A DE MOIVRE LIKE FORMULA FOR FIXED POINT THEORY

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1. Introduction.

Suppose V is a vectorfield on a compact manifold M with or without boundary ∂M. There is a formula, originally due to Marston Morse [Mo] in 1929, which is not as widely known as it should be. It was rediscovered in 1968 by C. Pugh [P] and by the author [G] in 1985. The formula related the index of the vectorfield V with the index of a vectorfield on part of the boundary. This formula, like de Moivre's $e^{i\theta} = \cos\theta + i\sin\theta$, contains a large amount of information.

So let V be a continuous vectorfield on M with no zeros on ∂M. Then

$$Ind(V) + Ind(\partial_- V) = \chi(M). \tag{1}$$

Here $\chi(M)$ denotes the Euler-Poincare number of M, and $\partial_- V$ is a vectorfield defined on part of ∂M as follows. Let $\partial_- M$ be the subset of ∂M containing all m so that $V(m)$ points inward. Then $\partial_- V$ is the vectorfield on $\partial_- M$ given by first restricting V to ∂M, and then projecting $V \mid \partial M$, using the outward pointing vectorfield N, onto its component field tangent to ∂M. This is denoted ∂V. Then $\partial_- V = \partial V \mid \partial_- M$.

Observe that the vectorfield $\partial_- V$ is defined on a space of one dimension lower than V. Thus the formula may be thought of as an inductive definition of index. For example, if $M = I$ and V is a vectorfield on I, then $Ind\,V = 1 - Ind\,\partial_- V$. Now $\partial_- M$ consists of two points, one point, or no points. Correspondingly, we let the corresponding $Ind\,(\partial_- V)$ equal 2, 1, or 0 respectively. Then the respective $Ind\,V$ equal -1, 0, or 1. In the case M is a circle we have $Ind\,(\partial_- V) = 0$ since $\partial S^1 = \partial_- S^1 = \emptyset$, so $Ind\,V = 0$. Thus we have defined the index for all connected 1-manifolds, and since the sum of the indices over disjoint manifolds is the index of the union of the vectorfields, we can calculate the index for one dimension. Similarly, the information in one dimension permits the calculation in two dimensions. There is a technical difficulty in that $\partial_- M$ is not a compact manifold, but the formula (1) is morally a definition.

Thus one should expect that (1) should imply all formulas involving the index of a vectorfield. Also a systematic searching of vectorfield phenomena might yield new results when viewed in conjunction with (1). In the rest of this report we will massage (1) and obtain some well known results. Also we will produce a formula relating the Euler-Poincare number, the Lefschetz number and the coincidence number. This leads to an apparently unknown generalization to immersions of a famous theorem of Hopf: If $\alpha : M^n \to \mathbb{R}^n$ is an immersion, then the Gauss map on ∂M must have degree equal to $\chi(M)$.

2. Massaging the formula.

a) If $\partial_- M$ is empty, then $Ind\,(\partial_- V) = 0$ so $Ind\,(V) = \chi(M)$. This is Hopf's theorem for closed manifolds or when $V \mid \partial M$ always points outwards.
b) For nice enough vectorfields, the generic case, we get a sequence of manifolds $M \supseteq \partial_- M \supseteq \partial_- \partial_- M \supseteq \partial_-^3 M \dots$ where $\partial_- \partial_- M$ is the set of points on the boundary of $\partial_- M$ on which ∂V pointsinto $\partial_- M$, and etc. Then we can eliminate the index of the smaller dimensional vectorfields and show that

$$Ind\, V = \chi(M) - \chi(\partial_- M) + \chi(\partial_-^2 M) - \dots \tag{2}$$

Proving similar versions of (2) was the goal of [Mo] and [P].
c) On the other hand, we can try and eliminate the Euler-Poincare number. To do that, we first define $\partial_+ M$ and $\partial_0 M$ by letting $\partial_+ M$ be the manifold where $V \mid \partial M$ points outward and $\partial_0 M$ is the set where $V \mid \partial M$ is tangent to ∂M. Thus $\partial M = \partial_- M \bigcup \partial_+ M \bigcup \partial_0 M$. Then let $\partial_+ V = \partial V \mid \partial M_+$ and $\partial_0 V = \partial V \mid \partial M_0$. Now $Ind\,\partial_0 V = 0$ since $\partial_0 V$ has no zeros. So

$$Ind\,(\partial V) = \chi(\partial M) = Ind\,(\partial_- V) + Ind\,(\partial_+ V) \tag{3}$$

Thus for even dimensional M, since ∂M is odd dimensional and so $\chi(\partial M) = 0$, we have:

$$Ind\,(\partial_- V) = -Ind\,(\partial_+ V). \tag{4}$$

For odd dimensional M we have

$$Ind\,(\partial_- V) = 2\chi(M) - Ind\,(\partial_+ V). \tag{5}$$

Combining (5) and (1) in the odd dimensional case we get

$$Ind\, V = \frac{1}{2}(Ind\,(\partial_+ V) - Ind\,(\partial_- V)). \tag{6}$$

In the even dimensional case there is no such formula, so the theory is different in even and odd dimensions. This formula was used by Sue Goodman [Gd] in the case V is transversal to ∂M. Then $Ind\, V = \frac{1}{2}(\chi(\partial_+ M) - \chi(\partial_- M))$.
d) Now let us write (1) using $Ind\,(\partial_+ V)$.
Using (4) and (5) we get

$$Ind\, V = (-1)^n \chi(M^n) + Ind\,(\partial_+ V). \tag{7}$$

which is Morse's version of the formula, [Mo]
e) To prove that $Ind\,(-V) = (-1)^n Ind\,(V)$ is instructive. Here n denotes the dimension of M. We assume that the formula is true for vectorfields on $(n-1)$ - dimensional spaces. Then $Ind\,(\partial_-(-V)) = Ind\,(-\partial_+ V) = (-1)^{n-1}(\partial_+ V)$. Using this and (1) and (3) we have

$$\begin{aligned} Ind\,(-V) &= \chi(M) - Ind\,(\partial_-(-V)) \\ &= \chi(M) - (-1)^{n-1} Ind\,(\partial_+ V) \\ &= \chi(M) + (-1)^n(\chi(\partial M) - Ind\,(\partial_- V)) \end{aligned}$$

If n is even then

$$\begin{aligned} Ind\,(-V) &= \chi(M) + (0 - Ind\,(\partial_- V) \\ &= Ind\,V \end{aligned}$$

If n is odd

$$\begin{aligned} Ind\,(-V) &= \chi(M) - [2\chi(M) - Ind\,(\partial_- V)] \\ &= -(\chi(M) - Ind\,(\partial_- V)) = -Ind\,V \end{aligned}$$

f) Let V be the vectorfield on $\mathbb{R}^n$ given by $V(\overrightarrow{x}) = A\overrightarrow{x}$ where A is a linear transformation. If $detA \neq 0$, then $\overrightarrow{0}$ is the only zero of V. We will show, using (1), the well known fact

$$Ind\,(V) = \frac{detA}{\mid detA \mid}. \tag{8}$$

Proof. We let $M = \{\overrightarrow{x} \parallel \overrightarrow{x} \parallel \leq 1\}$. Then on M we see that $Ind\,V + Ind\,(\partial_- V) = 1$. Now A is homotopic through linear transformations A_t such that $det(A_t) \neq 0$, to a diagonal matrix D. The corresponding vectorfields V_t all have a single zero at $\overrightarrow{0}$ and so the index of V_t on M does not change. Now D can be chosen to be either the identity I or the diagonal matrix J with -1 in the $(1,1)$ position and 1's down the main diagonal. In the case of $detA > 0, I$ may be chosen for D and the vectorfield $V_1 : V_1(\overrightarrow{x}) = \overrightarrow{x}$ always points outward on ∂M. Thus $\partial_- M$ is empty and so $ind\,(\partial_- V_1) = 0$. Hence $Ind\,V = 1$. For the case $det(A) < 0$, the vectorfield is $V_1(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$. Then $\partial_- M$ consists of the set of vectors $\overrightarrow{x}$ such that $\overrightarrow{x} \cdot J\overrightarrow{x} < 0$. That is the set $\{(x_1, \ldots, x_n) \in \partial M \mid x_1^2 > \frac{1}{2}\}$. The boundary of $\partial_- M$ are the spheres given by $x_1 = \pm\frac{1}{\sqrt{2}}$. Now if $\overrightarrow{x}$ is on the boundary of $\partial_- M$ we see that $\overrightarrow{x} \cdot J\overrightarrow{x} = 0$. Now consider for $\epsilon > 0$

$$\begin{gathered} (\overrightarrow{x} + \epsilon J\overrightarrow{x}) \cdot J(\overrightarrow{x} + \epsilon J\overrightarrow{x}) = \\ \overrightarrow{x} \cdot J\overrightarrow{x} + \epsilon\,\overrightarrow{x} \cdot (J^2\overrightarrow{x} + J\overrightarrow{x} \cdot J\overrightarrow{x}) + \epsilon^2 J\overrightarrow{x} \cdot J^2\overrightarrow{x} = \\ \epsilon(\overrightarrow{x} \cdot \overrightarrow{x} + J\overrightarrow{x} \cdot J\overrightarrow{x}) > 0. \end{gathered}$$

Thus $J\overrightarrow{x}$ is tangent to the sphere at $\overrightarrow{x}$ and points out of $\partial_- M$. Hence by (1), $Ind\,\partial_- V = \chi(\partial_- M)$. But $\partial_- M$ can be deformed to $S^0 = \{(1, 0, \ldots, 0), (-1, 0, \ldots, 0)\}$. ∎ So $\chi(\partial_- M) = 2$. Hence from (1)

$$Ind\,V_1 + 2 = 1$$

so $Ind\,V = -1$.

g) Let $f : \mathbb{R}^n$ be given by $f(\overrightarrow{x}) = A\overrightarrow{x}$ where A is a linear transformation. If A has no eigenvalue $\lambda = 1$ then f has a single fixed point at $\overrightarrow{0}$ and the Lefschetz number

$$\Lambda_f = \frac{det(I - A)}{\mid det(I - A) \mid} \tag{9}$$

Proof. The fixed point index of f at $\overrightarrow{0}$ is the index of the vectorfield $V(\overrightarrow{x}) = \overrightarrow{x} - f(\overrightarrow{x}) = (I - A)\overrightarrow{x}$.

Comparing (f) and (g) we see that if the linear transformation A has a unique zero and fixed point, the vectorfield index and the fixed point index agree if and only if A has an *even* number of eigenvalues between zero and one for *even* n, and in the case of *odd* n, they agree if and only if there are an *odd* number of eigenvalues between zero and one.

3. $e^{i\pi} + 1 = 0$

De Moivre's formula has the famous special case which relates the five most common constants of calculus. Similarly formula (1) implies an equation which relates the three most common algebraic invariants of fixed point theory, the Euler-Poincare number, the Lefschetz number and the coincidence number.

If $\partial M^n \subset \mathbb{R}^{n+1}$, recall that the Gauss map $\hat{N} : \partial M \to S^n$ is given by translating the "outward pointing" unit normal vector at m, denoted $N(m)$ to the origin. Similarly for any nonzero vectorfield V on ∂M we define the Gauss map $\hat{V} : \partial M \to S^n$ by translating the unit vector $V(m)/ \parallel V(m) \parallel$ to the origin.

Now suppose that $f : M^n \to M^n \subset \mathbb{R}^{n+1}$ is a map such that f has no fixed points on ∂M. Let V be the vectorfield

$$V(m) = (m - f(m))_m$$

Then

$$\Lambda_f + \Lambda_{(-\hat{V}),\hat{N}} = \chi(M) \tag{10}$$

where Λ_f is the Lefschetz number of f and $\Lambda_{(-\hat{V}),\hat{N}}$ is the coincidence number of the Gauss maps for V and N.

Equation (10) follows from (1) and the following observation. We use the description of coincidence numbers found in the last chapter of Vick [V]. There we see that a coincidence number $\Lambda_{f,g}$ is defined which is equal to the sum of all the coicidence numbers of all the coincidence points of $f, g : M_1^n \to M_1^n$ where M_1^n and M_i^n are closed oriented manifolds. Then Vick shows that

$$\Lambda_{f,g} = \sum_i (-1)^i trace(g^! f^*)_i \tag{11}$$

where $g^! : H^i(M_1, \mathbb{Z}) \to H^i(M_2, \mathbb{Z})$ is the Umkehr map. Now it is not hard to show that

$$Ind\,(\partial_- V) = \Lambda_{-\hat{V},\hat{N}} \tag{12}$$

Since $Ind\,V = \Lambda_f$, (12) and (1) imply (10).

4. Normal degree.

Let $\alpha : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed oriented manifold. Let V be a vectorfield on $\mathbb{R}^{n+1}$ with no zeros on $\alpha(M)$. Then

$$Ind\,(\partial_- V) = deg(\hat{N}) - deg(\hat{V}) \tag{13}$$

where $\hat{N}$ is the Gauss map of the unit normal chosen so that the orientation of M^n and $\mathbb{R}^{n+1}$ agree and $\partial_- V$ consists of those vectors of V such that $N(m) \cdot V(m) < 0$. This follows from (12) and (11).

We define the *winding number* $W(p, \alpha)$ at $p \in \mathbb{R}^{n+1}$ of $\alpha : M \to \mathbb{R}^{n+1}$ of $p \notin \alpha(M)$ as follows. Consider a ray ρ starting at p. Assume that ρ intersects $\alpha(M)$ transversally. For each m so that $\alpha(m)$ meets ρ, assign $+1$ if ρ and $N(m)$

point to the same "side" at m and assign -1 if they point to opposite sides. Then $W(p,x)$ is defined to be the sum of these ± 1's. This $W(p,x)$ is equal to the degree of the map

$$\pi \circ \alpha : M^n \xrightarrow{\alpha} \mathbb{R}^{n+1} - p \xrightarrow{\phi} S^n$$

Now suppose that V has a finite number of zeros at points $p_i \notin \chi(M)$. For each p_i there is a local index $Ind\,(p_i, V)$. Then we have the following formula

$$\sum_i W(p_i, \alpha) \cdot Ind\,(p_i, V) + Ind\,(\partial_- V) = deg(\hat{N}) \tag{14}$$

Proof. Since M is a closed oriented manifold immersed in $\mathbb{R}^{n+1}$, we see that M must be the boundary of an oriented manifold W. Now the embedding $\alpha : \partial W \hookrightarrow \mathbb{R}^{n+1}$ extends to a smooth map $f : W \to \mathbb{R}^{n+1}$ for which the p_i are regular values. Then the vectorfield V may be considered as a map $V : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Then $deg(V \circ f) = \sum_i W(p_i, \alpha) \cdot Ind\,(p_i, V)$ because this is the local degree calculated using the preimage of $\vec{0}$. But $deg(V \circ f) = deg(\hat{V})$ where $\hat{V} : \partial W \xrightarrow{\alpha} (\mathbb{R}^{n+1} - p) \to \mathbb{R}^{n+1} - 0 \xrightarrow{\phi} S^n$. Then (l4) follows from (13).

Equation (l4) affords a means of studying the degree Gauss map of an immersion This is called the normal degree and is a classical topic begun by Hopf in [H1] and continued by Milnor in [Mi]. Equations (1) and (14) combine to give the following result.

Theorem:. *If $\alpha : M^{n+1} \to \mathbb{R}^{n+1}$ is an immersion, then the degree of the Gauss map $\hat{N} : \partial M \to S^n$ is equal to the $\chi(M)$.*

Proof. Since α is an immersion, the vectorfield V on $\mathbb{R}^n$ pulls back to a vectorfield $V*$ on M. Now $Ind\,V^* = \sum_i W(p_i, \alpha) Ind\,(p_i, V)$, so (1) and (l4) imply that $deg\hat{N} = \chi(M)$.

Hopf orginally proved this in [H2] for α an embedding. This is a famous and important theorem. Also Hopf showed in [H2] that $deg\hat{N} = \frac{1}{2}\chi(\partial M)$ for even immersions. The case of $\partial M = S^1$ is known and the immersions are classified.

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