MA 174 FINAL EXAM 5/7/99

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We will use the following three vector fields in this exam. The position vector field:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

The Coulomb vector field:

$$\mathbf{E} = \frac{\mathbf{r}}{\|\mathbf{r}\|^3} = \frac{1}{(\sqrt{x^2 + y^2 + z^2})^3}\ (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

The rotation vector field:

$$\mathbf{R} = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}.$$

1.

(5 pts) a) For suitable $\mathbf{F}$ and region D, state the divergence theorem.

(5 pts) b) For suitable $\mathbf{F}$ and surface S, state Stokes' theorem.

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2. Recall $\mathbf{r}$ is the position vector field.

(5 pts) e) If the volume enclosed by a closed surface S equals 3π, what is the flux $\displaystyle\iint_S \mathbf{r}\cdot\mathbf{n}d\sigma$ equal to?

(5 pts) f) Show there is no closed surface S whose normal vector field $\mathbf{n}$ is always perpendicular to $\mathbf{r}$.

An alternative problem for 5c), worth five points:

Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

3) Recall $\mathbf{E} := \dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}$ is the coulomb vector field.

(10 pts) a) Show $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$.

(10 pts) b) Show $\boldsymbol{\nabla}\left(\dfrac{-1}{r}\right) = \dfrac{\mathbf{r}}{r^3} = \mathbf{E}$ where $r = \sqrt{x^2+y^2+z^2} = \|\mathbf{r}\|$.

(5 pts) c) Find the flux $\displaystyle\iint_S \mathbf{E} \cdot \mathbf{n} d\sigma$ for a closed surface S not containing $\mathbf{0}$.

(5 pts) d) Find the flux for S containing $\mathbf{0}$.

(5 pts) e) Find $\displaystyle\int_C \mathbf{E} \cdot \mathbf{n} ds$ where C is a circle in the x–y plane about the origin with radius 1.

4. Suppose $\mathbf{E}_1 = \dfrac{(\mathbf{r}-\mathbf{a})}{\|\mathbf{r}-\mathbf{a}\|^3}$ where $\mathbf{a} = (1,0,0)$ and let $\mathbf{E}_2 = \dfrac{\mathbf{r}-\mathbf{b}}{\|\mathbf{r}-\mathbf{b}\|^3}$ where $\mathbf{b} = (0,2,0)$.

(0 pts) a) True or False: The fact that the divergence of $\mathbf{E}$ is zero leads you to believe the same for $\mathbf{E}_1$ and the fact that the flux of $\mathbf{E}$ through a surface S depends only on whether $\mathbf{0}$ is enclosed in S or not enclosed in S leads you to believe the flux of $\mathbf{E}_1$ through S only depends on whether $\mathbf{a}$ is enclosed by S or not.

(0 pts) b) True or False: Your calculus skills are such that you could verify the mathematical statements in part a).

(5 pts) c) Evaluate $\nabla \cdot (\mathbf{E} + \mathbf{E}_1 + \mathbf{E}_2)$.

(5 pts) d) What is the flux of $\mathbf{E} + \mathbf{E}_1 + \mathbf{E}_2$ through a sphere S of radius $1/2$ centered at $(0,0,1)$?

5.

(5 pts) a) Show $\boldsymbol{\nabla} \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\boldsymbol{\nabla} \cdot \mathbf{F}_1 + b\boldsymbol{\nabla} \cdot \mathbf{F}_2$ where $\mathbf{F}_1$ and $\mathbf{F}_2$ are plane vector fields and a and b are constants.

(5 pts) b) Show $\boldsymbol{\nabla}(f \cdot g) = g(\boldsymbol{\nabla} f) + f(\boldsymbol{\nabla} g)$ where f and g are differentiable functions of two variables x and y.

(5 pts) c) Suppose $F(x, y) = f(x, y)g(x, y)$ and suppose that $F(x, y)$ has a maximum at $(1, 2)$. If $g(x, y)$ has a minimum at $(1, 2)$ and if f has no critical point at $(1, 2)$, what must $g(1, 2)$ equal?

(5 pts) d) Evaluate $\boldsymbol{\nabla} \cdot (\mathbf{r} + 3\mathbf{E})$.

(10 pts) 6. Suppose the surface S is bounded by the curves C_1, C_2 and C_3. Suppose $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ on S and suppose the circulations $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 1$ and $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -2$ where the integrals are taken in the direction of the arrows in the diagram.

What is $\displaystyle\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$?

7.

(10 pts) a) Find the work done by moving a particle of mass 1 around a circle of radius 1 around the origin of the x–y plane opposite to the direction of the force field $\mathbf{R} = -y\mathbf{i} + x\mathbf{j}$.

(5 pts) b) Calculate the curl, $\boldsymbol{\nabla} \times \mathbf{R}$, of $\mathbf{R}$ and then use Stokes' theorem to get a circulation integral around the boundary of a bounded region A in the x–y plane which gives the area of A.

8. $\displaystyle\int_1^4\int_0^{\sqrt{x}} \frac{3}{2}\, e^{y/\sqrt{x}}\,dy\,dx.$

(10 pts) a) Sketch the region of integration.

(10 pts) b) Reverse the order of integration.

9.

(10 pts) a) What is the unit outward pointing normal vector at the $(1,1,1)$ point on the ellipsoid S given by $x^2 + 2y^2 + 3z^2 = 6$?

(10 pts) b) What is the equation of the tangent plane to S at $(1,1,1)$?

(10 pts) c) S intersects the sphere $(x-1)^2 + y^2 + z^2 = 2$ (of radius $\sqrt{2}$ and center at $(1,0,0)$), in a curve C. Show $(1,1,1)$ is on this curve C.

(10 pts) d) What is the unit tangent $\mathbf{T}$ to C at $(1,1,1)$?

10.

(10 pts) a) Consider the set of all vectors $\mathbf{x}$ in space so that $\|\mathbf{x}-\mathbf{a}\| = b$ where $\mathbf{a}$ is a constant vector and b is a positive number. What quadric surface does this set represent. Can you describe the roles of $\mathbf{a}$ and $\mathbf{b}$?

OR

(alt. 10 pts) b) Where does the line $t(1,-1,1)+(1,1,1)$ intersect the ellipsoid $3x^2+4y^2+2z^2 = 10$?