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Splitting off H-spaces and Conner-Raymond Splitting Theorem

dedicated to Akio Hattori

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Introduction

Conner and Raymond studied the action of the torus $T^k = S^1 \times S^1 \times \cdots \times S^1$ on a T^k-space. The *orbit map* $w: T^k \to X$ is induced by the action of T^k on a basepoint in X. The Conner-Raymond Splitting Theorem is first stated and proved in [CR, Theorem 3.1] and is restated and proved in [G3, Corollary 4] in the following form:

THE CONNER-RAYMOND SPLITTING THEOREM. *If a torus T^k acts on X so that evaluation at a point gives a map $w: T^k \to X$ so that $w_*(H_1(T^k))$ is a direct summand of $H_1(X)$ of rank k, then X is equivariantly homeomorphic to $T^k \times Y$ for some space Y where T^k acts on the product by $g(h, y) = (gh, y)$.*

The above statement will be referred to as the Conner-Raymond Splitting Theorem. This is a particular case of the original version [CR], in which the factor (L, ϕ) is the identity map of $\pi_1(X, x)$, the kernel H is trivial and π_1 is replaced by $H_1(\ , Z)$.

Oprea [O, Theorem 11] has independently of Gottlieb found a Conner-Raymond Splitting Theorem for the case that k equals 1.

In this paper we prove the following generalization of the Conner-Raymond Splitting Theorem. G is assumed to be a compact Lie group. X is a completely regular pathconnected G-space and $w: G \to X$ is the orbit map. X and G are assumed to be homotopy equivalent of CW complexes. $\underline{G}$ is the identity map of G. Let T_G be the functor from the homotopy category to the category of groups, sending a space X to the group $[X, G]$, and $w^\sharp: [X, G] \to [G, G]$ be the morphism induced by precomposing with w, sending a homotopy class $f: X \to G$ to $f \circ w: G \to G$.

THEOREM 3.1. *The following statements are equivalent:*

(i) *X is isomorphic as a G-space to $G\times(X/G)$, where G acts diagonally, by multiplication on the first factor and trivially on the second.*
(ii) *$w^\sharp$ has a right inverse, that is $w^\sharp$ is onto.*
(iii) *w has a left inverse $r: X\to G$.*

In the particular case $G=T^k$, Theorem 3.1 implies the Conner-Raymond Splitting Theorem, as explained in 3.2 below.

A proof of Theorem 3.1 follows from Theorem 1.3 below. This is a splitting theorem which characterizes when a given space is a cartesian product of an H-space. Again all spaces involved are assumed to be homotopy equivalent to CW complexes.

1.3. SPLITTING THEOREM. *Given spaces X, K and Y the following statements are equivalent:*
(i) *K is an H-space and there exists a space Y such that X is homotopy equivalent to $K\times Y$.*
(ii) *There is a class $i: K\to X$ in the generalized Gottlieb set $G(K, X)$ (as defined by Varadarajan [V]) such that $i^\sharp$ has a right inverse (that is $i^\sharp$ is onto).*
(iii) *There are classes i in $G(K, X)$ and $r: X\to K$ such that r is a left inverse for i (and has a fiber Y).*

1.7 below shows that when K equals $S^1\times\cdots\times S^1$, Theorem 1.3 implies the main theorem of [G3] (the theorem on page 216) and Theorem 10 of [O].

As a corollary, any of the above conditions and the fact that K is a finite non contractible H-space, both imply (Corollary 1.8 below) that the Euler characteristic of X is zero.

The proof of Theorem 1.3 involves the ideas of the evaluation subgroup, studied by Gottlieb [G1] [G2] and followed by the work of Varadarajan [V].

This work is organized as follows: Theorem 1.3 is the main result of Section 1. Section 2 contains the proof of Theorem 2.2, which is a dual of Theorem 1.3, and gives necessary and sufficient conditions for splitting co-H-spaces as wedge summands. Section 3 contains the proof of Theorem 3.1. Section 4 gives homology conditions for splitting Eilenberg-Maclane spaces off a space.

All spaces considered are assumed to be homotopy equivalent to CW complexes. In this paper (co-)H-spaces do not necessarily have (co-)products which are (co-)associative.

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§1. Splitting H-spaces off a space

This Section reviews the definition of the Gottlieb set as defined by Varadarajan [V], and uses it to give necessary and sufficient conditions for the splitting of an H-space K. From 1.3 on, every space is assumed to be homotopy equivalent to a CW complex.

1.1 The Gottlieb set

Given spaces K and X the *Gottlieb set* denoted $G(K, X)$ is the subset of all the homotopy classes f in $[K, X]$ such that the diagram

$$\begin{array}{ccc} K\times X & \dashrightarrow & X \\ \uparrow j & & \uparrow \nabla \\ K\vee X & \xrightarrow{f\vee \underline{X}} & X\vee X, \end{array}$$

(in which j is the inclusion and ∇ is the folding class) has a class $\mu: K\times X\to X$ making it commute. This definition of Varadarajan [V] specializes to the definitions of the evaluation group given by Gottlieb [G1, G2], when K is a sphere.

The following lemma is a corollary to [V, 2.2].

1.2 LEMMA. *A class f in the Gottlieb set has the property that the image of $\pi_1(f)$ lies in the center of $\pi_1(X)$.* □

Suppose X, K and Y are homotopy equivalent to CW complexes in the following Theorem.

A necessary and sufficient condition for splitting H-spaces

1.3 SPLITTING THEOREM. *The following conditions are equivalent:*
(i) *K is an H-space and there exists a space Y such that X is homotopy equivalent to $K\times Y$.*
(ii) *There is a class i in $G(K, X)$ such that $i^\sharp: [X, K]\to[K, K]$ has a right inverse, (that is $i^\sharp$ is onto).*

(iii) *There are classes i in $G(K, X)$ and $r: X\to K$ such that r is a left inverse for i (and has a fiber Y).*

PROOF. (i)$\Longrightarrow$(iii)

Assuming (i) there are classes $f: X\to K\times Y$ and $g: K\times Y\to X$ which are homotopy inverses of one another. The map i can be defined as the composition $K\xrightarrow{i_K}K\times Y\xrightarrow{g}X$ and r can be defined as $X\xrightarrow{f}K\times Y\xrightarrow{\pi_K}K$. Then $r\circ i$ equals $\pi_K\circ f\circ g\circ i_K$ which is $\underline{K}$. Thus i has r as a left inverse. Let $m: K\times K\to K$ be the product of K. There is the pairing $K\times K\xrightarrow{\underline{K}\times f}K\times K\times Y\xrightarrow{m\times \underline{Y}}K\times Y\xrightarrow{g}X$, which restricts to $i\vee \underline{X}$ on the wedge, and thus i is in $G(K, X)$.

(iii)$\Longrightarrow$(ii)

$r^\#$ is a right inverse for $i^\#$.

(ii)$\Longrightarrow$(i)

A right inverse $\bar{r}$ for $i^\#$ can be chosen, making $\bar{r}(\underline{K})=r$ a left inverse to i. The map $h: Y\to X$ denotes the inclusion of the homotopy fiber of r. By [M], ΩK is homotopy equivalent to a CW complex. Thus in the fiber sequence $\Omega K\to Y\to X$ the base and fiber are homotopy equivalent to a CW complex, and hence by [S], Y is homotopy equivalent to a CW complex.

The fact that i is in $G(K, X)$ implies that there is a class $\mu: K\times X\to X$ making diagram in Section 1.1 commute.

The composition $r\circ\mu\circ(\underline{K}\times i)$ is a class $K\times K\to K$ establishing the fact that K is an H-space.

The composition $\mu\circ(\underline{K}\times h)$ is a class $K\times Y\to X$ denoted g in the following diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{i_Y} & K\times Y & \xrightarrow{\text{projection}} & K \\ \| & & \downarrow{\scriptstyle g} & & \| \\ Y & \xrightarrow{h} & X & \xrightarrow{r} & K, \end{array}$$

in which i_Y denotes the composition $Y\hookrightarrow K\vee Y\xrightarrow{j}K\times Y$.

By the definition of g and the diagram in Section 1.1, the left square commutes, while the right square commutes after π_* is applied. Thus g induces an isomorphism of homotopy groups, and as all spaces involved are homotopy equivalent to CW complexes, it follows that g is a homo-

topy equivalence. □

In the next lemma n is an integer, $\pi_1, \pi_2, \cdots, \pi_k$ are abelian groups, Π is $\pi_1\oplus\cdots\oplus\pi_k$, T equals $K(\pi_1, n)\times K(\pi_2, n)\times\cdots\times K(\pi_k, n)$, X is a space homotopy equivalent to a CW complex, and i is a class $T\to X$.

1.4 LEMMA. *The following conditions are equivalent:*

(i) *There exists $r: X\to T$ which is the left inverse of i.*

(ii) *$H_n(i;\Pi)$ has a left inverse, (that is, it is a split injection).*

(iii) *$H_n(i;\pi_j)$ has a left inverse, for all j, $1\le j\le k$.*

PROOF. (i)$\Longrightarrow$(ii) is trivial by functoriality.

(ii)$\Longrightarrow$(i).

A left inverse $\bar{r}$ to $H_n(i)$ is an element in $\mathrm{Hom}(H_n(X), \Pi)$. There is the following sequence of homomorphisms:

$$\mathrm{Hom}(H_n(X), \Pi)\xrightarrow{un}H^n(X, \Pi)\cong[X, T],$$

in which the homomorphism denoted un is part of the universal coefficient formula. Thus there is $r: X\to T$ so that $H_n(r;\Pi)$ equals $\bar{r}$. The fact that $\bar{r}$ is an inverse for $H_n(i;\Pi)$ implies that r is an inverse for i.

(ii)$\Longleftrightarrow$(iii) is clear. □

In the following version of the splitting theorem, T is a finite product of Eilenberg-Maclane spaces $\times_{m=1}^{l} K(\pi_m, n_m)$.

1.5 SPLITTING THEOREM FOR T. *The following conditions are equivalent:*

(i) *X is homotopy equivalent to $T\times Y$ for some Y.*

(ii) *There is $i: T\to X$ in $G(T, X)$ such that for all m, $1\le m\le l$, $H_{n_m}(i;\pi_{n_m})$ is a split embedding.* □

In 1.6 and 1.7 below, T^k is $S^1\times\cdots\times S^1=(S^1)^k$.

1.6 The Hurewicz rank and the toral number

The following definitions appear in [G3].

"Let G be a subgroup of $\pi_1(X)$. Define the *Hurewicz rank of* G as follows. Consider the image of G under the Hurewicz homomorphism h in the homology group. Then $h(G)$ may contain free summands of $H_1(X)$. We say that the *Hurewicz rank of* G is the maximum rank of these free summands. If there is no free summand in $h(G)$ then we say the Hurewicz rank of G is zero and if there is no maximum we say the Hurewicz rank of G is infinite."

The *toral number of* X is the biggest non negative integer k such that X is homotopy equivalent to $T^k \times Y$ for some space Y.

The following theorem is the main theorem of [G3], for a space X homotopy equivalent to a CW complex. T^k is the torus $S^1 \times S^1 \times \cdots \times S^1$. A generalization of this theorem appears in Theorem 4.9 below.

1.7 THEOREM. *The Hurewicz rank of X equals the toral number of X.*

PROOF. By 1.5, the toral number of X equals to the biggest integer k such that there is an element i in $G(T^k, X)$ for which $H_1(i; Z)$ is a split injection. Thus it is left to show that this k equals the Hurewicz rank of X. Given that the Hurewicz rank of X is a number m, there exist m classes $i_\theta, 1 \leq \theta \leq m$, each lying in the Gottlieb group, and such that for all $\theta, H_1(i_\theta; Z)$ is a split injection into a different summand of $H_1(X; Z)$. There are classes $\mu_\theta, 1 \leq \theta \leq m$ which make the diagram in Section 1.1 commute. A class $\lambda: T^k \times X \to X$ can be defined as the composition $\mu_1 \circ (\underline{S^1} \times \mu_2) \circ \cdots \circ (\underline{(S^1)^{\times(k-1)}} \times \mu_m)$. This class restricts to a class p on $T^m \times \{*\}$, and establishes the fact that p is in $G(T^m, X)$. As λ extends all μ_θ on the axes, it follows that $H_1(p; Z)$ equals $\bigoplus_{\theta=1}^{m} H_1(i_\theta; Z)$. Thus k is greater than or equal to m. The restricting of λ to μ_θ shows the other inequality and the proof follows. □

Using the known fact that every finite non contractible H-space has Euler characteristic equal to zero, the following corollary is obtained:

1.8 COROLLARY. *In the case that K is a finite non contractible H-space and any of the conditions of* 1.3 *holds, then it follows that the Euler characteristic of X is zero.* □

§ 2. Splitting co-H-spaces off a space

This Section reviews the definition of the dual Gottlieb set as defined by Varadarajan [V], and uses it to give necessary and sufficient conditions for splitting of a connected simply connected co-H-space K, which is homotopy equivalent to a CW complex. The proofs are dual to those

of Section 1, and thus are only sketched. From 2.2 on, every space is assumed to be homotopy equivalent to a connected **simply connected** CW **complex.**

2.1 The dual Gottlieb set

Given spaces K and X the *dual Gottlieb set* denoted $DG(X, K)$ is the subset of all the homotopy classes f in $[X, K]$ such that the diagram

$$\begin{array}{ccc} X & & K \vee X \\ \Delta \downarrow & & \downarrow j \\ X \times X & \xrightarrow{f \times \underline{X}} & K \times X \end{array}$$

(in which j is the inclusion and Δ is the diagonal class) has a class $\rho: X \to K \vee X$ making it commute.

X and K are homotopy equivalent to connected and simply connected CW complexes in the following Theorem.

A necessary and sufficient condition for splitting co-H-spaces

2.2 THEOREM. *The following conditions are equivalent:*

(i) *K is a co-H-space and there exists a simply connected space Y such that X is homotopy equivalent to $K \vee Y$.*

(ii) *There is a class i in $DG(X, K)$ such that $i_\sharp: [K, X] \to [K, K]$ has a right inverse, that is $i_\sharp$ is onto.*

(iii) *There are classes i in $DG(X, K)$ and $r: K \to X$ such that r is a right inverse of i (and has a cofiber Y).*

PROOF. (i)$\Longrightarrow$(iii).

The assumptions in (i) provide $f: X \to K \vee Y$, $g: K \vee Y \to X$ and $\rho: K \to K \vee K$. The proof is dual to the proof of (i)$\Longrightarrow$(iii) in 1.3.

(iii)$\Longrightarrow$(ii)

$r_\sharp$ is a right inverse for $i_\sharp$.

(ii)$\Longrightarrow$(i)

The one sided inverse of $i_\sharp$ implies the existence of r. The fact that i is in $DG(X, K)$ implies the existence of the following diagram cofiber sequences:

$$\begin{array}{ccccc} Y & \xleftarrow{\text{projection}} & K\vee Y & \xleftarrow{i_K} & K \\ \| & & \uparrow g & & \| \\ Y & \xleftarrow{h} & X & \xleftarrow{r} & K. \end{array}$$

Thus g implies homology isomorphism, and as all spaces are simply connected, the assertion follows. □

§ 3. The Conner-Raymond Splitting Theorem

From now on G is a compact Lie group acting on a completely regular path connected space X when both are homotopy equivalent to CW complexes. The orbit map $w: G\to X$ is the composition $G\to G\vee X\to G\times X\to X$ which by definition is in $G(G, X)$. In this Section a Theorem is proved which generalizes the Conner-Raymond Splitting Theorem appearing in the introduction.

3.1 THEOREM. *The following statements are equivalent:*

(i) *X is isomorphic as a G space to $G\times(X/G)$, where G acts diagonally, by multiplication on the first factor and trivially on the second.*

(ii) *$w^\sharp$ has a right inverse, that is $w^\sharp$ is onto.*

(iii) *w has a left inverse $r: X\to G$.*

PROOF. The proof of (i)⟹(iii) is trivial, as in this case w is the inclusion of G in $G\times(X/G)$ and r is the projection of $G\times(X/G)$ on G.

The proof of (iii)⟹(ii) is trivial by applying $[\ \ , G]$.

(ii)⟹(i)

The right inverse of $w^\sharp$ implies the existence of a left inverse r for w, forming the following diagram:

$$\begin{array}{ccccc} & & G & & \\ & & \uparrow r & & \\ G & \xrightarrow{w} & X & \xrightarrow{q} & X/G, \\ & & \uparrow h & & \\ & & Y & & \end{array}$$

in which $h: Y\to X$ is the inclusion of the homotopy fiber of r and q maps X to the quotient under the action of G. As $r\circ w$ is the identity

class, and G is a compact group it follows that the isotropy group at the basepoint is the trivial subgroup, because if the isotropy subgroup were the non trivial H, then $H_n(w): H_n(G, Z)\to H_n(G/H, Z)$ would have had no inverse, where n is the dimension of G. The fact that X is path connected implies that the isotropy group is trivial at every point. Thus by [B, II, 5.8] the quotient map $X\xrightarrow{q}X/G$ is a principal bundle with fiber G.

Two continuations of the proof are presented. One that uses Theorem 1.3 above and one that does not. Suppose we have a short exact sequence of abelian groups $G\xrightarrow{w}X\xrightarrow{q}X/G$. Then a one sided inverse to either w or q implies that the group X is the direct sum of the groups G and X/G. In the topological setup, the proof using 1.3 has an inverse to q while the other proof uses an inverse to w in some sense.

Using the fact that w is in the Gottlieb set and that $r\circ w$ is the identity class, the assumptions of Theorem 1.3 (iii) hold, and thus X is homotopy equivalent to $G\times Y$. Thus $\pi_*(X)$ equals $\pi_*(G)\oplus\pi_*(Y)$. This can be plugged into the long exact sequence of the fibration $q: X\twoheadrightarrow X/G$. It follows that $q\circ h: Y\to X/G$ induces an isomorphism on homotopy groups. As X and G are homotopy equivalent to CW complexes, so is Y, by [M] and by [S]. Also it follows that X is an ANR. By [H], being an ANR is a local property so that this property is preserved in X/G, which is therefore homotopy equivalent to a CW complex. Thus $q\circ h$ is a homotopy equivalence. Let β be a homotopy inverse, then $h\circ\beta: X/G\to X$ is a homotopy section of q, and by the covering homotopy property, there is a cross section $\gamma: X/G\to X$. Thus the principal bundle X is isomorphic to trivial principal bundle $G\times(X/G)$.

Another way to see that $q: X\twoheadrightarrow X/G$ is a trivial principal bundle consists of observing that X homotopy retracts into its fiber creating the following diagram

$$\begin{array}{ccccc} G & \xrightarrow{w} & X & \xrightarrow{q} & X/G \\ \| & & \downarrow r\times q & & \| \\ G & \xrightarrow{i_G} & G\times X/G & \xrightarrow{\pi} & X/G, \end{array}$$

in which i_G is the inclusion and π is the projection. Then $r\times q$ can be shown to be a fiber homotopy equivalence, by using the homotopy exact sequence and Dold's theorem [Do, Theorem 6.3] on fiber homotopy equivıces. This implies isomorphism between the two bundles. □

In the following corollary, π is an abelian group, n is an integer G is the group $K(\pi, n)^{\times k}$, denoted T^k. The proof is trivial using 1.3 1.4.

3.2 COROLLARY. *The following statements are equivalent:*

(i) *X is isomorphic as a T^k space to $T^k \times (X/T^k)$, where T^k acts gonally, by multiplication on the first factor and trivially on the nd.*

(ii) *X is homotopy equivalent to $T^k \times (X/T^k)$.*

(iii) *$H_n(w;\pi)$ induces a split embedding of $H_n(T^k;\pi)$ into $H_n(X;\pi)$.* □

Remarks about conditions for 3.1

The classical Conner-Raymond Splitting Theorem, follows from 3.2 he case that n equals one and π equals Z.

The restriction that G is a compact Lie group can not removed. For mple the group of the real numbers acts on S^1, and any map $r: S^1 \to R$ n inverse for w, but it is not true that S^1 is homeomorphic to $R^1 \times Y$ some Y.

The rank theorem

The purpose of this section is to generalize Theorem 1.7 above so $K(\pi, n)$ replaces S^1. Thus an integer k is determined for X, such k is biggest with the property that X is homotopy equivalent to Y for some Y, where T^k is the kth fold cartesian power of $K(\pi, n)$ π is an abelian group. All spaces are assumed to be homotopy valent to CW complexes.

The extending group

For a given compact space X homotopy equivalent to a CW complex, space of self maps of X forms an associative monoid, denoted X^X. space is homotopy equivalent to a CW complex by [M]. The sub- e of all self maps of X which are homotopy equivalences is a sub- oid denoted X^X_{he}, and the subset of all self maps homotopic to $\underline{X}$ is connected component of $\underline{X}$ in X^X, which is also a submonoid homotopy valent to a CW complex, denoted $X^X_{\underline{X}}$. It follows that $X^X_{\underline{X}}$ is a group-space, [W, p. 461].

It follows that the set of homotopy classes $[K, X^X_{\underline{X}}]$ is a group, for

every connected space K. Since X is compactly generated, this group the is isomorphic to the subset of $[K \times X, X]$ which we denote $E(K, X)$ and call the *extending group*, consisting of all classes $\mu: K \times X \to X$ such that μ restricts on $\{*\} \times X$ to the class $\underline{X}$, with $\mu + \nu$ equaling:

$$K \times X \xrightarrow{\Delta \times \underline{X}} K \times K \times X \xrightarrow{K \times \nu} K \times X \xrightarrow{\mu} X.$$

The existence of the extending group does not imply that the Gottlieb set is a group, because of the fact that a given class f in $G(K, X)$ may have many classes μ in $E(K, X)$ extending $f \vee \underline{X}$.

In the following, the image of the Gottlieb set under functors is considered, as in [Du].

4.2 The image of the Gottlieb set under a functor

Given a cofunctor $\mathcal{F}: HOM \to C$ from the homotopy category, the subset $\{\mathcal{F}(f); f \in G(K, X)\}$ of $HOM_C(\mathcal{F}(K), \mathcal{F}(X))$ is called the *$\mathcal{F}$ image of the Gottlieb set* and will be denoted by $G(K, X, \mathcal{F})$.

4.3 LEMMA. (i) *If $\mathcal{F}$ carries products to products, it carries $E(K, X)$ to $E(\mathcal{F}(K), \mathcal{F}(X))$.*

(ii) *If for any two classes μ, ν in $E(K, X)$ extending $f \vee \underline{X}$ it holds that $\mathcal{F}(\mu)$ equals $\mathcal{F}(\nu)$, then it follows that $G(K, X, \mathcal{F})$ is a monoid subset of $HOM_C(\mathcal{F}(K), \mathcal{F}(X))$.*

(iii) *If in addition to the assumptions of (i) and (ii), $\mathcal{F}(X)$ is a group, and for every μ extending $f \vee \underline{X}$ it holds that $\mathcal{F}(\mu)$ equals $\mathcal{F}(f) + \underline{\mathcal{F}(X)}$ where the addition is taken in $\mathcal{F}(X)$, then it follows that $G(K, X, \mathcal{F})$ is a subgroup of $HOM_C(\mathcal{F}(K), \mathcal{F}(X))$.*

PROOF. Proof of (i) is trivial.

Proof of (ii). By the assumption for any f in $G(K, X)$, $\mathcal{F}(f)$ can be presented as $\mathcal{F}(K \xrightarrow{i_K} K \times X \xrightarrow{\mu} X)$ and this presentation does not depend on the choice of μ in $E(K, X)$ extending $f \vee \underline{X}$. The operation in $G(K, X, \mathcal{F})$ can be defined by $\mathcal{F}(f) + \mathcal{F}(g) = \mathcal{F}(K \xrightarrow{i_K} K \times X \xrightarrow{\mu+\nu} X)$, where $\mu + \nu$ is the class resulting by adding μ and ν in the monoid $E(K, X)$ (mentioned in 4.1). It is easy to check that $G(K, X, \mathcal{F})$ is an associative monoid.

Proof of (iii). By definition $\mathcal{F}(f+g)$ equals $\mathcal{F}(K) \xrightarrow{\mathcal{F}(i_K)} \mathcal{F}(K) \times \mathcal{F}(X) \xrightarrow{\mathcal{F}(\Delta) \times \mathcal{F}(\underline{X})} \mathcal{F}(K) \times \mathcal{F}(K) \times \mathcal{F}(X) \xrightarrow{\mathcal{F}(\underline{K}) \times \mathcal{F}(\nu)} \mathcal{F}(K) \times \mathcal{F}(X) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(X)$, which can be presented as $\mathcal{F}(K) \xrightarrow{\mathcal{F}(\Delta)} \mathcal{F}(K) \times \mathcal{F}(K) \xrightarrow{\mathcal{F}(i)} \mathcal{F}(K) \times \mathcal{F}(K) \times$

$X) \xrightarrow{\mathcal{F}(\underline{K})\times(\mathcal{F}(g)+\mathcal{F}(\underline{X}))} \mathcal{F}(K)\times\mathcal{F}(X) \xrightarrow{\mathcal{F}(f)+\mathcal{F}(\underline{X})} \mathcal{F}(X)$, which is the same as $K) \xrightarrow{\mathcal{F}(\Delta)} \mathcal{F}(K)\times\mathcal{F}(K) \xrightarrow{\mathcal{F}(f)\times\mathcal{F}(g)} \mathcal{F}(X)\times\mathcal{F}(X) \xrightarrow{\text{action}} \mathcal{F}(X)$, which is the ition in $HOM_c(\mathcal{F}(K), \mathcal{F}(X))$. The inverse of $\mathcal{F}(f)$ is the composition $K) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(X) \xrightarrow{\text{inverse}} \mathcal{F}(X)$. □

4.4 *Examples.* (i) Let $\mathcal{F}$ be π_n. Then for every μ extending $f\vee\underline{X}$ have that $\pi_n(\mu)$ equals $\pi_n(f)\oplus\underline{\pi_n(X)}$. Thus it follows that $G(K, X, \pi_n)$ ι subgroup of $\text{Hom}(\pi_n(K), \pi_n(X))$.

(ii) Let K be an n-connected space and $\mathcal{F}$ be $H_n(\ ;\pi)$. Then for ry μ extending $f\vee\underline{X}$, $H_n(\mu;\pi)$ equals $H_n(f;\pi)\oplus\underline{H_n(X;\pi)}$. Thus it ows that $G(K, X, H_n(\ ;\pi))$ is a subgroup of $\text{Hom}(\underline{H_n(K;\pi)}, H_n(X;\pi))$.

4.5 COROLLARY. *The union HI of the images of all $H_n(f;\pi)$ as f ies in $G(K(\pi,n), X)$ is a subgroup of $H_n(X;\pi)$.*

PROOF. Given a subgroup S of $\text{Hom}(H_n(K;\pi), H_n(X;\pi))$, the union ıll the images of all maps in S is a subgroup of $H_n(X;\pi)$. □

The Hurewicz image of $G(K(\pi,n), X)$

HI is called the *Hurewicz image of* $G(K(\pi,n), X)$.

From now on the image of $H_n(f;\pi)$ will be denoted $\text{Im}(f)$.

4.7 LEMMA. *Given a subgroup SG of HI which is of the form* $f_1)\oplus\cdots\oplus\text{Im}(f_k)$, *then*

(i) *There is a class g in $G(T^k, X)$ so that* $\text{Im}(g)$ *equals SG.*

(ii) *If all $H_n(f_i;\pi)$ are one to one maps, so is $H_n(g;\pi)$.*

(iii) *If all $H_n(f_i;\pi)$ are injective maps, so is $H_n(g;\pi)$.*

PROOF. Each f_i has a class μ_i in $E(K(\pi,n), X)$ establishing the fact ; f_i is in $G(\pi, n, X)$. Thus $\mu: K(\pi,n)^{\times k}\times X\to X$ can be defined as $\mu_2+\cdots+\mu_k$ in $E(K(\pi,n)^{\times K}, X)$. It has a restriction $g: K(\pi,n)^{\times k}\times$ $\to X$ which is in $G(K(\pi,n)^{\times k}, X)$, and $H_n(g;\pi)$ equals $\bigoplus_{i=1}^{k} H_n(f_i;\pi)$. A ct sum of maps is one to one, provided that the direct summands Given left inverses for the summands, they imply the existence ɜft inverses r_i for f_i respectively. The sum of the r_i's in $[X, K(\pi,n)]$ ces a left inverse for $H_n(g;\pi)$. □

4.8 The injective Hurewicz rank of $G(K(\pi,n), X)$ and the (π,n) splitting number of X

The *injective Hurewicz image* is the biggest subgroup of HI which is a direct sum of $\pi\otimes\pi$'s, all of which split in $H_n(X;\pi)$. The number of copies of $\pi\otimes\pi$ is called *the injective Hurewicz rank of* $G(K(\pi,n), X)$. By 4.7 there is some g in $G(K(\pi,n)^{\times k}, X)$ so that $H_n(g;\pi)$ has a left inverse, and $\text{Im}(g)$ equals $(\pi\otimes\pi)^{\oplus \text{the injective Hurewicz rank}}$.

In the case when $n=1$ and $\pi=Z$, this specializes to the Hurewicz rank of $G(X)$ as in [G3], mentioned in 1.6 above. A similar invariant using fundamental group and 4.4 (i) above is mentioned in [L].

The (π,n) *splitting number of* X is the biggest integer k such that there exists a space Y and a homotopy equivalence of X with $K(\pi,n)^{\times k}\times Y$.

In the case when $n=1$ and $\pi=Z$, this specializes to the toral number of [G3], as mentioned in 1.7 above.

The following rank theorem generalizes the main theorem of [G3].

4.9 THE RANK THEOREM. *The (π,n) splitting number of X equals the injective Hurewicz rank of $G(K(\pi,n), X)$.*

PROOF. By 1.3 the (π,n) splitting number of X is the biggest number k such that there exists a class $i: T^k=K(\pi,n)^{\times k}\to X$ in $G(T^k, X)$ with a left inverse. By 1.4 this equals the biggest number k such that there exists i in $G(T^k, X)$ and $H_n(i;\pi)$ has a left inverse. By 4.7 this is the injective Hurewicz rank of $G(K(\pi,n), X)$. □

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