the Lefschetz fixed point theorem that CP^{2n} admits no fixed point free map. So if M is 0-dimensional it must be a point, which would yield a trivial submersion.

Now $2n+1 = \chi(CP^{2n}) = \chi(B) \cdot \chi(M)$. So $\chi(M)$ is odd. By the transfer theorem, $p^*: H^*(B; Z_2) \to H^*(CP^{2n}; Z_2)$ is a monomorphism. We may assume that M is connected by covering spaces arguments. Then by Theorem 2, there is a $\chi \in H^k(CP^{2n}; Z_2)$ such that $i^*(\chi) = \mu$. Here k is the dimension of M and $\mu \in H^k(M; Z_2)$ is the generator. Thus $\chi = \alpha^{(k/2)}$ and this implies that i^* is a monomorphism. Since $\pi_1(B) = 0$, the Serre exact sequence $\cdots \longrightarrow H^i(B) \xrightarrow{p^*} H^i(CP^{2n}) \xrightarrow{i^*} H^i(M) \xrightarrow{\tau} \cdots$ holds for $i < s+t$ where $H^j(B; Z_2) = 0$ for $0 < j < s$ and $H^j(M; Z_2) = 0$ for $0 < j < t$. The fact that both p^* and i^* are monomorphisms implies by an inductive argument that $H^{k+2}(B; Z_2) \neq 0$, where k is the dimension of M, and $H^i(B) = 0$ for $0 < i < k+2$. Thus there is a $\beta \in H^{k+2}(B)$ such that $p^*(\beta) = \alpha^{(k/2)+1}$. In addition i^* must be an isomorphism. Thus $\chi(M) = (k/2)+1$. The set $\{1, \beta, \beta^2, \ldots\}$ consists of $(2n+1)/(k/2+1) = \chi(CP^{2n})/\chi(M) = \chi(B)$ elements. Thus there can be no other elements in $H^*(B; Z_2)$, so $H^*(B; Z_2)$ is a truncated polynomial ring generated by an element β of dimension $2\chi(M) = 2 \cdot$ (odd number) and of height greater than 1. This is impossible since truncated polynomial rings of height greater than 1 are realized only when they are generated by elements of dimension 1, 2, 4, or 8.

A similar argument proves that QP^{2n} admits no submersion. Also the Cayley plane admits no submersion.

For the odd dimensional real and complex projective spaces there are the well known submersions $RP^{2n+1} \to CP^n$ and $CP^{2n+1} \to QP^n$. We do not know whether QP^{2n+1} admits a nontrivial submersion.

4.2. We shall demonstrate how the evaluation map may be used to compute the homology of total spaces for some fibre bundles. The connected classical groups are $SO(n)$, $U(n)$, $\mathrm{Sp}(n)$, $\mathrm{Spin}(n)$.

As an example, we shall consider fibre bundles with fibre CP^{2n}.

Theorem 8. *Let $CP^{2n} \xrightarrow{i} E \xrightarrow{p} B$ be a fibre bundle with structural group G a classical connected Lie group. Then the spectral sequence collapses and $H^*(E; Z) \cong H^*(B; Z) \otimes H^*(CP^{2n}; Z)$ as groups.*

Proof. Since G is a classical group, $H_*(G; Z)$ has only two torsion and a free part. Since $\chi(CP^{2n}) = 2n+1$, we see that $\omega^* = (2n+1)\omega^* = 0$. Now consider $\alpha \in H^2(CP^{2n}; Z)$, the generator of the cohomology ring. We shall prove that α is in the image of i^*. Hence $H^*(CP^{2n}; Z)$ is in the image of i^*. Hence CP^{2n} is totally non-homologous to zero and the result follows.

To show that α is in the image of i^*, we consider the Serre exact sequence $H^2(E; Z) \xrightarrow{i^*} H^2(CP^{2n}; Z) \xrightarrow{\tau} H^3(B; Z)$. Now Theorem 4 of [12] states that τ must factor through ω^* in the lowest non trivial dimension. But $\omega^* = 0$. Hence $\tau = 0$. Hence i^* is onto.

Remarks. 1) For a $U(n)$ bundle with CP^r as a fibre, the spectral sequence must collapse regardless of whether r is even or odd. This follows since $H^*(U(n); Z)$ has no torsion so $\omega^* = 0$ for all r, etc. This is a well known and important fact.

2) This type of an argument may be used when the cohomology of the fibre is generated by its second cohomology group. For example, if $M = CP^{2a} \times CP^{2b}$.

4.3. In this section we shall use the Lefschetz transfer to study $G_1(B(0(n)))$.

Recall that the first evaluation subgroup $G_1(X)$ is the image of $\omega_*: \pi_1(X^X; 1_X) \to \pi_1(X, *)$. In [10], Corollary 7.6, it is shown that $G_1(BO(2n+1)) = \pi_1(BO(2n+1)) \cong Z_2$. Now we shall show that $G_1(BO(2n)) \cong 1$.

Theorem 9. $G_1(BO(2n)) \cong 1$.

Proof. Consider the fibre bundle

$$S^{2n-1} \xrightarrow{i} BO(2n-1) \xrightarrow{p} BO(2n).$$

Let $\alpha \in \pi_1(BO(2n)) \cong Z_2$ be the generator. Then α corresponds to the map on the fibre $r: S^{2n-1} \to S^{2n-1}$ given by reflection about some axis. If we assume that $\alpha \in G_1(BO(2n))$, that implies that there is a "cyclic" homotopy

$$h_\tau: BO(2n) \to BO(2n)$$

such that $h_0 = h_1 =$ identity and the trace $t \to h_t(*)$ represents α. By the covering homotopy property, we have a homotopy $\tilde{h}_\tau: BO(2n-1) \to BO(2n-1)$ such that $h_0 =$ identity and h_1 maps each fibre into itself by a map homotopic to the reflection $r: S^{2n-1} \to S^{2n-1}$.

Now $\Lambda_r = 1 + (-1)^{2n-1} \deg r = 2$. Thus by the "Lefschetz number transfer theorem",

$$p^*: H^*(BO(2n); Q) \to H^*(BO(2n-1); Q)$$

is a monomorphism. But it is well known that this is not the case.

Corollary. $G_1(BG_{2n}) \cong 1$ *where* G_{2n} *is the space of self-homotopy equivalences of* S^{2n-1}.

Proof. If $G_1(BG_{2n}) \cong \pi_1(BG_{2n}) \cong Z_2$, then there would be a fibre preserving map $f: E_\infty \to E_\infty$ on the total space of the universal fibration $S^{2n-1} \to E_\infty \xrightarrow{p} BG_{2n}$ which induces the identity on the base space and restricts to a fibre S^{2n-1} as a map of degree -1. Thus every fibration with fibre S^{2n-1} must admit such an f. But by the proof of Theorem 9 $S^{2n-1} \to BO(2n-1) \xrightarrow{p} BO(2n)$ cannot admit such a map.

Corollary. $O(n)$ *is a normal subgroup of a connected topological group if and only if n is odd.*

Proof. a) Let n be odd. Consider the group $S^1 \times SO(n)$.

Then $\{1, -1\} \times SO(n)$ is a normal subgroup of $S^1 \times SO(n)$ and $S^1 \times SO(n)$ is clearly connected. Since n is odd, $\{1, -1\} \times SO(n)$ is isomorphic to $O(n)$ by the isomorphism given by $(\pm 1, A) \mapsto \pm A$. Note that if n is even, this homomorphism is not an isomorphism since $-I \in SO(n)$.

b) Assume n is even. Assume that $O(n)$ is a normal subgroup of a connected group G. Then the exact sequence $O(n) \to G \to G/O(n)$ gives rise to a fibration $BO(n) \xrightarrow{i} BG \xrightarrow{p} B(G/O(n))$. Since G is connected, we have $\pi_1(B_G) = 1$. Now we have the exact sequence

$$\pi_2(B(G/O(n))) \xrightarrow{d} \pi_1(BO(n)) \to 1 .$$
$$\wr\!\|$$
$$Z_2$$

Since d factors through ω_* we have $G_1(BO(n)) \cong Z_2$. This contradicts Theorem 9.

Remark. Note that a) proves that $G_1(BO(2n-1)) \cong Z_2$.

4.4. Suppose $S^1 \to E \to B$ is a principal bundle. Suppose G is a space of homotopy equivalences of B acting continuously on B. Let G^* denote the space of principal bundle maps (or equivalently the space of S^1-equivariant maps $E \to E$) which induce on B maps in G. What is the homotopy type of G^*?

In [12] we have an answer. If $k \in H^2(B; Z)$ is the element which classifies the principal bundle, then G^* is determined by $\hat{\omega}^*(k)$ where $\hat{\omega}: G \times B \to B$ is the action. In particular, if $H^1(B; Z) = 0$, then G^* has the homotopy type of the total space of the S^1-bundle over G classified by $\omega^*(k) \in H^2(B; Z)$.

In view of Theorem A of [13], we may make specific computations. For example:

a) Let $O(n+1)$ act on RP^n in the obvious way (or in any way for that matter). Since $\chi(RP^{2n}) = 1$, we see that $\omega^* = 0$. Thus $O(2n+1)^*$ is homotopy equivalent to $O(2n+1) \times S^1$ for all principal S^1-bundles.

b) Let $U(n)$ act on any compact manifold M such that $\pi_1(M)$ is finite. Then $U(n)^*$ is homotopy equivalent to $U(n) \times S^1$ for all principal S^1-bundles over M.

If G is a compact simply connected Lie group acting on B, then $G^* \simeq G \times S^1$. In addition, there is a cross-section to the obvious map $\Phi: G^* \to G$ which is a homomorphism of groups. This is proved by Stewart in [17], (also see [15]). In [12], we have a necessary and sufficient condition for the existence of a cross-section to $\phi: G^* \to G$; namely $\hat{\omega}^*(k)=0$. But we say nothing about whether the cross-section is a homomorphism.

Note added in Proof: R. Schultz can show that QP^{2n+1} does not admit a non-trivial submersion for $n \geqq 1$.

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