21

FIBRATIONS WITH COMPACT FIBRES.

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§1. Introduction. Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a Hurewicz Fibration. Let $f: E \to E$ be a fibre preserving map inducing the identity, 1_B, on B. Let $g: F \to F$ be the restriction to F of f. We shall let Λ_g denote the Lefschetz number of g. ($\Lambda_g = \Sigma(-1)^i$ (trace $(g^i{}_*)$) where $g^i{}_*: H_i(F;\ Q) \to H_i(F;\ Q)$). We shall let $d: \Omega B \to F$ denote the *transgression map* which arises in the Puppe sequence. In the case where the fibre F is homotopy equivalent to a compact CW complex, we obtain the following five theorems with the above notation holding throughout.

Transfer Theorem. *Let (B,A) be a CW complex pair.*

a) *There exists homorphisms $\tau^f: H^*(E, E_A) \to H^*(B,A)$ and $\tau_f: H_*(B,A) \to H_*(E, E_A)$ for any group of coefficiants, such that $p_* \circ \tau_f = \Lambda_g$ and $\tau^f \circ p^* = \Lambda_g$ (where Λ_g denotes multiplication by Λ_g)*

b) *If (B,A) is homotopy equivalent to a compact CW-pair, then there is an S-map $\hat{\tau}_f: B/A \to E/E_A$ which induces τ_f and τ^f.*

Here $\pi^{-1}(A) = E_A$. We shall call τ^f and τ_f *transfer homomorphisms* and $\hat{\tau}_f$ a *transfer map*.

Transgression Theorem. *$\Lambda_g d^*: H^i(F) \to H^i(\Omega B)$, $i > 0$, is the zero homomorphism for any coefficients.*

fibre inclusion theorem. *Let F be a Poincare space and let B be a locally compact CW complex. Let $\pi_1(B)$ act trivially on $[\bar{F}] \in H^F(F;\ Z) \cong Z$. Then there exists an element $\Lambda \in H^F(E;\ Z)$ such that $i^*(\Lambda) = \Lambda_g \cdot [\bar{F}]$.*

We denote dim X by X where no confusion results. Also $[F] \in H_F(F;\ Z)$ and $[\bar{F}] \in H^F(F;\ Z)$ denotes the fundamental classes when F is a Poincare space.

These three theorems are extensions to Hurewicz fibrations of analogous theorems for fibre bundles, [1], [2], [9]. In the case of the transfer map $\hat{\tau}_f$ the construction was made only for fibre bundles with structural group a compact

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Lie group acting smoothly on F, a compact manifold with or without boundary, and $f=1_E$.

The extension of the transfer and transgression theorems to fibrations from fibre bundles follows from the next two theorems which show that Hurewicz fibrations with compact fibres are essentially smooth fibre bundles.

OPEN FIBRE SMOOTHING THEOREM. *Let B be finite dimensional. $F \to E \to B$ is fibre homotopy equivalent to $V \to E' \to B$ where V is a smooth oriented manifold homotopy equivalent to F and $E' \to B$ is a fibre bundle with structural group* Diff (V).

CLOSED FIBRE SMOOTHING THEOREM. *Let B be finite dimensional. Let T^n be a product of n circles. For $n = \dim B$ we have $F \times T^n \xrightarrow{i \times 1} E \times T^n \xrightarrow{\pi \times (\text{Proj.})} B$ is fibre homotopy equivalent to a fibre bundle $W \times T^n \to E' \to B$ where W is a compact, oriented, smooth manifold with boundary, homotopy equivalent to F. The structural group is* Diff $(W \times T^n)$.

The open fibre smoothing theorem is used to prove the closed fibre smoothing theorem, which in turn is used to establish the transgression and transfer theorems. It was already known that every fibration is homotopy equivalent to a fibre bundle, [7], but the fibre turned out to be a function space. The advantage of the above fibre smoothing theorems is that the techniques of differential topology may be used on Hurewicz fibrations with compact fibres. That is exactly what happens in this paper.

The Transgression, Transfer, and Fibre Inclusion Formulas must give rise to numerous applications since they are such simple formulas relating well known invariants to information about fibrations, which occur in numerous branches of mathematics. We give applications in §2. Many of our applications are extensions of theorems in [2] from homeomorphisms and manifolds to homotopy equivalences and finite complexes. The most important is theorem 1 which states that $\Lambda_g d_* = 0 : \pi_{i+1}(B) \to \pi_i(F)$ for $i \leqslant$ (twice the connectivity of F).

The Transfer and Fibre Inclusion Formulas can be used to decide whether a map $X \to Y$ has a compact fibre. For example even dimensional projective spaces can never map to a compact space with a compact fibre, and maps of suspensions to compact Y do not have compact fibres if the map represents a torsion element in $[\Sigma X, Y]$ unless $\tilde{H}_*(Y; Q) \cong 0$.

For transformation groups, we obtain the following results. (1) If Z_n acts freely on a finite dimensional space X homotopy equivalent to a compact CW complex and if f is equivariant, then n divides Λ_f.

(2) If G is a compact Lie group acting on an X which is homotopy

equivalent to a compact connected CW complex, then $\omega_*:\tilde{H}_*(G;\ Z)\to\tilde{H}_*(X;\ Z)$ is trivial if $\chi(X)\neq 0$ and $\omega: G\to X$ is the orbit map for some point.

The transfer theorem is useful in studying the following question. Given a compact CW complex X, when does $\pi_*(X)$ have finite type? When $\chi(X)\neq 0$, we find some necessary conditions.

Finally we apply the transfer theorem to $K(\pi,1)$'s and thus obtain results about group theory. The transfer for fibrations gives a transfer in the homology and cohomology of groups. This in turn allows us to discover elementary facts about groups. For example; if G is a group with the free group of rank 2 as a normal subgroup and if H is the factor group, then H made abelian is a direct summand of G made abelian.

In §3 we state survey results about umkehr maps which we shall need in defining the transfer. In §4 we prove the Fibre Inclusion Theorem. The construction of the transfer is made in §5 and the transfer theorem is proved in §6 modulo the Fibre Smoothing Theorems. In §7 the Transgression Theorem is proved modulo the Fibre Smoothing Theorems. Finally in §8 the Fibre Smoothing Theorems are proved.

Remark. Another construction of the transfer map for fibrations using Spanier-Whitehead Duality will appear in a paper by J. C. Becker and D. H. Gottlieb. In this paper a proof of the transgression theorem is given using the transfer theorem, and thus the transgression theorem holds for all cohomology and homology theories.

§2. **Applications.** In this section we shall give diverse applications resulting from the transgression, transfer and fibre inclusion theorems. The extension of these theorems to Hurewicz fibrations permits many of the results of [2] to be extended to fibrations and spaces of homotopy equivalences. In addition, some completely new applications are presented involving group theory, transformation groups and fiberings of suspensions.

1). Let $F\xrightarrow{i}E\xrightarrow{p}B$ be a fibration as in the introduction.

THEOREM 1. *Let F be k-connected. Then*

$$0=\Lambda_g\cdot d:\pi_{i+1}(B)\to\pi_i(F)\quad\text{for}\quad i\leqslant 2k.$$

Proof. The proof is an obvious extension of the proof of theorem 6 of [2].

COROLLARY. a) *$\chi(F)\omega_*:\pi_i(F^F;\ 1)\to\pi_i(F)$ is trivial for $i\leqslant 2k$ for compact CW complexes F.*

b) $\Lambda_f \omega_* : \pi_i(G) \to \pi_i(F)$ *is trivial when* $i \leqslant 2k$ *for a group* G *acting on* F *such that* $f : F \to F$ *is equivarient.*

2). *The fibering question.* The fibering question is the following: Given a space E, is there a (nontrivial) fibre bundle such that E is the total space of a fibre bundle? Such questions have been studied by several people, among them A. Borel for E a Euclidean space and W. Browder [3] for E a sphere. In [2] the fibering question was studied for RP^n and CP^n and QP^n. The extension of the transfer and related theorems to fibrations will allow us to improve the results of [2] as well as add some new ones. In fact, we will consider a more general question, which we shall call the *fibration question.*

Question. Given a compact space E, can we find a Hurewicz fibration $F \xrightarrow{i} E' \xrightarrow{p} B$ with B a compact CW complex, with E' homotopy equivalent to E and F homotopy equivalent to a compact CW complex, which is nontrivial (neither i or p is a homotopy equivalence)? To put it more succinctly, can we find a map (not the constant or a homotopy equivalence) from E to a compact B with compact fibre?

In addition to the theorems in the introduction, we shall find the following facts useful. We suppose $F \to E \xrightarrow{p} B$ is a fibration with F, E, and B homotopy equivalent to compact CW complexes.

(1) $\chi(E) = \chi(B) \cdot \chi(F)$.

(2) If $f : E \to E$ covers 1_B as usual, then $\Lambda_f = \chi(B) \cdot \Lambda_g$ if F is connected.

(3) Quinn [13]. F and B are Poincare spaces if and only if E is a Poincare space.

THEOREM 2. *There are no (nontrivial) maps from* RP^{2n}, CP^{2n} QP^{2n} *and Cay* P^2 *to a compact CW complex with compact fibre. There are such maps for* RP^{2n+1} *and* CP^{2n+1}.

Proof. The proof is similar to theorem 7 of [2].

THEOREM 3. *Suppose* $\alpha : \Sigma X \to Y$ *has a compact homotopy theoretic fibre and suppose* Y *is a compact CW complex. If* $[\alpha] \in [\Sigma X, Y]$ *has finite order, then* $\tilde{H}_*(Y; Q) = 0$.

Proof. In fact we shall show that if $[\alpha]$ has order d^2 then every element in $\tilde{H}_*(Y, Z)$ has order dividing d. Let d also denote the map $\Sigma X \to \Sigma X$ corresponding to multiplication by d. Then $\alpha \circ d$ is homotopic to a constant map. Hence every element in the image of α_* has order dividing d. In addition, $\alpha \sim \alpha \circ (d+1)$. So $(d+1) : \Sigma X \to \Sigma X$ is, up to homotopy, a fibre preserving map over the identity on the base Y. We compute the Lefschetz number of $d+1$ and we have $\Lambda_{d+1}=(d+1)(\chi(\Sigma X)-1)+1=(d+1)(\chi(\Sigma X))-d$. Thus $\Lambda_{d+1}\neq 0$ since $d \neq 0$. Then $\Lambda_{d+1}=\chi(Y)\Lambda_g\neq 0$, where F is a fibre of α and g is the restriction of $d+1:\Sigma X\to\Sigma X$ to F. Thus $\Lambda_g\neq 0$. Hence there is a transfer such that $\alpha_*\circ\tau=\Lambda_g$. Hence every element of $\tilde{H}_*(Y;\, Z)$ has order dividing $d\Lambda_g\neq 0$ and so $\tilde{H}_*(Y;\, Q)=0$.

To see that every element of $\tilde{H}_*(Y;\, Z)$ has order dividing d^2 we observe that $\tilde{H}_*(Y;\, Q)=0$ implies that $\chi(Y)=1$. Thus every element $x\in\tilde{H}_*(Y;\, Z)$ satisfies $d\Lambda_{d+1}x=0$. Now $(2d+1):\Sigma X\to\Sigma X$ satisfies $\alpha\circ(2d+1)\sim\alpha$ and proceeding as above we see that $d\Lambda_{2d+1}x=0$ in the same way. Then $0=d^2x$.

Remark. It is natural to ask if Y must be contractible in the above theorem? It need not be, for let $\Sigma X=Y=$ Moore space $M(Z_n,m)$. Then $1:\Sigma X\to\Sigma X$ has finite order and has fibre a point. Also note that the Hopf fibration $S^1\to S^3\to S^2$ shows that the condition that $[\alpha]$ is torsion is necessary.

As another example of the technique used in theorem 3, we have the following corollary of the proof.

COROLLARY. *Suppose that $p:E\to B$ is homotopic to the constant map and B and the homotopy theoretic fibre F are compact, then B is contractible.*

Proof. Since $p\sim p\circ c$ where $c:E\to E$ is constant, we can find a fibre preserving map $f:E\to E$ which is homotopic to the constant map. Then arguing as above, we find that $\tilde{H}_*(B;\, Z)=0$. Now to show that B is contractible we consider the universal covering $\tilde{B}$ of B. We note that $p:E\xrightarrow{p'}\tilde{B}\to B$ where p' is homotopic to a constant map and p' has as fibre a path component of F. Thus $\tilde{H}_*(\tilde{B};\, Z)=0$ and so $\tilde{B}$ is contractible. Hence B is a $K(\pi,1)$ and π is finite since F is compact. But since B is compact, π is trivial. Hence B is contractible.

3). *Transformation groups.* By a finite dimensional space X we shall mean that X is paracompact and Hausdorff with finite covering dimension. Then $\check{H}^*(X;\, Z)$ is finite dimensional. If G is a finite group acting on a finite dimensional space X, then X/G is finite dimensional.

THEOREM 4. *Suppose Z_n acts freely on a finite dimensional space X and suppose that $g:X\to X$ is equivariant. If X is homotopy equivalent to a compact CW-complex, then n divides Λ_g.*

Proof. Let $G=Z_n$. Let $G\to E_G\to B_G$ be the universal bundle for G. The projection $E_G\times X\to X$ is a G-map, and so it induces a map $\varphi:E_G\times_G X\to X/G$.

The inverse image of any point under φ is the classifying space for the isotropy sub-group of some point of X, but since Z_n acts freely this subgroup is trivial, hence the inverse image is contractible. Using the Vietoris-Begle Mapping Theorem, we see that $\check{H}^*(E_G \times_G X) \cong \check{H}^*(X/G)$. (This standard argument may be found on page 371 of Glen E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press (1972)).

Now consider the fibre bundle

$$X \to E_G \times_G X \xrightarrow{p} B_G.$$

The equivariant map g gives rise to a map $f: E_G \times_G X \to E_G \times_G X$ which covers the identity on B_G and restricts to g on some fibre X. Applying the transfer theorem we have $\tau^f \circ p^* = \Lambda_g$ on singular cohomology. Now $E_G \times_G X$ is homotopy equivalent to a CW complex since it is the total space of a fibration where the fibre and base are homotopy equivalent to CW complex. Thus

$$H^*(E_G \times_G X) = \check{H}^*(E_G \times_G X) = \check{H}^*(X/G)$$

Now X/G is finite dimensional, so $H^i(E_G \times_G X) = 0$ for large i. Also $H^i(B_G) = Z_n$ for all odd $i > 0$. Thus for large odd i, $p^* = 0$ and so $0 = \Lambda_g = \tau^f \circ p^*: Z_n \to Z_n$. This can only occur if n divides Λ_g.

THEOREM 5. *Suppose a torus T acts as group of transformations on a finite dimensional space X homotopy equivalent to a compact CW complex. Let $f: X \to X$ be equivarient. If $\Lambda_f \neq 0$, then T must have a fixed point.*

COROLLARY. *Let G be a compact Lie group acting on a finite dimensional X which is homotopy equivalent to a compact connected CW complex. Let $\omega: G \to X$ be the evaluation at the base point of X. Then $0 = \omega_*: \tilde{H}_*(G; Z) \to \tilde{H}_*(X; Z)$.*

Proof. Essentially the same as the proofs of theorem 5 and its following proposition in [2].

4). *Finite type of $\pi_*(X)$.*

THEOREM 6. *Suppose that X is homotopy equivalent to a compact CW complex, and suppose that $\pi = \pi_1(X)$ has a compact $K(\pi, 1)$, and that $\pi_*(X)$ has finite type (finitely generated in each dimension). Then $\chi(\pi)$ divides $\chi(X)$ and there exists homomorphisms $\tau: H_*(\pi) \to H_*(X)$ and $\tau: H^*(X) \to H^*(\pi)$ such that $f_* \circ \tau = \chi(X)/\chi(\pi)$ and $\tau \circ f^* = \chi(X)/\chi(\pi)$ where $f: X \to K(\pi, 1)$ is the map given by the condition that $f_*: \pi_1(X) \to \pi$ is the identity.*

Proof. Let $\tilde{X}$ be the universal covering for X. Then we obtain a Hurewicz fibration $\tilde{X} \to X \xrightarrow{f} K(\pi, 1)$ where the fibre inclusion is the homotopy type of the covering projection. Since $\pi_i(X) \cong \pi_i(\tilde{X})$ for $i > 1$, we see that $\pi_*(\tilde{X})$ is of finite type, hence $H_*(\tilde{X})$ is of finite type since $\pi_1(\tilde{X}) = 0$. But $\tilde{X}$ is finite dimensional since X is, hence $H_*(\tilde{X})$ is finitely generated. Hence $\tilde{X}$ must be homotopy equivalent to a compact CW complex. Thus $\chi(X) = \chi(\tilde{X}) \cdot \chi(\pi)$ and the transfer theorem yields the result.

COROLLARY. *Suppose X is a compact CW complex.*

a) *Suppose $\pi_1(X)$ is a free group of rank r. If $r-1$ does not divide $\chi(X)$, then $\pi_*(X)$ is not of finite type.*

b) *Suppose $\pi_1(X)$ is a free abelian group. If $\chi(X) \neq 0$, then $\pi_*(X)$ is not of finite type.*

c) *Suppose that $\pi_1(X)$ is isomorphic to the direct sum of n fundamental groups of orientable surfaces (not S^2). If $\chi(X) \neq 0$ and $H_{2n}(X; Q) = 0$, then $\pi_*(X)$ does not have finite type.*

d) *If $\pi_*(X)$ has finite type and $\chi(X) \neq 0$ and $\pi_1(X)$ has a nontrivial center, then $K(\pi_1(X), 1)$ is not homotopic to a compact space.*

Proof of d). Let $\pi = \pi_1(X)$. If $K(\pi, 1)$ is compact, then $\chi(\pi) = 0$ if π has a nontrivial center; Corollary IV.3 [8]. But $0 \neq \chi(X) = \chi(\tilde{X})\ \chi(\pi)$.

5). *Group theory.* Group theory is imbedded in topology by a functor from the category of groups and homorphisms to the category of spaces and maps. A group π corresponds to the Eilenberg-MacLane space $K(\pi, 1)$. A homomorphism $h: G \to H$ gives rise to a map $K(h): K(G, 1) \to K(H, 1)$. The homomorphisms from G to H are in one to one correspondance the homotopy classes of maps from $K(G, 1)$ to $K(H, 1)$ which leave a base point fixed. Since $\pi_1(K(G, 1)) \cong G$ and $h = K(h)_* : \pi_1(K(G, 1)) \to \pi_1(K(H, 1))$, the Eilenberg-MacLane functor K composed with the fundamental group functor π_1 is the identity on the category of groups and homorphisms.

Thus no information is lost by considering $K(\pi, 1)$'s instead of groups. Of course it is certain that the topological viewpoint cannot recover the elementary facts of group theory very easily, if at all. On the other hand, topology is rich in constructions and concepts which may be applied to $K(\pi, 1)$'s, hopefully to yield theorems of a purely group theoretical nature which are not obvious, or even obtainable, by purely group theoretic means. The transfer theorem affords us a means to discover such group theoretic results.

Consider the topological concept of compactness, we may consider the class of groups G whose $K(G, 1)$'s are homotopy equivalent to a compact CW

complex. We shall call any such group a *compact* group (hopefully not to be confused with a compact topological group or Lie group). The class of compact groups is a very unnatural class from the point of group theory, yet it contains a large number of important groups. For example: the trivial group; the integers, Z; any free group of finite rank r, F_r; any free abelian group of rank r, T_r; knot groups; link groups; surface groups. If a group has an element of torsion, it is not a compact group. Compact groups must be finitely presented.

Now suppose we have a homorphism $h:G \to H$. This gives us a map $K(h):K(G,1) \to K(H,1)$. We may replace $K(h)$ by a homotopy equivalent fibration

$$F \to K(G,1) \xrightarrow{K(h)} K(H,1).$$

This is an example of a construction in topology which makes no sense in group theory. What is F? It is easy to see that F is a disjoint union of spaces all of which have the same homotopy type of K (Ker $(h), 1)$. The connected components of F are in one to one correspondence with coker (h).

Now F is homotopy equivalent to a compact CW complex if and only if Ker (h) is a compact group and coker (h) is a finite set.

THEOREM 7. *Given a homomorphism $h:G \to H$ such that Ker (h) is a compact group and* coker (h) *is a finite set, and given an endomorphism $k:G \to G$ such that $hk = h$, there exists a homomorphism $\tau : H_*(H) \to H_*(G)$ such that $h_* \circ \tau = |\text{coker } (h)| \cdot \Lambda_g$. Similarly for cohomology.*

Proof. Here $|\text{coker } (h)|$ is the number of elements in coker (h), and g is the restriction of h to Ker (h). Since $H_*(G) \cong H_*(K(G,1))$, this is the transfer theorem.

Let G' denote the commutator subgroup of G. Since $H_1(G) \cong {}^{G}/_{G'}$ and $\chi(F_r) = 1 - r$, we obtain the following.

COROLLARY. *Let F_2 be a normal subgroup of G. Let $H = {}^{G}/_{F_2}$. Then ${}^{H}/_{H'}$ is a direct summand of ${}^{G}/_{G'}$.*

Remark. This fact is false for any free group of rank different than 2. Consider $F_r \to F_2 \xrightarrow{\rho} Z_{r-1}$ where ρ maps one generator of F_2 to the generator of Z_{r-1} and the other generator to the identity. The kernel of ρ is F_r by Schrier's theorem.

Let Z be a normal subgroup of G and let $H = {}^{G}/_{Z}$. Suppose that $k:G \to G$ is an endomorphism such that $hk = h$ where $h:G \to H$ is the quotient. Then $k(Z) \subset Z$. Let $d = h(1)$ for $1 \in Z$. (Here Z is the group of the integers).

COROLLARY. *There is a homomorphism* $\tau : {}^{H}/_{H'} \to {}^{G}/_{G'}$ *such that* ${}^{H}/_{H'} \xrightarrow{\tau} {}^{G}/_{G'} \xrightarrow{h_*} {}^{H}/_{H'}$ *is multiplication by* $(1-d)$. *If* $d=0$ *or* 2 *then* ${}^{G}/_{G'} \cong ({}^{H}/_{H'}) \oplus ?$.

Proof. Use the transfer theorem. Note $\Lambda_g = 1-d$. If $\Lambda_g = \pm 1$, then τ is injective, hence ${}^{G}/_{G'}$ splits.

§3. The Umkehr Map. In this section we record some facts about Umkehr maps. Our reference will be Dold, [6] chapter VIII, especially section §10 entitled "Transfers". Note: Dold calls his Umkehr maps "transfers."

Let M be a closed smooth oriented manifold. Consider $A \subset B \subset M$ where A and B are compact subspaces which have neighborhood deformation retracts. Let A^* and B^* be compact deformation retracts of M-A and M-B respectively, so $A^* \supset B^*$. Then we have the Poincare duality isomorphism $D_M : H^i(B,A) \xrightarrow{\cong} H_{M-i}(A^*,B^*)$.

Let $f : M' \to M$ be a map between closed oriented manifolds. Let $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$. We define the Umkehr homorphisms $f_!$ and $f^!$ by

(1) $$f^! : H^i(B',A') \underset{\cong}{\xrightarrow{D_{M'}}} H_{M'-i}(A'^*,B'^*) \xrightarrow{f_*} H_{M'-i}(A^*,B^*) \underset{\cong}{\xrightarrow{D_M^{-1}}} H^{(M-M')+i}(B,A)$$

(2) $$f_! : H_i(B,A) \underset{\cong}{\xrightarrow{D_M^{-1}}} H^{M-i}(A^*,B^*) \xrightarrow{f^*} H^{M-i}(A^{*\prime},B^{*\prime}) \underset{\cong}{\xrightarrow{D_M}} H_{(M-M')+i}(B',A')$$

Now $f_!$ and $f^!$ are independent of the choice of (A^*,B^*). The Umkehr maps satisfy the following formulas,

(3) $(id)^! = id$, $(id)_! = id$
$(fg)^! = f^! g^!$; $(fg)_! = g_! f_!$
(4) $f_!(x \cap \xi) = (f^*(x)) \cap f_!(\xi)$
(5) $f^!(f^*(x) \cup y) = x \cup f^!(y), x \in H^*(B,A), y \in H^*(f^{-1}B, f^{-1}A)$.
(6) $f_*(y \cap f_!(\xi)) = (-1)^d (f^!(y) \cap \xi)$ where $d = (M - |\xi|)(M - M')$.

Here $x \in H^*(B,A)$, $\xi \in H_*(M, M-B)$ for (4) and (6). (We may form a cap product $\cap : H^*(B,A) \times H_*(M,M-B) \to H_*(A^*,B^*)$ by using the usual cap product and inclusion and excisions, [6], p. 239; 12.6. This cap product gives rise to D_M.)

Cond P 293 ; #(7.6)

(7) If $i:(\tilde{K},\tilde{L})\to(K,L)\subset M$ is an inclusion

$$\begin{array}{ccc} H^*(f^{-1}(K),f^{-1}(L)) & \xrightarrow{f^!} & H^*(K,L) \\ \downarrow i^* & & \downarrow i^* \\ H^*(f^{-1}(\tilde{K}),f^{-1}(\tilde{L})) & \xrightarrow{f^!} & H^*(\tilde{K};\tilde{L}) \end{array}$$

commutes. Similarly for $f_!$. Dold's Book Prop. 10.?, p. 3?

(8) Consider $A\subset B\subset N^n\subset M^{n+k}$ where B is compact and N and M are oriented manifolds. Define isomorphisms

$$T:H_q(M-A,M-B)\underset{\cong}{\xrightarrow{D_M^{-1}}} H^{n+k-q}(B,A)\underset{\cong}{\xrightarrow{D_N}} H_{q-k}(N-A,N-B)$$

$$T:H^{q-k}(N-A,N-B)\underset{\cong}{\xrightarrow{D_N}} H_{n+k-q}(B,A)\underset{\cong}{\xrightarrow{D_M^{-1}}} H^q(M-A,M-B).$$

We let T stand for both the homology and cohomology isomorphisms. If M is the double of the total space of a k-plane disk bundle over N, then T agrees with the Thom isomorphism times $(-1)^{k(n+k-q)}$.

Now let α and β be k-vector bundles over manifolds N and M. Let N^α and M^β denote their respective Thom spaces. Suppose there is a bundle map $\alpha\to\beta$ which covers $f:N\to M$. This induces a map $\bar{f}:N^\alpha\to M^\beta$. Suppose α and β are orientable.

(9) $f^*=(-1)^d T^{-1}\bar{f}^* T, f_*=(-1)^d T\bar{f}_* T^{-1}$. This holds in the relative case as well. Here $d=k(M-N)$.

Suppose $s:N^n\to M^m$ is an embedding and $M\subset L^l$. Assume N,M,L are orientable and let ν be the normal bundle of $s(N)$ in L and μ be the normal bundle of M in L. Then let $\hat{s}:M^\mu\to N^\nu$ denote the collapsing map. In the relative case, let $s:(B,A)\to(\tilde{B},\tilde{A})$. We denote $\hat{s}:(\tilde{B},\tilde{A})^\mu={}^{\tilde{B}^\mu}/_{\tilde{A}^\mu}\to{}^{B^\nu}/_{A^\nu}=(B,A)^\nu$. Then,

(10) $s^!=T^{-1}\hat{s}^*T, s_!=T\hat{s}_*T^{-1}$.

We may see this in the absolute cohomology case by considering the commutative diagram for closed M and N.

$$\begin{array}{ccccccc}
T: H^i(N) & \underset{\cong}{\xrightarrow{D_N}} & H_{N-i}(N) & \underset{\cong}{\xrightarrow{D_L^{-1}}} & H^{L-N+i}(L, L-N) & \xrightarrow{\cong} & H^{L-N+i}(N^\nu) \\
\downarrow s^! & & s_* \downarrow & & \downarrow & & \downarrow \hat{s}^* \\
T: H^{M-N+i}(M) & \underset{\cong}{\xrightarrow{D_M}} & H_{N-i}(M) & \underset{\cong}{\xrightarrow{D_L^{-1}}} & H^{L-N+i}(L, L-M) & \xrightarrow{\cong} & H^{L-N+i}(M^\mu)
\end{array}$$

The homology and relative cases follow similarly.

(11) We can improve (10) as follows. Let $s: N \to M$ be a map which is homotopic to an imbedding $j: N \to L$ which lies within a tubular neighborhood of M in L. Then $s^! = T^{-1}\hat{s}^* T$, $s_! = T\hat{s}_* T^{-1}$.

Now suppose we have a commutative diagram

$$\begin{array}{ccc}
N^n & \xrightarrow{f} & M^{n+k} \\
\cap & & \cap \\
\bar{N}^m & \xrightarrow{\bar{f}} & \bar{M}^{m+k}
\end{array}$$

where $N = \bar{f}^{-1}(M)$ and $N, \bar{N}, M$, and $\bar{M}$ are oriented manifolds.

(12) $(-1)^d f^! = \bar{f}^! : H^i(B', A') \to H^{i+k}(B, A)$ where $d = (m-n)k$.
$(-1)^d f_! = \bar{f}_! : H_i(B, A) \to H_{i+k}(B', A')$

for $A \subset B \subset M$ and $(B', A') = (f^{-1}(B), f^{-1}(A))$. This is seen to be true by first showing $\pm f^! = \bar{f}^!$ for N, M and open sets of $\bar{N}, \bar{M}$. Then the diagram

$$\begin{array}{ccccc}
T: H_*(N-A', N-B') & \underset{\cong}{\xrightarrow{D_N^{-1}}} & H^*(B', A') & \underset{\cong}{\xrightarrow{D_{\bar{N}}}} & H_*(\bar{N}-A', \bar{N}-B') \\
f_* \downarrow & & \swarrow f^! \swarrow \bar{f}^! & & \bar{f}_* \downarrow \\
T: H_*(M-A, M-B) & \underset{\cong}{\xrightarrow{D_M^{-1}}} & H^*(B, A) & \underset{\cong}{\xrightarrow{D_{\bar{M}}}} & H_*(\bar{M}-A, \bar{M}-B)
\end{array}$$

commutes up to sign around the outside by (9). Hence $\pm f^! = \bar{f}^!$.

§4. The Fibre Inclusion Theorem. In this section we prove the fibre inclusion theorem. The following pullback diagram and notation will be used throughout this paper.

$$(1)\qquad \begin{array}{ccccc} F\times F & \xrightarrow{\tilde{i}} & P & \longrightarrow & E \\ \Delta\uparrow\downarrow p_1 \uparrow \tilde{s} & & \Delta\uparrow\downarrow p\uparrow s & & \downarrow \\ F & \longrightarrow & E & \xrightarrow{\pi} & B \end{array}$$

Here $P=\{(e,e')\in E\times E|\pi(e)=\pi(e')\}$ and $p(e,e')=e$. Let $\Delta(e)=(e,e)$ and let $s(e)=(e,f(e))$ (since $\pi(f(e))=\pi(e)$ we have $s(e)\in P$). Thus Δ and s are cross-sections to p.

Now assume that B is a closed oriented manifold. We shall first prove the fibre inclusion theorem in this case. By Quinn's theorem, both E and P are oriented Poincare spaces since $\pi_1(B)$ acts trivially on $H^F(F)$. Now pick a contractible neighbourhood U about some point $x\in B$. Then let $V=\pi^{-1}(U)\subset E$. We may assume that $V=U\times F$. Consider

$$H_E\left((\bar{U},\dot{U})\times F\right)\xrightarrow{\cong}H_E(E,E-V)\longleftarrow H_E(E)$$

where $\bar{U}$ is the closure of U and $\dot{U}$ is the boundary of $\bar{U}$. We regard $(\bar{U},\dot{U})$ as homeomorphic to (D^B,S^{B-1}). Then it is easy to see by spectral sequence arguments that

$$[(\bar{U},\dot{U})]\times[F]=[(\bar{U},\dot{U})\times F]\longleftarrow[(E,E-V)]\longleftarrow[E]$$

under the above inclusion maps. Consider

$$\begin{array}{ccccccc} H^*(\bar{U}\times F) & \xrightarrow[a]{} & H_*((\bar{U},\dot{U})\times F) & \xrightarrow{s_*} & H_*((\bar{U},\dot{U})\times F\times F) & \xleftarrow[c]{} & H^*(\bar{U}\times F\times F) \\ \uparrow & & \downarrow\cong & & \downarrow & & \uparrow \\ H^*(E) & \xrightarrow[b]{} & H_*(E,E-V) & \xrightarrow{s_*} & H_*(P,P-p^{-1}(V)) & \xleftarrow[d]{} & H^*(P) \\ \uparrow & & \uparrow & & \uparrow & & \updownarrow \\ s^!:H^*(E) & \xrightarrow[\cong]{\cap[E]} & H_*(E) & \xrightarrow{s_*} & H_*(P) & \xleftarrow[\cong]{\cap[P]} & H^*(P) \end{array}$$

where a is cap product with $[(\bar{U},\dot{U})]\times[F]$ and b is cap product with $[(E,E\text{-}V)]$ and c is cap with $[\bar{U},\dot{U}]\times[F\times F]$ and d is cap with $[P,P-p^{-1}(V)]$.

This diagram commutes. Now the top horizontal line is essentially

$$\tilde{s}^{!}:H^{*}(F)\underset{\cong}{\overset{\cap[F]}{\to}}H_{*}(F)\overset{\tilde{s}_{*}}{\to}H_{*}(F\times F)\underset{\cong}{\overset{\cap[F\times F]}{\to}}H^{*}(F\times F)$$

times $(-1)^{B\cdot F}$ where $\tilde{s}$ equals s restricted to $F\subset E$. This is so because

$$1\times x\to(1\times x)\cap\big([\bar{U},\dot{U}]\times[F]\big)=(-1)^{x\cdot B}[\bar{U},\dot{U}]\times(x\cap[F])$$

$$\overset{s_{*}}{\longrightarrow}(-1)^{x\cdot B}[\bar{U},\dot{U}]\times\tilde{s}_{*}(x\cap[F])=(-1)^{x\cdot B}[\bar{U},\dot{U}]\times\big(\tilde{s}^{!}(x)\cap[F\times F]\big)$$

$$=(-1)^{xB+(x+F)B}\big(1\times\tilde{s}^{!}(x)\big)\cap\big([\bar{U},\dot{U}]\times[F\times F]\big).$$

The commuting diagram then establishes the following formula.

LEMMA 1. $(-1)^{BF}\tilde{i}^{*}s^{!}=\tilde{s}^{!}i^{*}$.

Now we need the following lemma:

LEMMA 2. $\tilde{\Delta}^{*}\tilde{s}^{!}(1)=\Lambda_{g}\cdot[\bar{F}]$.

Proof. We shall show this by showing that $\langle\tilde{\Delta}^{*}\tilde{s}^{!}(1),[F]\rangle=\Lambda_{g}$.

$$\begin{aligned}\langle\tilde{\Delta}^{*}\tilde{s}^{!}(1),[F]\rangle&=\langle(p_{1}^{!}\tilde{\Delta}^{!})(\tilde{\Delta}^{*}\tilde{s}^{!}(1)),[F]\rangle\\&=\langle\tilde{\Delta}^{!}(\tilde{\Delta}^{*}\tilde{s}^{!}(1)),p_{1!}([F])\rangle=\langle\tilde{\Delta}^{!}(\tilde{\Delta}^{*}\tilde{s}^{!}(1)),[F\times F]\rangle\\&=\langle\tilde{s}^{!}(1)\cup\tilde{\Delta}^{!}(1),[F\times F]\rangle.\end{aligned}$$

Now this last expression is equal to Λ_{g} by a computation similar to that of Robert F. Brown ([4], Theorem 2.2). In the special case that F is a manifold, the last expression is the intersection number of $\tilde{\Delta}(F)$ and $\tilde{s}(F)$ in $F\times F$ which is equal to the algebraic sum of the fixed point indicies which is Λ_{g}, (see p. 337 (13.5) of [6]).

Thus we have the following Lemma.

LEMMA 3. $i^{*}(\Delta^{*}s^{!}(1))=(-1)^{B\cdot F}\Lambda_{g}[\bar{F}]$.

Proof. $i^{*}(\Delta^{*}s^{!}(1))=\tilde{\Delta}^{*}\tilde{i}^{*}s^{!}(1)=\tilde{\Delta}^{*}\tilde{s}^{!}i^{*}(1)=\tilde{\Delta}^{*}\tilde{s}^{!}(1)=(-1)^{BF}\Lambda_{g}[\bar{F}]$ using Lemmas 1 and 2.

Now setting $\Lambda=(-1)^{BF}\Delta^{*}s^{!}(1)$, we see that $i^{*}(\Lambda)=\Lambda_{g}[\bar{F}]$ when B is a closed oriented manifold.

Now suppose that B is a compact oriented manifold with boundary. Let $D(B)$ be the double of B. Then we have the fibre square

$$\begin{array}{ccc} F & \longrightarrow & F \\ \downarrow{\scriptstyle \bar{i}} & & \downarrow{\scriptstyle i} \\ D(E) & \underset{j'}{\overset{r'}{\rightleftarrows}} & E \\ \downarrow & & \downarrow \\ D(B) & \underset{j}{\overset{r}{\rightleftarrows}} & B \end{array}$$

where r and r' are retractions and j and j' are inclusions. Since $D(B)$ is a closed oriented manifold and since $\pi_1(D(B))$ operates trivially on $H^F(F;Z)$, we have a $\Lambda' \in H^F(D(E))$ such that $\bar{i}^*(\Lambda') = \Lambda_g[\bar{F}]$. Define $\Lambda \in H^F(E)$ by $\Lambda = j'^*(\Lambda')$. Then $i^*(\Lambda) = i^*(j'^*(\Lambda')) = \bar{i}^*(\Lambda') = \Lambda_g[\bar{F}]$, and the fibre inclusion formula is established in this case.

Now suppose B is a finite CW complex. Then we can imbed B in some Euclidean space and find a closed regular neighbourhood N of B. Let $r: N \to B$ be the retraction. Then we have

$$\begin{array}{ccc} F & \overset{1}{\longrightarrow} & F \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ r^*E & \overset{\tilde{r}}{\longrightarrow} & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ N & \overset{r}{\longrightarrow} & B\,. \end{array}$$

Since r is a homotopy equivalence, $\tilde{r}$ is a homotopy equivalence. Define $\Lambda = (\tilde{r}^*)^{-1}(\Lambda')$ where $i'^*(\Lambda') = \Lambda_g[\bar{F}]$. Then

$$i^*(\Lambda) = (\tilde{r}\,i')^*(\Lambda) = i'^*\tilde{r}^*(\tilde{r}^*)^{-1}(\Lambda') = \Lambda_g[\bar{F}].$$

Finally, suppose that B is a locally compact CW complex. Let B^n be the n skeleton of B and let $n \gg \dim F$. Then $H^F(E) \cong H^F(\pi^{-1}(B^n))$ under the inclusion map. Now the $\Lambda \in H^F(\pi^{-1}(B^n))$ will give rise to the $\Lambda \in H^F(E)$ under this isomorphism. This proves the Fibre Inclusion Theorem.

Remark. (1) The fibre inclusion formula is true for Z_2 coefficients where F is a closed unoriented manifold and there is no condition on $\pi_1(B)$.

(2) The action of $\pi_1(B)$ on $[\bar{F}]$ must be trivial in order to insure that E and P are oriented Poincare complexes.

§5. The Construction of the Transfer Map. We shall construct the transfer for a special case in this section. Let F and B be closed, smooth, oriented, connected, manifolds and we shall let $F \to E \xrightarrow{\pi} B$ be a fibre bundle with structural group acting smoothly on F and preserving the orientation of F (that is, $\pi_1(B)$ acts trivially on $H^F(F)$).

Then E and P (from the pullback diagram (1)) are closed, smooth, oriented, connected manifolds. We may assume that the cross-sections s, $\Delta : E \to P$ are smooth since any cross-section can be approximated by a smooth cross-section.

Let $E \xrightarrow{j'} R^N$ be an embedding. We define an embedding $j : E \to B \times R^N$ by $j(e) = (p(e), j'(e))$. Let ν be the normal bundle of this embedding. Let α be the bundle of tangents along the fibre for $E \xrightarrow{\pi} B$.

LEMMA 5. *$\nu \oplus \alpha = N_E$, the trivial bundle on E of dimension N.*

Proof. j^* (tangent bundle of $B \times R^N) = \pi^*(\tau_B) \oplus N_E$, also j^* (tangent bundle of $B \times R^N) = \nu \oplus \tau_E = \nu \oplus \alpha \oplus \pi^*(\tau_B)$. Thus $\nu \oplus \alpha = N_E$.

LEMMA 6. *Let β be the bundle of tangents along the fibre for $P \xrightarrow{p} E$. Then $\beta = p^*\alpha$. Hence $\beta | \Delta(E) = \alpha$ and $\mu \oplus \beta = N_P$, the trivial N bundle on P, where $\mu = p^*(\nu)$.*

Now define $\hat{\pi} : \Sigma^N B^+ = (B \times R^N)^c \to E^\nu$ to be the collapsing map, from the suspension of B union a point, to the tubular neighborhood of E in $B \times R^N$. Here $(B \times R^N)^c = {}^{B \times S^N}/_{B \times D^{N'}}$ the one point compactification of $B \times R^N$. Let $\bar{s} : E^\nu \to P^\mu$ be the inclusion map induced by the inclusion $\nu = \mu | s(E) \to \mu$.

Define $\hat{\Delta} : P^\mu \to E^{\mu|\Delta(E)}$ to be the collapsing map. Now

$$\begin{aligned}\mu|\Delta(E) &= \nu \oplus \text{ (normal bundle of } \Delta(E) \text{ in } P) \\ &= \nu \oplus \alpha = N_E.\end{aligned}$$

So $\hat{\Delta}$ is in fact a map $P^\mu \to \Sigma^N E^+$.

Definition of the Transfer.

$$\hat{\tau}_f : \Sigma^N(B^+) \xrightarrow{\hat{\pi}} E^\nu \xrightarrow{\bar{s}} P^\mu \xrightarrow{\hat{\Delta}} \Sigma^N E^+.$$

Note that we may define a tubular neighborhood, L, of E in $B\times R^N$ so that if $A\subset B$, then $L\cap(E_A\times R^N)$ is the total space of $\nu|E_A$. Thus for $A\subset B$, we may define $\hat\tau_f:\Sigma^N A^+\to\Sigma^N E_A^+$ by restricting $\hat\tau_f$ to $\Sigma^N A^+$. If $A\subset X\subset B$ and X are closed, we may define $\hat\tau_f:\Sigma^N({}^X/_A)\to\Sigma^N({}^{E_X}/_{E_A})$.

The theorem we aim to prove states the existence of the transfer in situations more general than our definitions here. Also many choices are made, especially that of the ambient fibre bundle $E\to B$. So the main import of the above paragraph is that we can define some transfers which are natural with respect to one another in the particular situation we are studying.

§6. Proof of the Transfer Theorem. Consider the following commutative diagram.

$$\begin{array}{ccc} F & \underset{l}{\overset{t}{\rightleftarrows}} & F' \\ \downarrow i & & \downarrow i' \\ E & \underset{k}{\overset{s}{\rightleftarrows}} & E' \\ \downarrow \pi & & \downarrow \pi' \\ B & \underset{j}{\overset{r}{\rightleftarrows}} & B' \end{array}$$

where π and π' are Hurewicz fibrations and $rj=1_B$, and $sk=1_{E'}$ and $tl=1_{F'}$. Here r, s, t are retracts and j, k, l are inclusions. We say that π' is a *retract* of π if either of the following two conditions holds:

(i) t and l are identity maps and $F=F'$ and E is a pullback of E' by r;

(ii) $B=B'$ and r and j are identity maps. The reason that care is taken by introducing conditions (i) or (ii) is the following property which we shall need: If $f:E'\to E'$ is a fibre preserving map covering $1_{B'}$, then there exists a fibre preserving map $h:E\to E$ covering 1_B such that $shk=f$.

Let C consist of the set of fibrations $F\to E\overset{\pi}{\to}B$ for which the following property holds: For every map $f:E\to E$ and subcomplex $A\subset B$, there exists a transfer map $\hat\tau_f:{}^B/_A\to{}^E/_{E_A}$ satisfying the transfer theorem (b).

Retraction Lemma. *Any fibration belongs to C if it is a retract of a fibration belonging to C.*

Proof. We define $\hat\tau_f':{}^B/_A\to{}^E/_{E_A}$ for any map $f:E'\to E'$ by letting $\hat\tau'_f=s\hat\tau_h i$ where $h:E\to E$ satisfies $shk=f$ and τ_h is the transfer from ${}^B/_{j(A)}\to{}^E/_{k(E'_A)}$.

Note that $\Lambda_{lgt}=\Lambda_g$ and that $lgt=h|F$. Now $\hat{\tau}'_f$ induces the appropriate formulas in homology and cohomology. For example,

$$\pi'_*(\hat{\tau}'_f)_*=\pi'_*(s_*\tau_h j_*)=r_*\pi_*\tau_h j_*=\Lambda_{lgt}(r_* j_*)=\Lambda_g.$$

We now prove the transfer theorem by proving it for a special type of fibration (the initial lemma), and then showing that every fibration for which the theorem is true is a retract of one of the initial fibrations.

INITIAL LEMMA. *Suppose $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a fibre bundle with B and F closed connected oriented smooth manifolds, and with structural group the group of diffeomorphisms of F which preserves orientation. Then it belongs to C.*

Proof. Let $\tau_f=\Sigma(\hat{\tau}_f)_*\Sigma^{-1}$ and $\tau^f=\Sigma^{-1}(\hat{\tau}_f)^*\Sigma$ where Σ is N-fold suspension and $\hat{\tau}_f$ is defined in the previous section. We must show that $\pi_*\tau_f=\Lambda_g$ and $\tau^f\pi^*=\Lambda_g$ in homology and cohomology respectively.

$$\begin{aligned}\tau_f&=\Sigma(\hat{\tau}_f)_*\Sigma^{-1}=\Sigma(\hat{\Delta}\bar{s}\hat{\pi})_*\Sigma^{-1}\\ &=\Sigma\hat{\Delta}_*T^{-1}T\bar{s}_*T^{-1}T\hat{\pi}_*\Sigma^{-1}\\ &=(-1)^a\Delta_!\circ(-1)^b s_*\circ(-1)^c\pi_!\\ &=(-1)^{a+b+c}\Delta_! s_*\pi_!\end{aligned}$$

where $a=N(E+N-q)$, $b=NF-F^2$, $c=N(B+N-q)$, hence

$$\tau_f=(-1)^F\Delta_! s_*\pi_!$$

Now

$$\begin{aligned}\tau_f(x)&=(-1)^F\Delta_! s_*\pi_!(x)=(-1)^F p_*\circ\Delta_*\big(1\cap(\Delta_! s_*\pi_!(x))\big)\\ &=(-1)^F p_*\big((-1)^{(P-x)F}\Delta^!(1)\cap s_*\pi_!(x)\big)\\ &=(-1)^{F+(P-x)F}p_*\big(s_*(s^*\Delta^!(1)\cap\pi_!(x))\big)\\ &=(-1)^{F+(P-x)F}s^*\Delta^!(1)\cap\pi_!(x)\end{aligned}$$

So

$$\begin{aligned}\pi_*\tau_f(x)&=(-1)^{F+(P-x)F}\pi_*\big(s^*\Delta^!(1)\cap\pi_!(x)\big)\\ &=(-1)^{F+(P-x)F}\cdot(-1)^{(B-x)F}\big(\pi^! s^*\Delta^!(1)\big)\cap x\\ &=(-1)^F\big(\pi^! s^*\Delta^!(1)\big)\cap x=\Lambda_g\cdot 1\cap x=\Lambda_g x\end{aligned}$$

Here $\pi^!(s^*\Delta^!(1)) = (-1)^F \Lambda_g \cdot 1$ by lemma 3 and the fact that $(-1)^{BF}\tilde{\pi}^! i^* = i^* \pi^!$ for $\tilde{\pi}: F \to *$, which is true by an argument similar to lemma 3.

For cohomology we have for $x \in H^*(B,A)$

$$\begin{aligned}
\tau^f(x) &= \Sigma^{-1}\hat{\tau}_f^*\Sigma(x) = \Sigma^{-1}\hat{\pi}^*\bar{s}^*\hat{\Delta}^*\Sigma(x)\\
&= \Sigma^{-1}\hat{\pi}^* TT^{-1}\bar{s}^* TT^{-1}\hat{\Delta}^*\Sigma(x)\\
&= (-1)^F \pi^! s^* \Delta^!(x)\\
&= (-1)^F \pi^! s^* \big(\Delta^!(\Delta^* p^*)(x)\big)\\
&= (-1)^F \pi^! s^* \big(p^*(x) \cup \Delta^!(1)\big)
\end{aligned}$$

since $\Delta^!(\Delta^*(x)) = x \cup \Delta^!(1)$ (by §3, (5))

$$\begin{aligned}
&= (-1)^F \pi^! \big(s^* p^*(x) \cup s^*\Delta^!(1)\big)\\
&= (-1)^F \pi^! \big(x \cup s^*\Delta^!(1)\big)
\end{aligned}$$

As in the previous paragraph, $\pi^!(s^*\Delta^!(1)) = (-1)^F \Lambda_g$. Hence $\tau^f \pi^* = \Lambda_g$.

Now that we have shown that $F \xrightarrow{i} E \xrightarrow{\pi} B$ is in C, where F, B are smooth closed connected orientable manifolds and the bundle is orientable in the appropriate way, we must show that all fibrations are retracts of such a fibre bundle. Then the retraction lemma will give us the theorem.

Lemma a). *Suppose $F \xrightarrow{i} E \xrightarrow{\pi} B$ as above, except that F is a compact manifold with boundary. Then $F \xrightarrow{i} E \xrightarrow{\pi} B \in C$.*

Proof. $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a retract of $DF \xrightarrow{Di} DE \xrightarrow{D\pi} B$ where DE and DF are doubles.

Lemma b). *Suppose $F \xrightarrow{i} E \xrightarrow{\pi} B$ is as in previous lemma (a) except that B is a compact manifold with boundary. Then $F \to E \to B \in C$.*

Proof. $F \to DE \xrightarrow{D\pi} DB$ has $F \to E \to B$ as a retract.

Hence by (a) and retraction lemmas $F \to E \to B \in C$.

Lemma c). *Suppose $F \xrightarrow{i} E \xrightarrow{\pi} B$ as in lemma (b), except that B is a finite complex. Then $F \to E \to B \in C$.*

Proof. We can embed B into a closed regular neighborhood by $B \overset{i}{\subset} N \subset R^N$. Then $F \to E \to B$ is a retract of $F \to r^* E \to N$ where $r: N \to B$ is a deformation retract.

LEMMA d). *$F \to E \to B$ as in (c) except that F is not orientable. Let $\tilde{F} \xrightarrow{p} F$ be the oriented double covering of F and let Z be the mapping cylinder of p. Then Z is an orientable manifold and the structural group of $F \to E \to B$ acts on Z in an orientation preserving way. Thus we may find a fibre bundle $Z \to \bar{E} \to B$ which retracts to $F \to E \to B$. (See Lemma 3 of [9]). Hence $F \to E \to B \in C$.*

LEMMA e). *$F \to E \to B$ has properties of lemma (c) except that F is orientable but the structural group does not preserve orientation. Then it is a retract of $F \times RP^2 \to E \times RP^2 \to B$ which satisfies lemma (d).*

LEMMA f). *If $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a fibration with connected fibre, homotopy equivalent to a finite CW complex and finite dimensional base. Then it is a retract of $W \times T \to E' \to B$ where T is a torus and W is a manifold with boundary which is a regular neighbourhood of F. This is the closed fibre smoothing theorem which will be proved later.*

LEMMA g). *Now assume $F \to E \to B$ is a fibration with F possibly a disconnected fibre. Let F_0 be a path component of F.*

Then we have the following diagram. Suppose that F is the union of k copies of F_0 (k must be finite since F is compact)

$$\begin{array}{ccc} F_0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & E \\ \downarrow \pi & & \downarrow \pi \\ \tilde{B} & \underset{p}{\longrightarrow} & B \end{array}$$

According to Kahn and Priddy [11], or others, there exists an S-map $\hat{\tau}_k : {}^{B}/_{A} \to {}^{\tilde{B}}/_{\tilde{A}}$ which induces the transfer for covering spaces where $\tilde{B} \underset{p}{\to} B$ is a covering of index k.

Note that f induces the identity on B and hence a deck transformation on $\tilde{B}$. If the deck transformation is the identity then $\Lambda_g = k\Lambda_{g'}$ where $g' = g|F_0$. Define $\hat{\tau}_f : {}^{B}/_{A} \xrightarrow{\hat{\tau}_k} {}^{\tilde{B}}/_{\tilde{A}} \xrightarrow{\hat{\tau}_f} {}^{E}/_{E_A}$. This transfer must satisfy the correct formula. If the deck transformation is not the identity, on the other hand, then $\Lambda_g = 0$ and we may define $\hat{\tau}_f$ to be trivial. This proves transfer theorem (b).

In the case of transfer theorem (a), the transfers τ_f and τ^f are defined for arbitrary base B. From theorem [2], we already know that τ_f and τ^f can be defined for fibre bundles over an arbitrary base. The closed fibre smoothing theorem and the retraction lemma allow us to extend τ_f and τ^f to the case of fibrations when B is finite dimensional. We define τ_f and τ^f for arbitrary bases by letting $\tau_f = \tau_f | B^{2N} : H_N(B) \to H_N(E)$ for all N, where B^{2N} is the $2N$ skeleton of B. Similarly for cohomology. This proves part (a) of the transfer theorem.

Remark. We do not really need to consider the disconnected fibre case at the end of the proof of part (b), that is lemma (f). We could have included the disconnected case in the initial lemma by observing that the construction of the covering transfer map can be accomplished as in §4 with the same ideas.

§7. **Proof of the Transgression Theorem.** Let C be the class of fibrations for which $\Lambda_g d^* = 0$ for every map $f: E \to E$ covering the identity on B. We shall prove the transgression theorem in the same manner as the transfer theorem, by first proving a retraction lemma and an initial lemma, and then showing that all the relevant fibrations are retracts of the fibrations in the initial lemma.

RETRACTION LEMMA. *Let $F' \xrightarrow{i'} E' \xrightarrow{\pi'} B'$ be a retract of $F \xrightarrow{i} E \xrightarrow{\pi} B$. If $\pi \in C$, then $\pi' \in C$.*

Proof. The commutative diagram

$$\begin{array}{ccc} F & \underset{l}{\overset{t}{\rightleftarrows}} & F' \\ \downarrow i & & \downarrow i' \\ E & \underset{k}{\overset{s}{\rightleftarrows}} & E' \\ \downarrow \pi & & \downarrow \pi' \\ B & \underset{j}{\overset{r}{\rightleftarrows}} & B' \end{array}$$

gives rise to the homotopy commutative diagram,

$$\begin{array}{ccc} \Omega B & \underset{\Omega j}{\overset{\Omega r}{\rightleftarrows}} & \Omega B' \\ \downarrow d & & \downarrow d' \\ F & \underset{l}{\overset{t}{\rightleftarrows}} & F' \end{array}$$

Thus $d' \sim t \circ d \circ (\Omega j)$. Let $f: E' \to E'$ cover $1_{B'}$ and $g = f|F$. Then $\Lambda_g(d')^* = (\Omega r)^* \circ (\Lambda_g d^*) \circ t^* = (\Omega r)^* \circ (\Lambda_{(dgt)} \cdot d^*) \circ t^* = (\Omega r)^* \circ (0) \circ t^* = 0$.

INITIAL LEMMA. *Suppose $F \to E \xrightarrow{\pi} B$ is a fibre bundle which has a compact, closed, oriented, connected, topological manifold as fibre F. Suppose that B is an arbitrary CW complex with $\pi_1(B)$ acting trivially on $H^F(F;Z)$. Then $\pi \in C$.*

Proof. There exists a $\Lambda \in H^F(E;Z)$ such that $i^*(\Lambda) = \Lambda_g[\bar{F}]$, by theorem 1 of [2]. (This looks like the Fibre Inclusion Formula, except that the base need not be locally compact. On the other hand we are restricted to fibre bundles with F a manifold).

Now the argument is essentially the same as in [2], but we shall reproduce it here. We have a homotopy commutative diagram arising from standard considerations as follows:

$$\begin{array}{ccc} \Omega B \times F & \xrightarrow{\hat{d}} & F \\ \downarrow j \times 1 & & \downarrow i \\ P \times F & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ B & \xrightarrow{1_B} & B \end{array}$$

Here P is the space of paths on B and $\hat{d}|(\Omega B \times *) = d$ and $\hat{d}|(* \times F) = 1_F$.

So $\hat{d}^*(i^*(\Lambda)) = \Lambda_g(1 \times [\bar{F}])$. Now assume ΩB is connected for simplicity. (ΩB disconnected follows immediately from the connected case). $0 = \hat{d}^*(x \cup i^*(\Lambda)) = (d^*(x) \times 1 + \text{terms of different dimensions}) \cup (\Lambda_g \times [\bar{F}]) = (\Lambda_g d^*(x)) \times [\bar{F}] + \text{terms of different dimensions}$. Hence $0 = \Lambda_g d^*(x) \times [\bar{F}]$. Hence $\Lambda_g d^*(x) = 0$ for all $x \in \tilde{H}^*(F)$.

Now we proceed as in §5, showing that every fibre bundle with compact connected manifold F is a retract of a fibre bundle as in the Initial Lemma. We use the Closed Fibre Smoothing Theorem to show that fibrations with F fibre homotopy equivalent to connected compact *CW* complexes are retracts of Initial Lemma fibre bundles, provided that B is finite dimensional.

Now we assume that B is infinite dimensional. If B is infinite dimensional, let B^N be the N-skeleton of B where $N \gg$ (homological dimension of F). Since $j: B^N \subset B$ is N-1 connected, we see that $\Omega j: \Omega B^N \to \Omega B$ is N-2 connected.

Consider the homotopy commutative diagram:

$$\begin{array}{ccc}
\Omega B^N & \longrightarrow & \Omega B \\
\downarrow{\scriptstyle d_N} & & \downarrow{\scriptstyle d} \\
F & \xrightarrow{1} & F \\
\downarrow & & \downarrow \\
\pi^{-1}(B^N) & \longrightarrow & E \\
\downarrow & & \downarrow \\
B^N & \xrightarrow{j} & B
\end{array}$$

Then $0=\Lambda_g d_N^*=(\Omega j)^*(\Lambda_g d^*)$. Since $(\Omega j)^*$ is an isomorphism up to dimensions exceeding the dimension of F, we have $\Lambda_g d^*=0$.

Finally, we consider the case when F is disconnected. Let F_0 be a connected componant of F and consider the homotopy commutative diagram.

$$\begin{array}{ccc}
\Omega \tilde{B} & \longrightarrow & \Omega B \\
\downarrow{\scriptstyle \tilde{d}} & & \downarrow{\scriptstyle d} \\
F_0 & \longrightarrow & F \\
\downarrow & & \downarrow \\
E & \longrightarrow & E \\
\downarrow & & \downarrow \\
\tilde{B} & \longrightarrow & B
\end{array}$$

Here $\tilde{B}$ is the appropriate covering of B, so $\Omega\tilde{B}\xrightarrow{\Omega p}\Omega B$ is a homotopy equivalence on path-components; as is the inclusion $j:F_0\to F$. Thus d^* is the direct sum of copies of $\tilde{d}^*$ and the theorem follows immediately.

The Transgression Formula gives rise to Corollaries about evaluation maps. Let $\omega:M\to F$ be the evaluation map based on a base point $*\in F$, where M is a space of self maps of F. That is, $\omega(f)=f(*)$.

COROLLARY. *Let F_1^F be the identity componant of F^F where F is homotopy equivalent to a compact connected CW complex. Then $\chi(F)\omega^*=0:\tilde{H}^*(F)\to\tilde{H}^*(F_1^F)$.*

Proof. If we consider the universal fibration for F, denoted by $F\to E_\infty\to$

B_∞, we obtain the commutative diagram

$$\begin{array}{ccc} \Omega B_\infty & \xrightarrow{h} & F^F \\ \downarrow d & & \downarrow \omega \\ F & \xrightarrow{1} & F \end{array}$$

where h is an H-space homotopy equivalence.

COROLLARY. *Let G be a space of homeomorphisms of F, and let $f: F \to F$ be G-equivarient. Let F be homotopy equivalent to a compact connected CW complex.*

Then $\Lambda_f \omega^* = 0 : \tilde{H}^*(F) \to \tilde{H}^*(G)$.

Proof. Consider the fibre bundle $F \to E_G \times_G F \to B_G$. Define $\bar{f}: E_G \times_G F \to E_G \times_G F$ by $\bar{f}(\langle e, x \rangle) = \langle e, f(x) \rangle$. Then $\bar{f}$ covers the identity on B_G and restricts to $f: F \to F$ on some fibre. Hence $\Lambda_f d^* = 0$ by the transgression Formula. Thus $\Lambda_f \omega^* = 0$ by the commutative diagram

$$\begin{array}{ccc} \Omega B_G & \xrightarrow{h} & G \\ \downarrow d & & \downarrow \omega \\ F & \xrightarrow{1} & F \end{array}$$

where h is a homotopy equivalence.

Remark. The first Corollary extends theorem A of [9] from compact manifolds and homeomorphisms to compact CW complexes and homotopy equivalences, thus answering a question of the second author. The second Corollary extends a theorem in §3.2 of [2] from compact manifolds to compact CW complexes.

§8. **Proof of the Fibre Smoothing Theorems.** Theorem A below is a slightly sharper version of the open fibre smoothing theorem. Theorems A and B together imply the closed fibre smoothing theorem as stated in the introduction.

THEOREM A. *Let F be a finite CW complex of dimension k, and let V be an open regular neighbourhood of F in R^n (with $n \geqslant 5$). If B is a CW complex* *of dimension at most $n-2k$, then any Hurewicz fibration over B with fibre F is fibre homotopy equivalent to a bundle with fibre V and group* Diff (V).

Proof. Diff (V) is the group of diffeomorphisms of V with the C^∞ topology. We also use the C^∞ topology for the spaces Emb (V, V), Imm (V, V) of C^∞ embeddings and immersions of V in V. Let $H(V)$ be the space of homotopy equivalences from V to V with the C^o topology, and put $\text{Emb}_H(V,V) = \text{Emb}\,(V,V) \cap H(V)$ and $\text{Imm}_H(V,V) = \text{Imm}\,(V,V) \cap H(V)$. Let $\text{Rep}_H(\tau_V, \tau_V)$ be the space of vector bundle maps from τ_V to τ_V covering homotopy equivalences from V to V, with the C^o topology.

Let $f: \text{Diff}\,(V) \to \text{Emb}_H(V,V)$ and $e: \text{Emb}_H(V,V) \to \text{Imm}_H(V,V)$ be the inclusion maps. Let $d: \text{Imm}_H(V,V) \to \text{Rep}_H(\tau_V, \tau_V)$ be the derivative map, and let $p: \text{Rep}_H(\tau_V,\tau_V) \to H(V)$ be the projection. Observe that p, d, e, f are all homomorphisms of associative H-spaces, so they induce maps of classifying spaces.

The composite $g = pdef: \text{Diff}(V) \to H(V)$ is just the inclusion, and it is required to show that $(Bg)_*: [B, B\text{Diff}(V)] \to [B, BH(V)]$ is surjective. Since τ_V is trivial, $p: \text{Rep}_H(\tau_V, \tau_V) \to H(V)$ has a cross-section which is a homomorphism of H-spaces, so $(Bp)_*: [B, B\text{Rep}_H(\tau_V, \tau_V)] \to [B, BH(V)]$ is surjective.

Since V is open, $d: \text{Imm}_H(V,V) \to \text{Rep}_H(\tau_V, \tau_V)$ is a homotopy equivalence. This follows from the Smale-Hirsch theory of immersions: see [10] for a more recent exposition.

Let $X = \{h \in \text{Imm}_H(V,V) : h|F \text{ is an embedding}\}$. General position theorems imply that the pair $(\text{Imm}_H(V,V), X)$ is $(n-2k-1)$-connected. Since V is a regular neighborhood of F, and any immersion in X embeds some neighborhood of F, X deformation retracts onto $\text{Emb}_H(V,V)$. It follows that $e: \text{Emb}_H(V,V) \to \text{Imm}_H(V,V)$ is $(n-2k-1)$ connected, so

$$(Be)_*: [B, BEmb_H(V,V)] \to [B, BImm_H(V,V)]$$

is surjective.

Let W be a compact regular neighbourhood of F in V, so $V - W \cong \partial W \times [0,\infty)$. The restriction map $\rho: \text{Diff}(V) \to \text{Emb}_H(W,V)$ has the covering homotopy property, by results of Thom and Palais [12]. To see that ρ is surjective, let $h \in \text{Emb}_H(W,V)$ and put $U = \overline{V - h(W)}$. The restrictions on n and k ensure that $n-k \geqslant 3$, so U has the proper homotopy type of $\partial U \times [0,\infty)$. By the Stallings engulfing theorem [14], U is diffeomorphic to $\partial U \times [0,\infty)$, so h extends to a diffeomorphism of V. The fibre of ρ is the group $\text{Diff}(V/W) = \{h \in \text{Diff}(V) : h|W = 1\}$. By an elementary "combing" argument, $\text{Diff}(V/W)$ is contractible, so ρ is a homotopy equivalence. A similar argument shows that the restriction map $\mathrm{Emb}_H(V,V) \to \mathrm{Emb}_H(W,V)$ is a homotopy equivalence; it follows that $f:\mathrm{Diff}(V) \to \mathrm{Emb}_H(V,V)$ is a homotopy equivalence.

Together, these results on p,d,e,f show that $(Bg)_*:[B,B\mathrm{Diff}(V)] \to [B,BH(V)]$ is surjective, as required.

If W is a compact smooth manifold then diffeomorphisms of W are simple homotopy equivalences, so $\pi_0(\mathrm{Diff}(W)) \to \pi_0(H(W))$ is not in general surjective. This shows that Hurewicz fibrations over S^1 with fibre W are not in general equivalent to bundles with fibre W and group $\mathrm{Diff}(W)$.

THEOREM B. *Let W be a compact C^∞ manifold and let $V=\mathrm{Int}\ W$. If $p:E \to B$ is a fibre bundle with fibre V and group* $\mathrm{Diff}(V)$, *and B is an n-dimensional CW complex, then $pp_1:E \times T^n \to B$ is isomorphic to a bundle with group* $\mathrm{Diff}(W \times T^n)$.

Proof. Let M be the boundary of W. It is more convenient to regard V as the union of W and an open collar $M \times [0,\infty)$. Let

$$\mathrm{Diff}(V,W) = \{h \in \mathrm{Diff}(V) : h(W) = W\},$$

and put

$$K(W) = \mathrm{Diff}(V)/\mathrm{Diff}(V,W).$$

Here $\mathrm{Diff}(V,W)$ acts on $\mathrm{Diff}(V)$ on the right, so $h_1,h_2 \in \mathrm{Diff}(V)$ represent the same element of $K(W)$ if and only if $h_1(W)=h_2(W)$. It is easily proved that the projection $\mathrm{Diff}(V) \to K(W)$ is locally trivial, with fibre $\mathrm{Diff}(V,W)$.

Let $s:K(W) \to K(W \times S^1)$ be induced by the map $\mathrm{Diff}(V) \to \mathrm{Diff}(V \times S^1)$ sending h to $h \times 1$.

LEMMA 1. *$s:K(W) \to K(W \times S^1)$ is null-homotopic.*

Proof of Theorem B Assuming Lemma 1. It is enough to reduce the group of the bundle $pp_1:E \times T^n \to B$ to $\mathrm{Diff}(V \times T^n, W \times T^n)$. Suppose the group of $pp_1|p^{-1}(B^r) \times T^r \to B^r$ has been reduced to $\mathrm{Diff}(V \times T^r, W \times T^r)$ (this is clearly possible if $r=0$). The obstruction to extending this reduction over B^{r+1} is an element of $H^{r+1}(B^{r+1},B^r;\pi_r K(W \times T^r))$, say α. By Lemma 1, $s_*(\alpha) \in H^{r+1}(B^{r+1},B^r;\pi_r K(W \times T^{r+1}))$ vanishes; it follows that the group of the bundle $pp_1|p^{-1}(B^{r+1}) \times T^{r+1} \to B^{r+1}$ can be reduced to $\mathrm{Diff}\,(V \times T^{r+1}, W \times T^{r+1})$. By induction, Theorem 2 is proved assuming Lemma 1.

Proof of Lemma 1. Since M is invariant under $\mathrm{Diff}\,(V,W)$, we can put

$$\mathrm{Diff}'(V,W) = \{h \in \mathrm{Diff}(V,W) : h(x,u) = (hx,u)(x \in M, u \in [0,\infty))\}.$$

Let $\mathrm{Diff}'(V)$ be the space (not group) of diffeomorphisms $h: V \to V$ such that;

$$h(M \times (u+2)) \subset M \times (u+1, \infty) \text{ for all } u \in [0,4], \tag{1}$$

$$h(M \times (3, \infty)) \supset M \times 4. \tag{2}$$

Put $K'(W) = \mathrm{Diff}'(V)/\mathrm{Diff}'(V,W)$, and let $i: K'(W) \to K(W)$ be induced by inclusion maps.

LEMMA 2. *There is a map* $j: K(W) \to K'(W)$ *such that* $ij \simeq 1: K(W) \to K(W)$.

Proof. Let $K''(W) = \mathrm{Diff}(V)/\mathrm{Diff}'(V,W)$, and let $i': K'(W) \to K''(W)$, $i'': K''(W) \to K(W)$ be the inclusion and projection maps. By a theorem of Palais [12], the restriction map $\mathrm{Diff}(V,W) \to \mathrm{Diff}(M \times [0,\infty))$ is a fibre map. The map $\mathrm{Diff}(M \times [0,\infty)) \to \mathrm{Diff}(M \times [0,\infty)/M \times 0)$ sending g to $g((g|M) \times 1)^{-1}$ is the projection of a (trivial) fibre bundle. By composing these two fibre maps, we obtain a fibration $\mathrm{Diff}(V,W) \to \mathrm{Diff}(M \times [0,\infty)/M \times 0)$ with fibre $\mathrm{Diff}'(V,W)$. It follows that $i'': K''(W) \to K(W)$ is a fibre map with fibre $\mathrm{Diff}(M \times [0,\infty)/M \times 0)$. Since $\mathrm{Diff}(M \times [0,\infty)/M \times 0)$ is contractible [5] p. 337, i'' is a homotopy equivalence. It will suffice to construct a map $j': K''(W) \to K'(W)$ such that $i'j' \simeq 1$.

Define continuous maps $\alpha, \beta, \gamma: \mathrm{Diff}(V) \to [4, \infty)$ by

$$\alpha(h) = \inf\{a \in [4,\infty): h(M \times (a-1)) \subset M \times (3,\infty)\},$$

$$\beta(h) = \inf\{b \in [4,\infty): h(M \times (\alpha(h)+1,\infty)) \supset M \times (b-1)\},$$

$$\gamma(h) = \inf\{c \in [4,\infty): h(M \times (c-1)) \subset M \times (\beta(h)+1,\infty)\}.$$

Then $\gamma(h) > \alpha(h)+2$ and $h(M \times \alpha(h)) \subset M \times (3,\infty)$,

$$h(M \times (\alpha(h)+1,\infty)) \supset M \times \beta(h), \quad h(M \times \gamma(h)) \subset M \times (\beta(h))+1,\infty).$$

Choose a fixed homotopy $\theta: [4,\infty) \times I \to \mathrm{Diff}[0,\infty)$ such that, for all $b \in [4,\infty)$, $\theta(b) = 1$, $\theta_t(b)|[0,3] = 1$, $\theta_1(b)(4) = b$ and $\theta_1(b)(5) = b+1$. Let P be the space

$$\{(a,c): a,c \in [4,\infty), c \geqslant a+2\},$$

and choose a fixed homotopy $\phi: P \times I \to \mathrm{Diff}[0,\infty)$ such that, for all $(a,c) \in P$, $\phi_0(a,c) = 1$, $\phi_t(a,c)|[0,1] = 1$,

$$\phi_1(a,c)(2) = a, \quad \phi_1(a,c)(3) = a+1$$

and

$$\phi_1(a,c)(4)=c.$$

Define homotopies $\bar{\theta},\bar{\phi}:\mathrm{Diff}(V)\times I\to\mathrm{Diff}(V)$ by

$$\bar{\theta}_t(h)|W=1,\quad \bar{\theta}_t(h)(x,u)=\big(x,\theta_t(\beta(h))(u)\big),$$
$$\bar{\phi}_t(h)|W=1,\quad \bar{\phi}_t(h)(x,u)=\big(x,\phi_t(\alpha(h),\gamma(h))(u)\big).$$

Define homotopy $H:\mathrm{Diff}(V)\times I\to\mathrm{Diff}(V)$ by

$$H_t(h)=\big(\bar{\theta}_t(h)\big)^{-1}h\big(\bar{\phi}_t(h)\big).$$

Then $H_0(h)=h$,

$$H_1(h)(M\times 2)\subset M\times(3,\infty),\quad H_1(h)(M\times(3,\infty))\supset M\times 4$$

and $H_1(h)(M\times 4)\subset M\times(5,\infty)$. These conditions imply that

$$H_1(h)(M\times(u+2))\subset M\times(u+1,\infty)$$

for all $u\in[0,4]$, so $H_1(h)\in\mathrm{Diff}'(V)$.

Since α,β,γ are invariant under $\mathrm{Diff}'(V,W)$, H induces a homotopy $H':K''(W)\times I\to K''(W)$. Define

$j':K''(W)\to K'(W)$ by $j'(h)=H_1(h)$: then $i'j'\cong 1$ as required.

To complete the proof of Lemma 1, it is enough to show that si: $K'(W)\to K(W\times S^1)$ is null-homotopic.

We shall form $V\times S^1$ from $V\times[0,4]$ by glueing $V\times 0$ to $V\times 4$. Put $Z=M\times[0,1]\cup M\times[2,\infty)$ and define

$$G:\mathrm{Diff}'(V)\to\mathrm{Emb}\,\big(Z\times[0,4],\,V\times[0,4]\big)$$

by

$$G(h)(x,u,a)=(x,u+a,a)\qquad(0\leqslant u\leqslant 1)$$

and

$$G(h)(x,u,a)=(h(x,u+a),a)\qquad(u\geqslant 2),$$

where

$$h \in \mathrm{Diff}'(V), \quad x \in M, \quad u \in [0,\infty), \quad a \in [0,4].$$

$G(h)$ is an embedding since $M \times [a, a+1]$ and $h(M \times [a+2,\infty))$ are disjoint for all $a \in [0,4]$.

Let $\mathfrak{F}(V \times [0,4])$ be the space of C^∞ vector fields on $V \times [0,4]$. There is a map

$$\xi : \mathrm{Diff}'(V) \to \mathfrak{F}(V \times [0,4]) \qquad \text{such that:}$$

(1) $dp(\xi(h)) = \dfrac{d}{da}$, where $p : V \times [0,4] \to [0,4]$ is the projection.

(2) $\xi(h)$ agrees with $dG(h)(0 \times \dfrac{d}{da})$ over $G(h)(Z \times [0,4])$.

By integrating $\xi(h)$ for each $h \in \mathrm{Diff}'(V)$, we obtain a map $H : \mathrm{Diff}'(V) \to \mathrm{Diff}(V \times [0,4])$ such that $H(h)|V \times 0 = 1$ and $dH(h)(0 \times \dfrac{d}{da}) = \xi(h)$. It follows that

$$pH(h) = p : V \times [0,4] \to [0,4]$$

and

$$H(h)(x,u,a) = (x, u+a, a) \quad (0 \leqslant u \leqslant 1),$$
$$H(h)(h(x,u),a) = (h(x,u+a),a) \quad (u \geqslant 2).$$

Let $\phi : \mathbf{R} \to [0,4]$ be a fixed C^∞ function such that $\phi(u) = 4$ if $u \leqslant 0$ and $\phi(u) = 0$ if $u \geqslant 1$. Define $\alpha_0, \alpha_1 : \mathrm{Diff}'(V) \times V \to [0,4]$ by

$$\alpha_0(h,x,u) = \phi(u) \qquad (u \geqslant 0),$$
$$\alpha_0(h,y) = 4 \qquad (y \notin M \times [0,\infty)),$$
$$\alpha_1(h,h(x,u)) = \phi(u-2) \qquad (u \geqslant 2),$$
$$\alpha_1(h,h(y)) = 4 \qquad (y \notin M \times [2,\infty)).$$

Put $\alpha_t = (1-t)\alpha_0 + t\alpha_1$ for all $t \in I$.
Define $e : V \times [0,4] \to V \times S^1$ by $e(y,a) = (y, e^{\pi i a/2})$. Define homotopy

$$E' : \mathrm{Diff}'(V) \times I \to \mathrm{Emb}\,(M \times [0,\infty), \quad V \times [0,\infty))$$

by

$$E'_t(h)(x,u) = H(h)(x,u,\alpha_t(h,x,u)) \qquad \text{for } x \in M, \quad u \in [0,\infty).$$

E' induces a homotopy $E: \mathrm{Diff}'(V) \times I \to \mathrm{Emb}(M \times S^1, V \times S^1)$ by

$$E_t(h)(e(x,u)) = eE'_t(h)(x,u) \qquad (x \in M, u \in [0,4]);$$

this is well-defined since

$$H(h)(x,0,\alpha_t(h,x,0)) = (x,4,4) \quad \text{and} \quad H(h)(x,4,\alpha_t(h,x,4)) = (x,4,0).$$

A similar computation shows that $E_t(h)$ is a C^∞ embedding. Put

$$N_t(h) = E_t(h)(M \times S^1) \subset V \times S^1; \; e^{-1}(N_t(h)) \subset V \times [0,4]$$

is shown schematically in Fig. 1.

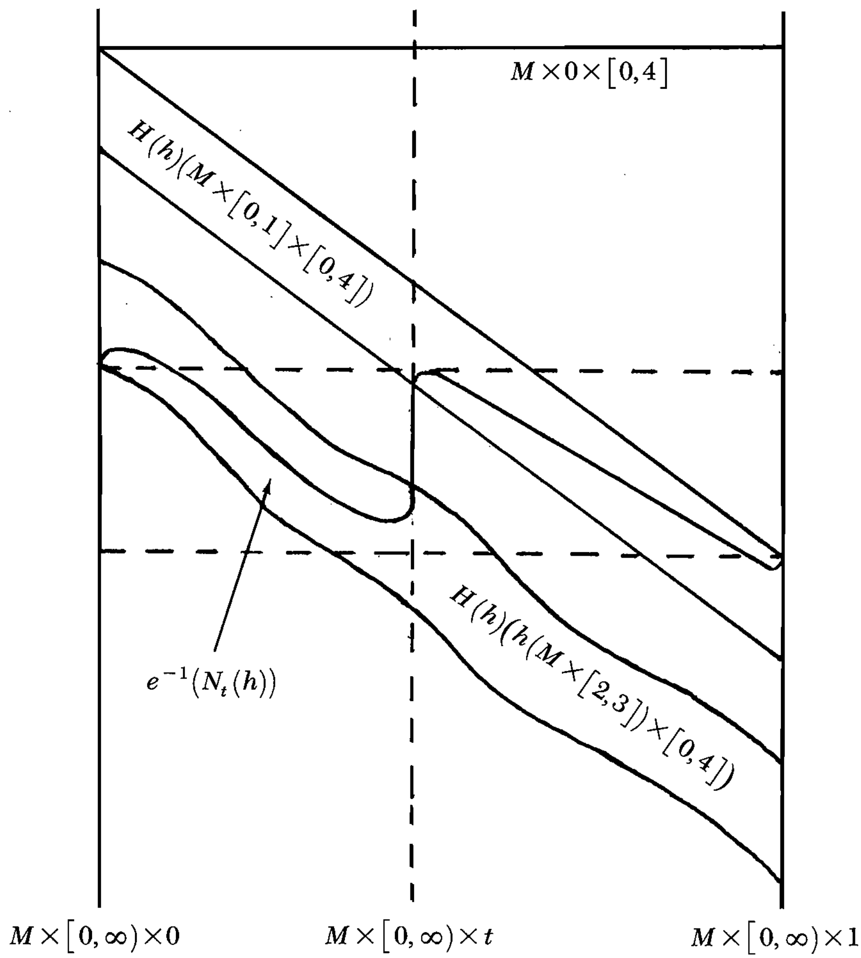

Fig. 1

Define isotopy $L: I \to \mathrm{Emb}(M \times S^1, V \times S^1)$ by

$$L_t(e(x,u)) = e\big(x, 4(1-t) + t(u+\phi(u)), (1-t)(4-u) + t\phi(u)\big)$$
$$(x \in M,\ t \in I,\ u \in [0,4]).$$

L_t is a C^∞ embedding for all t, and $L_0(M \times S^1) = M \times S^1$. It is easily checked that $L_1 = E_0(h)$ for all $h \in \mathrm{Diff}'(V)$.

Therefore $E_t(h)$ is isotopic to L_0 for all h, t. By the isotopy extension theorem, there is a diffeomorphism g of $V \times S^1$ such that $g(M \times S^1) = N_t(h)$. If also $g'(M \times S^1) = N_t(h)$, then $g(W) = g'(W)$ implying that g, g' represent the same element of $K(W \times S^1)$. In this way E induces a homotopy

$$\mathrm{Diff}'(V) \times I \to K(W \times S^1).$$

If $f \in \mathrm{Diff}'(V, W)$ then $N_t(hf) = N_t(h)$ (for if $(x,u) \in M \times [1,\infty) - h(M \times (2,\infty))$ then

$$\alpha_t(h,x,u) = \alpha_t(hf,x,u) = 4t).$$

So E induces a homotopy $\bar{E}: K'(W) \times I \to K(W \times S^1)$. Since

$$\begin{aligned} &eE_1'(h)\big(M \times [0,\infty) - h(M \times (2,\infty))\big) \\ &= e\big(M \times [4,\infty) - h(M \times (6,\infty)), 4\big) \\ &= e\big(M \times [4,\infty) - h(M \times (6,\infty)), 0\big) \\ &= eE_1'(h)\big(M \times [4,\infty) - h(M \times (6,\infty))\big), \end{aligned}$$

we have

$$\begin{aligned} N_1(h) &= eE_1'(h)(M \times [0,4]) \\ &= eE_1'(h)\big(h(M \times [2,6])\big). \end{aligned}$$

If $x \in M$, $u \in [2,6]$ then

$$\begin{aligned} E_1'(h)(h(x,u)) &= H(h)\big(h(x,u), \phi(u-2)\big) \\ &= \big(h(x, u+\phi(u-2)), \phi(u-2)\big) \\ &= (h \times 1)E_1'(1)(x,u), \end{aligned}$$

so $N_1(h) = (h \times 1)N_1(1)$.

There is an isotopy $L': I \to \text{Emb}(M \times S^1, V \times S^1)$ such that L'_1 is the inclusion and $L'_0(M \times S^1) = N_1(1)$. By the isotopy extension theorem, there are isotopies $\overline{L}, \overline{L}': I \to \text{Diff}(V \times S^1)$ such that $\overline{L}_0 = \overline{L}'_0 = 1$ and $L_t = \overline{L}_t | M \times S^1$, $L'_t = \overline{L}'_t | M \times S^1$. Now $L, E, (h \times 1)L'$ combine to give a homotopy

$$J: K'(W) \times I \to K(W \times S^1) \quad \text{with} \quad J_0(h) = M \times S^1$$

and $J_1(h) = (h \times 1)(M \times S^1)$. So J is a homotopy from a constant map to si, as required.

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