Math 401
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Vectors

I. Definitions and Properties

Let $\mathbb{E}$ be the space studied by Euclid and the Greeks in 200 BC. Let x be a point in $\mathbb{E}$. Let $\mathbb{E}_x$ be the set of all possible velocities of a particle at x.

Definition 1. The *exponential map* associates to every velocity $\vec{v}_x \in \mathbb{E}_x$, the point in $\mathbb{E}$ which a particle at x reaches after one "second" after traveling in a straight line with velocity $\vec{v}_x$.

Proposition 2. The exponential map is a one to one correspondence between $\mathbb{E}_x$ and $\mathbb{E}$.

Definition 3. Let $a \in \mathbb{R}$ be a real number and $\vec{v} \in \mathbb{E}_x$. We define *scalar multiplication* $a\vec{v}$ to be that velocity so that a particle at x travelling at velocity $a\vec{v}$ reaches a distance $|a|$ times the distance the particle travelling at velocity $\vec{v}$. The two particles travel on the same straight line, in the same direction if $a > 0$, in the opposite direction if $a < 0$, and if $a = 0$, the particle moving with velocity $0\vec{v} = \vec{0}$ does not move.

Definition 4. Let $\vec{v}$ and $\vec{w}$ be in $\mathbb{E}_x$. We define *vector addition* $\vec{v} + \vec{w}$ as that velocity which takes a particle to the point $z \in \mathbb{E}$ in one second, where z the point reached in two seconds by a particle which travels with velocity $\vec{v}$ for one second to reach the point y, and then travels for one second along a velocity $\vec{w} \in \mathbb{E}_y$ for one second to reach z. Here $\vec{w} \in \mathbb{E}_x$ is *equivalent* to $\vec{w} \in \mathbb{E}_y$ by the following definition.

Definition 5. $\vec{v} \in \mathbb{E}_x$ is *equivalent* to $\vec{w} \in \mathbb{E}_y$ if the line segments swept out by the two particles are parallel, of the same length, and are directed in the same way. That means it is possible to construct a parallelogram with the two line segments as opposite sides and the line segment from x to y. If the two line segments lie on a straight line, the motion of the two particles should be in the same direction.

Proposition 6. Let $a, b \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{E}_x$.

a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ — commutivity of addition

b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ — associativity of addition

c) $\vec{u} + \vec{0} = \vec{u}$

d) $\vec{u} + (-\vec{u}) = \vec{0}$

e) $0\vec{u} = \vec{0},\ 1\vec{u} = \vec{u},\ (-1)\vec{u} = -\vec{u}$

f) $a(b\vec{u}) = (ab)\vec{u}$ — associativity of scalar multiplication

g) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ — distributivity

h) $(a + b)\vec{u} = a\vec{u} + b\vec{u}$ — distributivity

Theorem 7. *(The Fundamental Theorem)*

If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{E}_x$ do not lie in a plane, then every vector $\vec{r} \in \mathbb{E}_x$ can be written uniquely as a linear combination, $a\vec{u} + b\vec{v} + c\vec{w} = \vec{r}$.

Definition 8. Three velocity vectors in $\mathbb{E}_x$ which do not lie in a plane are a *basis* for $\mathbb{E}_x$.

II. Using Vectors in Euclidean Geometry

To use vectors to solve Euclidean Geometry problems, the first thing is to *choose* a point x for your $\mathbb{E}_x$. You can choose it off the plane if you want to preserve generality and if you are looking for symmetry. You choose x at a point on the figure if you want to eliminate an equation.

You think of your vectors as velocity vectors in $\mathbb{E}_x$ or as points in $\mathbb{E}$, whichever seems more suitable. You pass from one point of view to the other by the exponential map.

The most important elements of Euclidean geometry are planes, lines, line segments and triangles.

Planes: Let Π be a plane passing through points A, B, C (which are not on a line) and suppose we choose a point x <u>not</u> in the plane. let $\vec{A}, \vec{B}, \vec{C} \in \mathbb{E}_x$ be the vectors associated with A, B, C under the exponential maps. Note $\vec{A}, \vec{B}, \vec{C}$ form a basis of $\mathbb{E}_x$. Then any point in the plane Π is associated under the exponential map *uniquely* to a vector of the form

$$a\vec{A} + b\vec{B} + c\vec{C} \qquad \text{where } a + b + c = 1. \tag{1}$$

Those points on or inside the triangle ABC correspond *uniquely* (under the exponential map) to a vector of form (1) with the condition

$$a \geq 0, b \geq 0, c \geq 0. \tag{2}$$

If x were chosen <u>on</u> the plane, then $\vec{A}, \vec{B}$, and $\vec{C}$ would no longer form a basis.

Lines: Let ℓ be a line passing through two distinct points A and B, and suppose we choose a point x <u>not</u> on the line. Then any point on the line ℓ corresponds uniquely to

$$a\vec{A} + b\vec{B} \qquad \text{where } a + b = 1. \tag{3}$$

If the point lies on the line segment between A and B, then

$$a \geq 0, \ b \geq 0. \tag{4}$$

Note that if $a = 0$, then the corresponding point is B, and if $b = 0$, then the corresponding point is A.

Another way to describe the line ℓ is by the set of points which corresponds to

$$\vec{A} + t\vec{v} \qquad \text{for arbitrary } t \text{ and} \tag{5}$$

$\vec{v} \in \mathbb{E}_x$ which is parallel to the line ℓ. A useful choice of $\vec{v}$ is $\vec{B} - \vec{A}$. Then we can regard the particle as being at A when $t = 0$ and at B when $t = 1$. Thus ℓ is given by

$$\vec{A} + t(\vec{B} - \vec{A}). \tag{6}$$

Let x be chosen at a key point in the diagram, then that point corresponds to the zero vector $\vec{0}$. Then the above equations lose their uniqueness clauses and their symmetry. On the other hand an extra symbol is not needed. The choice of x is the first step in solving the problem.

Another approach is to label the sides of the diagram as vectors. Then the diagrams correspond to relations.

For example figure 1 corresponds to $\vec{A} + \vec{B} = \vec{C}$ and figure 2 corresponds to $\vec{A} + \vec{B} + \vec{C} = \vec{0}$.

Figure 1

Figure 2

To interpret the results of the algebra we note that $\vec{A} + \vec{B} + \vec{C} = \vec{0}$ means the sides and lengths can form a triangle. Also $\vec{A}+\vec{B}+\vec{C}+\vec{D} = \vec{0}$ means that the sides labeled by A, B, C, D can form a quadrilateral. Also $\vec{A} = \dfrac{-1}{3}\vec{C}$ means side C is parallel to side A and 3 times bigger. The minus sign can sometimes be ignored. It means that the vector is directed in the opposite direction, which sometimes isn't important in Euclidean Geometry.

III. Linear Transformations

Mathematicians have learned in the last 50 years that functions are the fundamental concept of Mathematics. Via functions, we can describe how two things can be the same and yet different at the same time.

A *function* (alias map, mapping, transformation, ...) is a rule which assigns to each element in a set, called the *source* (or domain), an element in another set, which is called the *target* (or range) set. The notation $f : A \to B$ is read: The function f which maps the source A into the target B.

The key class of functions to use with vectors are linear transformations.

Definition 9. A *linear transformation* is a mapping $T : \mathbb{E} \to \mathbb{E}$ such that

$$\begin{aligned} T(a\vec{v}) &= aT(\vec{v}) \\ T(\vec{v} + \vec{w}) &= T(\vec{v}) + T(\vec{w}). \end{aligned} \tag{7}$$

Proposition 10. Let T be a linear transformation.

a) $T(a\vec{A} + b\vec{B} + c\vec{C}) = a\vec{A}' + b\vec{B}' + c\vec{C}'$ where

$$\vec{A}' = T(\vec{A}),\ \vec{B}' = T(\vec{B}) \text{ and } \vec{C}' = T(\vec{C}).$$

b) If two linear transformations agree on a basis, then they agree for every vector. That is, if $\vec{A}, \vec{B}, \vec{C}$ is a basis and if $T(\vec{A}) = S(\vec{A}),\ T(\vec{B}) = S(\vec{B})$ and $T(\vec{C}) = S(\vec{C})$, then $T(\vec{v}) = S(\vec{v})$ for any vector.

c) Let $\vec{A}, \vec{B}, \vec{C}$ be a basis, and let $\vec{A}', \vec{B}', \vec{C}'$ be any three vectors. There is a linear transformation T so that $\vec{A}' = T(\vec{A}),\ \vec{B}' = T(\vec{B}),\ \vec{C}' = T(\vec{C})$. If $\vec{A}', \vec{B}', \vec{C}'$ also are a basis, then T is an *isomorphism*.

d) $T(\vec{0}) = \vec{0}$ if T is a linear transformation.

Definition 11. $T : \mathbb{E} \to \mathbb{E}'$ is an *isomorphism* if

a) For every $\vec{w} \in \mathbb{E}'$, there is a $\vec{v} \in \mathbb{E}$ so that $T(\vec{v}) = \vec{w}$, and

b) If $T(\vec{v}) = T(\vec{w})$, then $\vec{v} = \vec{w}$.

If there is an isomorphism between two spaces, we say they are *isomorphic*. The exponential map is an isomorphism, so $\mathbb{E}_x$ and $\mathbb{E}$ are isomorphic. Parallel translation gives an isomorphism between $\mathbb{E}_x$ and $\mathbb{E}_y$.

IV. Using Isomorphism to Solve Geometry Problem

To prove a theoretical theorem involving a triangle, ΔABC, choose a particularly nice triangle $\Delta A'B'C'$ for the problem and define the unique linear transformation which takes $A \to A',\ B \to B'$, and $C \to C'$. Suppose the theorem is easily true for triangle $\Delta A'B'C'$. Then it may be true for ΔABC under the *right* circumstances.

The *wrong* circumstances involve equal angles or equal lengths. The *right* circumstances include midpoints, parallel lines, intersecting lines.

The reason this works follows from the following Proposition.

Proposition 12. Let $T : \mathbb{E} \to \mathbb{E}$ be an isomorphism.

a) T sends a plane not passing through $\vec{0}$ into a plane not passing through $\vec{0}$.

b) T sends a line into a line.

c) T sends line segments into line segments, and midpoints into midpoints, and preserves any ratio of lengths along the line.

d) T sends parallel lines into parallel lines and intersecting lines into intersecting lines.

e) T sends triangles into triangles, quadrilaterals into quadrilaterals, pentagons into pentagons.

To show how this method works, consider the proposition that the medians of a triangle meet in a common point. Let ΔABC be an arbitrary triangle and $\Delta A'B'C'$ be an equilateral triangle. The medians obviously meet in a point in a equilateral triangle. Now the isomorphism which takes ΔABC into $\Delta A'B'C'$ takes medians to medians and intersections to intersections. So the medians of ΔABC also meet in a point.

We summarize the method: Any proposition whose hypotheses and conclusions are invariant under linear transformations is true for all cases if it is true for one case.

V. Axiomatics: Level 0 of Notation

Euclid wrote down his 10 axioms, after much thought and trial and error by Euclid himself and probably by other mathematicians as well. He knew, and his contemporaries knew, that the choice of axioms was arbitrary, subject only to the conditions that they were obviously true, and that they implied most things known about geometry.

Over the centuries, though, the axioms became set in stone. No one dreamed of tampering with them. Only the parallel axiom was played with, to try to show that it was implied by the other nine.

Nowadays mathematicians can propose all kinds of axioms. We do it as follows: We define objects, such as *vector spaces*, by listing their properties. For example, Proposition 6 is transformed into a list of axioms for a vector space. Two vector spaces are the "same" (*isomorphic*) via Definition 11.

Now the description of vectors as velocities and the choice of an x makes $\mathbb{E}$ isomorphic to a vector space in which Theorem 7 holds as an additional axiom. So in effect, we have chosen a different set of axioms for Euclidean Geometry (Proposition 6, Theorem 7). Does this choice imply all the theorems of Euclidean Geometry?

No! Because we haven't defined the idea of a circle or sphere. That is, we haven't defined the notion of the "length" of a vector. Similarly, we haven't defined Area or Volume. We will add these concepts to our vector spaces by means of two choices: A standard length and a standard orientation.

The choice of a standard length, which is called a *metric*, leads to the dot product in Level -1 of notation. The additional choice of orientation leads to the cross product in Level -2 of notation.

The Greeks, by the way, never chose a unit of length or area or volume in their theoretical work. On the one hand this is very *good*, since different choices couldn't cause confusion. A bee knows what it means for one flower to be twice as far or twice as big as another, without the aid of inches or centimeters. So length does not depend on a metric.

On the other hand, picking a metric, and thus expressing length, area, or volume as numbers leads to very convenient ways of expressing information.

For example, the famous formula $A = \pi r^2$ expresses the relation of the area of a circle to its radius much better (even in words) than Euclid's "The areas of two circles are proportional to the squares of their radii." On the other hand, replacing $a : b = c : d$ by $\frac{a}{b} = \frac{c}{d}$ loses something too. For example, the statement "A man is to a woman as a bull is to a cow" makes more sense than "A man divided by a woman is equal to a bull divided by a cow".

VI. The Dot Product: Level -1

We choose a unit of length. Then every vector $\vec{v} \in \mathbb{E}_x$ has a length $\|\vec{v}\| = v \geq 0$. We define the *dot product* of two vectors $\vec{v}$ and $\vec{w} \in \mathbb{E}_x$ as follows:

Definition 13. $\vec{v} \cdot \vec{w} = vw \cos\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$.

Proposition 14.

a) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ — commutativity

b) $a(\vec{v} \cdot \vec{w}) = (a\vec{v}) \cdot \vec{w} = \vec{v} \cdot (a\vec{w})$

c) $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$ — distributivity

d) $\vec{v} \cdot \vec{v} = v^2 \geq 0$, and if $\vec{v} \cdot \vec{v} = 0$, then $\vec{v} = \vec{0}$ — positive definiteness

Definition 15. If $\vec{v} \cdot \vec{w} = 0$, we say that $\vec{v}$ and $\vec{w}$ are *orthogonal*. If $\|\vec{v}\| = 1$, we say that $\vec{v}$ is a *unit vector*. We say $\vec{i}, \vec{j}, \vec{k}$ is an *orthonormal basis* if each vector is a unit vector and $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$.

Proposition 16.

a) Let $\vec{v}$ and $\vec{w}$ be expressed in terms of an orthonormal basis by $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{w} = x'\vec{i} + y'\vec{j} + z'\vec{k}$. Then $\vec{v} \cdot \vec{w} = xx' + yy' + zz'$.

b) If the basis in 16a were just an ordinary basis, then

$$\vec{v} \cdot \vec{w} = (x, y, z) \begin{pmatrix} \vec{i} \cdot \vec{i} & \vec{i} \cdot \vec{j} & i \cdot \vec{k} \\ \vec{j} \cdot \vec{i} & \vec{j} \cdot \vec{j} & \vec{k} \cdot \vec{k} \\ \vec{k} \cdot \vec{i} & \vec{k} \cdot \vec{j} & \vec{k} \cdot \vec{k} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Proposition 16a gives us the Pythagorean Theorem. It forms the foundation of matrix notation.

From the point of view of axiomatics, we let Proposition 14 be additional axioms which are added to the axioms of Proposition 6. The result is what we call an "inner product space."

Two inner product spaces are *isometric* (the same) if there is an *isometry* between them. An *isometry* $T : \mathbb{E}_x \to \mathbb{E}_x$ is an isomorphism which preserves the metric. That is,

$$(T\vec{v}) \cdot (T\vec{w}) = \vec{v} \cdot \vec{w}. \tag{7}$$

This implies that T preserves angles and lengths. If we choose an $x \in \mathbb{E}$ to be our origin, and choose a unit length in $\mathbb{E}$, then $\mathbb{E}$ is isometric to an innerproduct space satisfying Theorem 7. Thus an axiom system comprising Proposition 6, Theorem 7, and Proposition 14 would give us Euclidean Geometry.

VII. The Cross Product: Level -2

We choose an *orientation* of $\mathbb{E}$, either the right hand or the left hand.

Definition 17. Let $\vec{v}$ and $\vec{w} \in \mathbb{E}_x$. We define the cross product $\vec{v} \times \vec{w}$ to be a vector of length $vw|\sin\theta|$ orthogonal to the plane defined by $\vec{v}$ and $\vec{w}$ with the direction fixed by the orientation. (Line the fore finger along $\vec{v}$ and the middle finger along $\vec{w}$, then $\vec{v} \times \vec{w}$ points in the direction of the thumb.)

Unlike the properties in propositions 6 and 14, the cross product is not susceptible to generalization to different dimensions directly. It is a creature of $\mathbb{E}$.

Proposition 18. Elementary Algebra.

a) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ anti–commutivity

b) $a(\vec{A} \times \vec{B}) = (a\vec{A}) \times \vec{B} = \vec{A} \times a\vec{B}$

c) $(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$ distributivity

d) $\vec{A} \times \vec{B} = \vec{0}$ if and only if $\vec{A}$ and $\vec{B}$ are parallel.

Proposition 19. Geometrical Facts.

a) $\vec{A} \times \vec{B}$ is orthogonal to $\vec{A}$ and to $\vec{B}$.

b) $\vec{A} \times \vec{B} =$ (area of $\vec{A}\vec{B}$ parallelogram) (oriented normal unit vector)

c) $(\vec{A} \times \vec{B}) \cdot \vec{C} = \pm$ (volume of the parallelopiped with sides $\vec{A}, \vec{B}, \vec{C}$)

The $+$ sign is taken if $\vec{A}, \vec{B}, \vec{C}$ agree with orientation convention.

Proposition 20. Oriented Orthonormal Basis.

Let $\vec{i}, \vec{j}, \vec{k}$ be an orthonormal basis so that $\vec{i} \times \vec{j} = \vec{k}$. Then suppose

$$\begin{aligned} \vec{A} &= A_1\vec{i} + A_2\vec{j} + A_3\vec{k} \quad \text{and} \\ \vec{B} &= B_1\vec{i} + B_2\vec{j} + B_3\vec{k} \quad \text{and} \\ \vec{C} &= C_1\vec{i} + C_2\vec{j} + C_3\vec{k} \end{aligned}$$

(a)

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = (A_2B_3 - A_3B_2)\vec{i} - (A_1B_3 - A_3B_1)\vec{j} + (A_1B_2 - A_2B_1)\vec{k}$$

(b)

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \det \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Proposition 21. Vector Identities.

a) $(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot (\vec{B} \times \vec{C})$

b) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ (replaces associativity)

c) $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [(\vec{A} \times \vec{C}) \cdot \vec{D}]\vec{B} - [(\vec{B} \times \vec{C}) \cdot \vec{D}]\vec{A}$

d) $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$

VIII. Level -3

In level -2 we have all we need to solve any problem or prove any proposition in Euclidean Geometry (with the exception of constructions with ruler and compass). So in level -2 we have algebracized Geometry. In the next Levels we will arithmetize Geometry. That is we will introduce notation which gives names to every point in $\mathbb{E}$, and in such a way that the vector operations of $+, \cdot$, and $\times$ become arithmetic calculations.

In this level we choose a point $x \in \mathbb{E}$ and call it $\vec{0}$, the origin. With the origin fixed, we can identify the points of $\mathbb{E}$ with unique vectors, and thus we can do all the vector operations directly on the points of $\mathbb{E}$.

IX. Level -6

We choose a basis $\vec{b}, \vec{\beta}, \vec{B}$ for $\mathbb{E}$. Then every point x can be written as a unique linear combination of this basis. For example $\vec{x} = 3\vec{b} - 4\vec{B} + 2\vec{\beta}$. Thus x can be completely described by giving its $\vec{\beta}$ number 2, its $\vec{B}$ number -4, and its $\vec{b}$ number 3.

X. Level -8

We introduce an ordering to the basis. Let us choose $\vec{B}$ to be the first vector, $\vec{b}$ to be the second vector, and $\vec{\beta}$ to be the third vector. Let us change our notation to reflect this by denoting $\vec{B}$ as $\vec{B}_1$, $\vec{b}$ as $\vec{B}_2$, $\vec{\beta}$ as $\vec{B}_3$.

Then $\vec{x} = -4\vec{B}_1 + 3\vec{B}_2 + 2\vec{B}_3$. We can condense this into $\vec{x} = (-4, 3, 2)$. In this notation

$$\vec{x} + \vec{y} = (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3) \tag{8}$$

$$a \cdot \vec{x} = (ax_1, ax_2, ax_3). \tag{9}$$

As a surprise extra benefit, linear transformations are described by means of matrices. This follows from Proposition 10.

Let

$$\begin{aligned} T(\vec{B}_1) &= \vec{B}_1 + 2\vec{B}_2 + 3\vec{B}_3 \\ T(\vec{B}_2) &= 4\vec{B}_1 + 5\vec{B}_2 + 6\vec{B}_3 \\ T(\vec{B}_3) &= 7\vec{B}_1 + 8\vec{B}_2 + 9\vec{B}_3 \end{aligned} \tag{10}$$

Then T is completely described by the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \tag{11}$$

and $T(\vec{x})$ is given by

$$(-4, 3, 2) \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = (22, 23, 24) \tag{12}$$

So matrix arithmetic is a slave of Level -8!!

XI. Level -9

Let us choose our ordered basis to be an orthonormal basis. Let us denote it by $\vec{i}, \vec{j}, \vec{k}$. Then Proposition 16a allows us to express the dot product arithmetically by

$$(x, y, z) \cdot (x', y', z') = xx' + yy' + zz'. \tag{13}$$

XII. Level -10

We require $\vec{i}, \vec{j}, \vec{k}$ to be an *oriented* basis. That means that

$$\vec{i} \times \vec{j} = \vec{k}. \tag{14}$$

Then Proposition 20 allows us to express the cross product arithmetically

$$(x, y, z) \times (x', y', z') = (yz' - zy',\ zx' - xz', xy' - yx') \tag{15}$$

XIII. Level -13 The Standard

We choose symbols x, y, z to represent a variable vector $\vec{X} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z)$. We call the coefficient of $\vec{i}$ *the x coordinate*, the coefficient of $\vec{j}$ *the y coordinate*, and the coefficient of $\vec{k}$ *the z coordinate*. The lines given by $t\vec{i}$ and $t\vec{j}$ and $t\vec{k}$ are called the x, y, and z axes. We can represent the x axis as the solution of two equations: $y = 0$ and $z = 0$.

In fact, that is the power of level -13. Many curves and surfaces can be expressed by equations. Since everybody knows the standard Level -13, information can be quickly and efficiently exchanged.

For example: the intersection of the cylinder $x^2 + y^2 = 1$ with the sphere $x^2 + y^2 + z^2 = 1$. The smallest distance between $(1, 2, 3)$ and the plane $x + 2y + 3z = 4$.

XIV. Calculus and Vectors

We assume that a vector $\vec{X}$ is a function of time t. Then as t varies we see that $\vec{X}$ travels over a path in space or on a plane or maybe on a line. We define the velocity vector as the derivative with respect to t of $\vec{X}$ and we denote it by $\vec{X}'$.

$$\vec{X}' = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(\vec{X}(t + \Delta t) - \vec{X}(t)) \tag{16}$$

Proposition 22. Differentiation of Vectors.

a) If $\vec{X} = x\vec{i} + y\vec{j} + z\vec{k}$ then $\vec{X}' = x'\vec{i} + y'\vec{j} + z'\vec{k}$

b) $(\vec{X} \times \vec{Y})' = \vec{X}' \times \vec{Y} + \vec{X} \times \vec{Y}'$

c) $(\vec{X} \cdot \vec{Y})' = \vec{X}' \cdot \vec{Y} + \vec{X} \cdot \vec{Y}'$

d) $(a\vec{X} + b\vec{Y})' = a\vec{X}' + b\vec{Y}'$

Vector Problems

1. Give a vector which bisects the angle between $\vec{A}$ and $\vec{B}$.
2. A 100 lb. weight is resting on an icy inclined plane of angle 45^o. What tension in the rope keeps the weight motionless?
3. If the wind $\vec{v}_1$ is blowing at 30 miles/hour from the west, and the plane $\vec{v}_2$ is flying NW at 100 mph. with respect to the air, find the resulting velocity $\vec{v}_1 + \vec{v}_2$ with respect to the ground.
4. a) A smuggler's ship is 5 miles due west of the lighthouse and the coast guard ship is 5 miles south west of the lighthouse. What is the vector $\vec{A}$ from the coast guard to the smuggler?

   b) What direction should the coast guard go to intercept the smuggler if the smuggler is travelling due north at 15 miles per hour and the coast guard can go at 20 mph?

   c) How long will it take the coast guard to catch the smuggler?

   d) What is the time and direction if the smuggler is heading due south instead of north?
5. Find the length of $2\vec{i} + \vec{j} + 2\vec{k}$ and of $3\vec{i} - 4\vec{k}$ and the angle between them.
6. Do the lines $t(\vec{i} + \vec{j} + \vec{k})$ and $s(\vec{i} + \vec{j}) + 4\vec{k}$ intersect?
7. Prove the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its sides.
8. Prove the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.
9. Prove the sum of the squares of the sides of any quadrangle minus the sum of the squares of the two diagonals, equals four times the square of the distance between the midpoints of the diagonals.
10. Show that the midpoints of a quadrilateral are the vertices of a parallelogram whose area is one half that of the quadrilateral.
11. What is the locus of points $\vec{x}$ so that $(\vec{x} - \vec{a}) \cdot (\vec{x} - \vec{a}) = 0$?
12. What is the locus of points $\vec{x}$ so that $(\vec{x} + \vec{a}) \cdot (\vec{x} - \vec{a}) = 0$?