EIGENBUNDLES, QUATERNIONS, AND BERRY'S PHASE

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April 15, 2003

Abstract. Given a parameterized space of square matrices, the associated set of eigenvectors forms some kind of a structure over the parameter space. When is that structure a vector bundle? When is there a vector field of eigenvectors? We answer those questions in terms of three obstructions, using a Homotopy Theory approach. We illustrate our obstructions with five examples. One of those examples gives rise to a 4 by 4 matrix representation of the Complex Quaternions. This representation shows the relationship of the Biquaternions with low dimensional Lie groups and algebras, Electro-magnetism, and Relativity Theory. The eigenstructure of this representation is very interesting, and our choice of notation produces important mathematical expressions found in those fields and in Quantum Mechanics. In particular, we show that the Doppler shift factor is analogous to Berry's Phase.

1. Introduction

This work was stimulated by the Gibbs Lecture of Sir Michael Berry given at the 2002 American Math. Soc. meeting in San Diego California. Berry's lecture discussed the discription of physical phenomina by means of slowly changing eigenvectors of relevant linear operators, usually Hamiltonians of Quantum Mechanics. This work was advanced by several mathematical physicists, such as Barry Simon, under the name of Berry's Phase. The original papers are [Berry(1984)] and [Simon(1983)]. A multitude of similar phenomena are found in [Berry(1990)].

Berry's Phase can be thought of in terms of *eigenbundles*, or *spectral bundles* as some mathematical physicists call them. These are vector bundles whose fibres are spaces of eigenvectors associated to linear operators which are parameterized by the base space.

There are two questions involving these spectral bundles. The first is: When do they exist? The second is: What is a relevant connection to put on a spectral bundle which results in physical descriptions?

The first question is topological, the second is more geometrical and of course physical. We will approach the first question from a homotopy theoretical point of view. Spectral bundles are related to an area of Analysis concerned with spectral

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1991 *Mathematics Subject Classification.* 57R45, 17B90, 15A63.

Key words and phrases. exponential map, singularity, electromagnetism, energy-momentum, vector bundles, Clifford Algebras, Doppler shift.

projections. Mathematical physicists have incorporated some homotopy concepts, such as homotopy groups, in their study of spectral bundles, [Avron, Sadun, Segert, Simon(1989)]. What we do here is study the existence of spectral bundles by means of a commutative diagram. This will characterize when spectral bundles exist in terms of three obstructions, and will organize the many variants under which the existence problem can be posed.

We illustrate the issues involved by giving a few simple examples and one sophisticated example. The sophisticated example consists of a set of 4×4 matrices which are a representation of the *biquaternions*, that is the quaternions complexified. We denote the quaternions by $\mathbb{H}$ and the biquaternions by $\mathbb{H} \otimes \mathbb{C}$.

The quaternions and biquaternions have been studied for over 150 years as a convenient language for physics, [Gsponer, Hurni(2002)] The generalization of quaternions, called Clifford Algebras, has also been extensively studied by physicists, especially by Dave Hestenes under the name of *Geometric Algebra*, [Hestenes, Sobcyk(1987)].

Our particular 4–dimensional representation of the biquaternions naturally gives rise to 4–dimensional representations of important low dimensional Lie groups and algebras. There is a conjugate representation also, and a "modulus square mapping", $\mathfrak{m}$, from these representations of the biquaternions gives well known relationships of low dimensional Lie groups, and electromagnetic energy-momentum tensors, as well as a cononical form of the eigenvectors of Lorentz transformations. This last feature allows us to see the Doppler shift factor as an analogue of Berry's phase. Finally in Section 8, we give two examples of probability distributions in Quantum Mechanics which can be expressed as inner products of eigenvectors .

2. Examples

In this section we set up our basic point of view and illustrate with 4 examples.

Let V be a vector space over the Real numbers $\mathbb{R}$ or the Complex numbers $\mathbb{C}$. Consider the space $Hom(V,V)$ of linear maps from V to V. We assume that V is a finite dimensional space so that we can describe the topology of $Hom(V,V)$ simply. If a basis is chosen for the n–dimensional space V, then we have automatically chosen an isomorphism from $Hom(V,V)$ to $M_n(\mathbb{K})$ where $\mathbb{K}$ stands for either the scalars $\mathbb{R}$ or $\mathbb{C}$. Here $M_n(\mathbb{K})$ denotes the space of $n \times n$ matrices with entries in $\mathbb{K}$. This space is given the Euclidean topology of $\mathbb{K}^{n^2}$.

Now let $\Phi : B \to Hom(V,V)$ be a continuous map where B is a topological space. We will call Φ a *field* of linear operators (or matrices) on B. In the physics literature, this is frequently called a system of linear operators parametrized by B. In Physics, B is usually an interval of the Real line and the parameter is frequently thought of as time. Another variant is the field is over a parameter space B, and a physical process is represented by a path in the parameter space B.

There is a trivial example where $\Phi : B \to M_n(\mathbb{K})$ is the constant map which maps every point to the identity matrix I. In this case, any subbundle of the trivial bundle is an eigenbundle.

At the opposite extreme we give an example for which no eigenbundle exists.

Let B be the rotation group in two dimensions, $SO(2)$, and let Φ be the inclusion map of $SO(2)$ into the space of 2×2 matrices $M_2(\mathbb{R})$. Every rotation except for the identity has imaginary eigenvectors and eigenvalues, hence there cannot be a real spectral bundle over $SO(2)$.

We will give four examples below which illustrate various issues which arise in the study of the existence of spectral bundles.

Example 1 : Let $B = \mathbb{R}$ and let $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ be given by

$$\Phi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Every $\Phi(t)$ has only one eigenvalue $\lambda = 1$ with corresponding eigenspace spanned by the vector $(1,0)^T$ when t does not equal 0, and at $t = 0$, $\Phi(0) = I$ so the eigenspace is all of $\mathbb{R}^2$. In this case the spectral line bundle exists and is trivial since there is a nonzero cross-section. For example, the map which takes $t \mapsto (t, (1,0)^T)$ is a cross-section. We regard this cross-section as a vector field of eigenvectors.

Example 2: Let $B = \mathbb{R}$ and let $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ be given by

$$\Phi(t) = \begin{pmatrix} 1 & f(t) \\ g(t) & 1 \end{pmatrix}$$

where $f(t)$ is a continuous real valued function which is greater than zero if t is positive and equal to zero if t is nonpositive; and $g(t)$ has the opposite property, for example $g(t) = f(-t)$. In this example again, there is only one eigenvalue $\lambda = 1$, but now the eigenspaces are spanned by $(1,0)^T$ for $t > 0$ and $(0,1)^T$ for $t < 0$, and at $t = 0$ the eigenspace is $\mathbb{R}^2$. Thus there is no continuous choice of eigenvectors over $\mathbb{R}$ and so there is no eigenbundle. However, if we were willing to change the field Φ slightly, by letting $f(t)$ be zero in a small interval about 0, then we can connect up the $(1,0)^T$ vector field continously with the $(0,1)^T$ vector fields through eigenvectors in $\mathbb{R}^2$ near 0. So example 2 shows that degenerate eigenspaces are an obstruction to eigenbundles, but under some circumstances, a slight change in Φ can eliminate the obstruction.

Example 3: Let $B = \mathbb{R}^3$ and let $\Phi : \mathbb{R}^3 \to M_2(\mathbb{R})$ be given by

$$\Phi(u, v, w) = \begin{pmatrix} u & v \\ v & w \end{pmatrix}$$

Then $\Phi(u, 0, u)$ has only one eigenvalue $\lambda = u$ and the associated eigenspace is the whole of $\mathbb{R}^2$. Off the line l given by $\{(u, 0, u)\}$ however, Φ has two distinct real eigenvalues and the corresponding eigenspaces are one– dimensional and orthogonal, because $\Phi(b)$ is a symmetric matrix. Let $B' = \mathbb{R}^3 - l$. Then there are two spectral line-bundles over B'. But neither of them is a trivial line bundle. So there is no eigenvector field over B'.

This is seen by moving around a loop which links the line l. The line bundle over the loop is not trivial, so it looks like a Mobius band. If we regard the map Φ as mapping into $M_2(\mathbb{C})$, the eigenbundles over B' are complex line bundles and must be trivial since complex line bundles are classified by the first Chern class which lives in the second cohomology group with integer coefficients. Since B' is homotopically equivalent to the circle, the second cohomology group, and hence the Chern class, and hence the line bundle, must be trivial.

This example was mentioned by M. V. Berry in [Berry(1990)] on Page 38, where he states that this phenomenon didn't seem to be widely known in matrix theory.

The fourth example is more complex, and it is related to the quaternions $\mathbb{H}$, the biquaternions $\mathbb{H} \otimes \mathbb{C}$, $SL(2,\mathbb{C})$, $SO(3,1)$, $\mathfrak{so}(3,1)$, $SU(2)$ and $\mathfrak{su}(2)$ and other topics.

Example 4: Let $B = \mathbb{C}^3$ and $\Phi : \mathbb{C}^3 \to M_4(\mathbb{C})$ so that $\Phi(A_1, A_2, A_3)$ is a matrix F such that

$$F = \left(\begin{array}{c|ccc} 0 & A_1 & A_2 & A_3 \\ \hline A_1 & 0 & -iA_3 & iA_2 \\ A_2 & iA_3 & 0 & -iA_1 \\ A_3 & -iA_2 & iA_1 & 0 \end{array}\right)$$

Or in block form,

$$F = \begin{pmatrix} 0 & \vec{A}^T \\ \vec{A} & \times(-i\vec{A}) \end{pmatrix}$$

where the notation $\times(-i\vec{A})$ symbolizes the 3×3 matrix which operates on a column vector v to produce the cross product $v \times (-i\vec{A})$.

Let $\cdot$ represent the usual Euclidean inner product extended linearly to the complex case. Thus $\vec{A} \cdot \vec{A} = A_1A_1 + A_2A_2 + A_3A_3$. Then the eigenspace structure of $\Phi(\vec{A})$ depends on $\vec{A} \cdot \vec{A}$.

Case 1: $\vec{A} \cdot \vec{A} \neq 0$. In this case there are two nonzero eigenvalues, one the negative of the other (since the square of the eigenvalue equals $\vec{A} \cdot \vec{A}$). Each eigenvalue corresponds to a two–dimensional eigenspace. Let B_1 denote the set of all vectors $\vec{A}$ such that $\vec{A} \cdot \vec{A} \neq 0$. Then there are no eigenbundles for Φ restricted to B_1.

Case 2: $\vec{A} \cdot \vec{A} = 0$ and $\vec{A} \neq 0$. In this case there is only one eigenvalue, 0, and it corresponds to a two–dimensional eigenspace. Let B_2 denote the set of all vectors $\vec{A}$ such that $\vec{A} \cdot \vec{A} = 0$ and $\vec{A} \neq 0$. Then there is an eigenbundle of rank two over B_2. It splits as a Whitney sum of two trivial line bundles. So there are two linearly independent eigenvector fields over B_2, and one of them consists of real eigenvectors.

Case 3: $\vec{A} = 0$. In this case $\Phi(\vec{0})$ is the zero matrix, so every vector in $\mathbb{C}^4$ is an eigenvector.

The above assertions are proved in [Gottlieb(1998), (2001)]. See section 7 of this paper.

3. Obstructions to the existence of eigenbundles

We will show that the obstruction to the existence of spectral bundles over B for the field $\Phi : B \to Hom(V,V)$ consists of two crossections which must be constructed over B. A *cross-section* to a continuous map $f : X \to Y$ is a map $s : Y \to X$ so that the composition $f \circ s$ is the identity map, 1_Y, on Y. This means that we are able to choose in a continuous way one element in each fibre $f^{-1}(y)$ of f. A cross-section is a homeomorphism of Y to its image $s(Y)$ in X. Thus we may regard Y as a subspace $s(Y)$ of X.

If the first two cross-sections, s_1 and s_2 exist, then the existence of a third, s_3, gives an eigenvector field.

Suppose we want to construct a spectral bundle whose fibres are k–dimensional eigenspaces over a field $\Phi : B \to Hom(V,V)$ where V is an n dimensional vector space. Then we first consider the product space $B \times \mathbb{K} \times G_{k,n} \times V$. Here $G_{k,n} = G(V)$ is the Grassmannian space of k–planes in V.

We define a subspace L_3 of $B \times \mathbb{K} \times G_{k,n} \times V$ as follows: L_3 consists of all the points $(b, \lambda, W, \vec{v})$ in $B \times \mathbb{K} \times G_{k,n} \times V$ so that λ is an eigenvalue of $\Phi(b)$, and W is a k–dimensional eigenspace associated to λ, and $\vec{v}$ is an eigenvector in W.

Now the projections

$$B \times \mathbb{K} \times G_{k,n} \times V \xrightarrow{\pi_3} B \times \mathbb{K} \times G_{k,n} \xrightarrow{\pi_2} B \times \mathbb{K} \xrightarrow{\pi_1} B$$

give rise to a sequence of mappings

$$L_3 \xrightarrow{\pi_3} L_2 \xrightarrow{\pi_2} L_1 \xrightarrow{\pi_1} B$$

where $L_2 := \pi_3(L_3)$ and $L_1 := \pi_2(L_2)$ are the images of the projections π_3 and π_2 respectively. That is: L_2 and L_1 are the subpaces of $B \times \mathbb{K} \times G_{k,n}$ and $B \times \mathbb{K}$ consisting of the points (b, λ, W) and (b, λ) respectively where λ is an eigenvalue of $\Phi(b)$, and W is a k–dimensional eigenspace associated to λ.

Now the map $\pi_3 : L_3 \to L_2$ is a k–plane vector bundle. In fact it is a k–spectral bundle with respect to the matrix field $L_2 \to M_n(\mathbb{K})$ defined by $(b, \lambda, W) \mapsto \Phi(b)$. Now this spectral bundle restricts to a subspace as a spectral bundle over the matrix field restricted to the subspace. So if $s : B \to L_2$ is a cross-section to the map $\pi_1 \circ \pi_2 : L_2 \to B$, then the restriction of the spectral bundle over L_2 to the spectral bundle over $s(B)$ gives a spectral bundle $\pi_3 : L_3' \to s(B)$ over B for the matrix field Φ.

The above paragraphs give the notation and the proof for the following classification theorem for spectral bundles:

Theorem 3.1. *The k-spectral bundles are in one to one correspondence with the cross-sections of the map $\pi_1 \circ \pi_2 : L_2 \to B$*

It is convenient to break the cross-section s into two cross-sections: $s_1 : B \to L_1$, and $s_2 : s_1(B) \to L_2'$ where L_2' denotes $\pi_2^{-1}(s_1(B))$, the preimage of $s_1(B)$ contained in L_2. Now the composition $s_2 \circ s_1$ is a cross-section to $\pi_1 \circ \pi_2 : L_2 \to B$. On the other hand, a cross-section $s : B \to L_2$ induces the cross-section $\pi_2 \circ s =: s_1$, and the cross-section s_2 is $s \circ \pi_1 : s_1(B) \to L_2'$.

The following diagram may be helpful in tracing the above notation in the theorem below. The horizontal arrows represent inclusion maps.

$$
\begin{tikzcd}
L_3'' \arrow[r] & L_3' \arrow[r] \arrow[d, "\pi_3"] & L_3 \arrow[r] \arrow[d, "\pi_3"] & B \times \mathbb{K} \times G_{k,n} \times V \arrow[d, "\pi_3"'] \\
s_2 s_1 B \arrow[r] \arrow[u, "s_3"] & L_2' \arrow[r] & L_2 \arrow[r] \arrow[d, "\pi_2"] & B \times \mathbb{K} \times G_{k,n} \arrow[d, "\pi_2"'] \\
 & s_1 B \arrow[r] \arrow[u, "s_2"] & L_1 \arrow[r] & B \times \mathbb{K} \arrow[d, "\pi_1"'] \\
 & & B \arrow[r, equal] \arrow[u, "s_1"] & B
\end{tikzcd}
$$

Theorem 3.2.

a) *The set of s_1 cross-sections is in one to one correspondence with the continuous functions $\lambda : B \to \mathbb{K}$ so that every every $\lambda(b)$ is an eigenvalue of $\Phi(b)$ whose associated eigenspace has dimension $\geq k$.*

b) *The set of s_2 cross-sections corresponds to the continuous selections of k–dimensional subspaces of eigenvectors with eigenvalues $\lambda(b)$.*

c) *The set of nowhere zero cross-sections s_3 of the spectral bundle $L_3'' \xrightarrow{\pi_3} s_2 s_1(B) = B$ corresponds to the set of nowhere zero eigenvector fields for the eigenbundle.*

Proof.

a) The cross-section $s_1(b) = (b, \lambda(b))$ is continuous if and only if $\lambda(b)$ is continuous.

b) $s_2(b) = (b, \lambda(b), W_b)$ where $b \mapsto W_b$ picks out a k–dimensional subspace of eigenvectors with eigenvalue $\lambda(b)$ contained in V, that is it is a function from $B \to G(V_k)$. Now s_2 is continuous if and only if the function $B \to G_k(V)$ is continuous.

c) s_3 is a cross-section to the vector bundle $L_3'' \xrightarrow{\pi_3} s_2 s_1(B) = B$, so $s_3(b)$ is an eigenvector for $\Phi(b)$. If $s_3(b) \neq 0$ for all b in B, then the spectral bundle has a trivial line bundle summand, or equivalently, a nonzero eigenvector field. $\square$

Now let us consider L_1 for complex spectral line bundles. This is the largest of the possible L_1's for a fixed Φ. Every other L_1 for higher dimensional complex spectral bundles, or for real spectral bundles associated to Φ, must be a subspace of the L_1 for complex spectral line bundles. In those cases it is possible that there are no eigenvalues for $\Phi(b)$ and hence there is no cross-section s_1. Examples like the real rotation matrices $SO(2)$ or the spectral 3-bundles of example 4 show that there is no s_1 because π_1 is not onto. But for complex spectral line bundles, not only must π_1 be onto, but L_1 is a topological branched covering of B, where we mean the following by *topological branched covering*: A space X which admits a continuous onto map $p : X \to B$ such that all fibres are discrete and so that the path lifting property holds. That is for every $x \in X$, and path σ in B starting at $\sigma(0) = p(x)$, there is a path $\overline{\sigma}$ in X so that $\sigma = p \circ \overline{\sigma}$ and $\overline{\sigma}(0) = x$.

Theorem 3.3. *For complex line bundles, $\pi_1 : L_1 \to B$ is a topological branched covering of B.*

Proof. Consider the mapping from B to the complex polynomials of degree n given by $b \mapsto det(\lambda I - \Phi(b))$ This is a continuous map from b to the characteristic polynomial of $\Phi(b)$. The Fundamental Theorem of Algebra tells us that there are n roots of this polynomial counting multiplicities, for any point b. The roots are of course, the eigenvalues of $\Phi(b)$. I like to think of it using vector fields. Over each b in $B \times \mathbb{C}$ is a fibre $\mathbb{C}$. On each fibre there is a vertical vector field on $\mathbb{C}$ given by attaching the vector $p_b(z)$ to z where p_b is the characteristic polynomial for $\Phi(b)$. Each zero has a positive vector field index, equal to the multiplicity of the corresponding root. The sum of the local indices adds up to a global index n for every fibre. The set of the the zeros is L_1. So every b is covered by at least one zero and at most n zeros. Hence π_1 is onto, and L_1 consists of at most n connected components over B. As we move from one b to a nearby point, there are zeros in the new fibre close to where they were at b, because no zero can be annihilated by another since there are no nonpositive indices to cancil out. This gives L_1 the branched covering structure. See [Gottlieb, Samaranayake(1994)] for a detailed discussion of the index of vector fields. □

In the case of real matrices, the real characteristic polynomial $det(\lambda I - \Phi(b))$ can be thought of as a vertical vector field on the fibres $\mathbb{R}$. Again the zeros of this vertical vector field on $B \times \mathbb{R}$ gives us L_1, but here it is not necessarily a branched cover over B. The reason is that the zeros of the characteristic polynomial on the real line have indicial values of 1, -1 or 0 . The opposite signs and zero indices allow the zeros on the Real line to annihilate each other, so that there may not be a nearby zero on a nearby fibre to continue the local covering of B by L_1.

The total index on each fibre $\mathbb{R}$ is 1 for odd order matrices and 0 for even order matrices, so the sum of the local indices of each zero add up to 0 in even dimensions and 1 in odd dimensions. Thus, for odd dimensional matrix fields, there is always a zero of index 1 in each fibre, so π_1 is always onto in that case. For the even dimensional matrix field however, there is no guarantee of a zero in every fibre, so π_1 may not be onto.

The real matrix field may be considered as acting on a complex vector space. In this case, the zeros on the real line in $\mathbb{C}$ still have their indices of positive integers as well as their indices ± 1 or 0 on the Real line. In this case, a real zero's annihilation actually is given by a splitting of the zero into two complex conjugate zeros, which of course are off the Real line. Thus a real zero doesn't disappear, it splits into two conjugate zeros which leave the Real line in the Complex plane.

Now we will reconsider our examples in light of the above considerations.

Example 1 has only one eigenvalue for each $b \in \mathbb{R}$, so s_1 exists. At each point b there is only one 1–dimensional eigenspace except at $b = 1$, where it is 2–dimensional. This potentially blocks the existence of s_2, but it happens that we may choose a 1–dimensional eigenspace in the 2–dimensional eigenspace so that the choice of 1–dimensional eigenspaces is continuous. So s_2 exists. There is an obvious eigenvector field, so s_3 exists. It is worth remarking that given a vector bundle over a contractible space such as $\mathbb{R}$, the vector bundle must be trivial and there are always nonzero vector fields; or to say it another way, we can always split off a trivial line bundle.

Example 2 is the same as Example 1, except that it is impossible to choose a 1–dimensional subspace at $b = 0$ in such a way to make a continuous selection of 1–dimensional eigenbundles. Hence s_2 does not exist. The possibility was mentioned of altering Φ slightly to eliminate this obstruction to s_2 existing. For 1–dimensional B's such as a line interval or a circle, this can always be done. Of course, since $Hom(V, V)$ is contractible,we can always homotopy Φ to a constant and obtain a new s_2, but this is too large a change for most purposes.

There are homotopy obstructions to changing Φ so as to eliminate the obstruction to s_2. Suppose that D is the unit disk in the plane. Let $B = D$, and let $\Phi(b)$ be a symmetric matrix of order 2 with eigenvalues ± 1 when $b \in S^1$, where S^1 is the boundary of D. Suppose that the $+1$ eigenvectors are pointing orthogonally outside of D. The it is impossible to extend Φ over D with values symmetric matrices such that every matrix has no 2–dimensional eigenspace. This follows since the outward pointing eigenvector field cannot be extended to a nonzero vector field over D, since such a vector field has $index = 1$. Since every symmetric matrix has a two frame of eigenvectors whenever the two eigenvalues are distinct, such an extension of Φ would give rise to a a nonzero vector field. Contradiction.

Example 3 exhibits some homotopy type features. Recall

$$\Phi(u, v, w) = \begin{pmatrix} u & v \\ v & w \end{pmatrix}$$

Since the matrices are symmetric, the eigenvalues are real and we can find continuous eigenvalue functions on $B = \mathbb{R}^3$. Hence s_1's exist. On the other hand, s_2 does not exist. We know that if an s_2 existed, there would be a eigenbundle over $\mathbb{R}^3$, which is contractible. Hence it would be a trivial line bundle. But we know that on a circle linking l, the restriction line bundle is not trivial. So that contradicts the triviality of a bundle over $\mathbb{R}^3$. If we consider the question over $B' = \mathbb{R}^3 - l$, we have eliminated degenerate eigenspaces, every eigenspace is 1–dimensional, so we can choose a continuous selection of eigenspaces, so s_2 exists, and we have a spectral line bundle over B'. But it is not a trivial bundle. Now real line bundles are classified by their Stiefel-Whitney class w_1, which lives in the first cohomology group of B' with $\mathbb{Z}_2$ coefficients, $H^1(B', \mathbb{Z}_2)$. Now B' is homotopy equivalent to S^1, and so there is only one nonzero $w_1 \in H^1(B', \mathbb{Z}_2) = \mathbb{Z}_2$.

If we consider the same field acting on a complex two–dimensional vector space, we again get a spectral line bundle over B', but this time the bundle is trivial in that is there is a nonzero eigenvector field, but it is not completely real. A complex line bundle is classified by its Chern class $c_1 \in H^2(B', \mathbb{Z})$, the two–dimensional cohomology group with integer coefficients. Since B' is homotopy equivalent to a circle, the two–dimensional cohomology must be zero and hence $c_1 = 0$, so the bundle is trivial.

Example 4 has the property that every eigenspace has complex dimension 2 except for the 0 matrix. If we remove the 0 matrix from consideration, we see that if s_1 exists, then s_2 would exist and we would have an eigen 2-bundle. If we restrict to Case 2, the set B_2 of all vectors $\vec{A}$ such that $\vec{A}\cdot\vec{A}=0$ and $\vec{A}\neq 0$, we get s_1 since the only eigenvalue is 0. Hence in this case there exists an eigenbundle of rank 2 over B_2. Let us write $\vec{A}:=\vec{E}+i\vec{B}$ where $\vec{E}$ and $\vec{B}$ are real vectors. In this case, where $\vec{A}\cdot\vec{A}=0$, we have $E=B$ and $\vec{E}\cdot\vec{B}=0$. We may describe the eigenspace by means of two linearly independent eigenvectors: $\vec{E}+i\vec{B}$ and $E^2u+\vec{E}\times\vec{B}$ where $u=(1,0,0,0)$. Here we are regarding the 3-vectors as living in the space orthogonal to u. These eigenvectors each give rise to an eigenvector field which shows that over B_2 the eigenbundle of rank 2 splits as a Whitney sum of two trivial spectral line bundles.

In Case 1 of Example 4, where B_1 is the set of vectors $\vec{A}$ such that $\vec{A}\cdot\vec{A}\neq 0$, we see that s_1 does not exist. In this case each matrix has two distinct eigenvector spaces. Recall that for complex line bundles, Theorem 3.3 states that $\pi_1: L_1\to B$ is a branched covering of B. If we restrict ourselves to matrices so that every eigenvalue is distinct, then the branching part of the branched covering is eliminated and we have a covering. Each connected component of the covering space is a connected covering space. A cross-section s_1 exists if and only if there is a connected component which is homeomorphic to B, that is, if and only if there exists a one to one covering of B. In situation at hand, the eigenvalues are are not distinct, but there are only two of them, one being the negative of the other. This gives rise to a two to one covering of B_1. Hence s_1 does not exist.

In this case, if we move around a closed curve in B_1 which loops B_2 one time, we arrive at the same matrix, but the eigenspace has been transported to the eigenspace corresponding to the opposite eigenvalue. This is a subtle effect when encountered without the aid of the double covering point of view.

We will add one more example to our list of four examples. This will actually be an extension of Example 4, and is a faithful 4–dimensional representation of the Biquaternions $\mathbb{H}\otimes\mathbb{C}$.

Example 5: Consider the set $I+S$ of all 4×4 matrices of the form $aI+F$ where a is any complex number and I is the identity matrix and F is any matrix from Example 4. That is $F\in S$ and so has the form

$$F=\begin{pmatrix} 0 & \vec{A}^T \\ \vec{A} & \times(-i\vec{A}) \end{pmatrix}$$

Here $B=\mathbb{C}^4$, and $\Phi(A_0,A_1,A_2,A_3)=A_0I+F$. That is:

$$\Phi(A_0,A_1,A_2,A_3)=\left(\begin{array}{c|ccc} A_0 & A_1 & A_2 & A_3 \\ \hline A_1 & A_0 & -iA_3 & iA_2 \\ A_2 & iA_3 & A_0 & -iA_1 \\ A_3 & -iA_2 & iA_1 & A_0 \end{array}\right)$$

Let $\langle\ ,\rangle$ represent the usual Minkowskian inner product extended linearly to the complex case. Thus, if $A:=(A_0,A_1,A_2,A_3)=:(A_0,\vec{A})$, then $\langle A,A\rangle=-A_0A_0+$

$A_1A_1 + A_2A_2 + A_3A_3 = -A_0A_0 + \vec{A}\cdot\vec{A}$. Then the eigenspace structure of $\Phi(A)$ depends on $\langle A, A\rangle$.

Case 1: $\langle A, A\rangle \neq 0$ and $\vec{A} \neq 0$. In this case there are two nonzero eigenvalues when $\vec{A}\cdot\vec{A} \neq 0$. Each eigenvalue corresponds to a two–dimensional eigenspace. Let B_1 denote the set of all vectors A such that $\langle A, A\rangle \neq 0$. Then there are no eigenbundles for Φ restricted to B_1.

Case 2: $\langle A, A\rangle = 0$ and $\vec{A} \neq 0$. In this case there is one or two eigenvalues, but one of them is equal to 0, and it corresponds to a two–dimensional eigenspace. Let B_2 denote the set of all vectors A such that $\langle A, A\rangle = 0$ and $\vec{A} \neq 0$. Then there is an eigenbundle of rank two over B_2. It splits as a Whitney sum of two trivial line bundles. So there are two linearly independent eigenvector fields over B_2, and one of them consists of real eigenvectors.

Case 3: $\vec{A} = 0$. In this case $\Phi(A)$ is a diagonal matrix, so every vector in $\mathbb{C}^4$ is an eigenvector.

We note that the cases of Example 5 seems to be very similar to the cases of Example 4, but now the eigenvalues are not each other's negatives, and in Case 2 there are one *or* two eigenvalues. But one of them is always zero, so s_1 exists in that case since the eigenvalue map is the constant zero. But then the nonzero eigenvalue also must form an eigenfunction over B_2, and so there is another spectral 2-bundle over B_2. Over the region where $A_0 = 0$, this second spectral 2-bundle is identical with the first.

4. Biquaternions

The set of matrices of Example 5

$$\Phi(A_0, A_1, A_2, A_3) = \left(\begin{array}{c|ccc} A_0 & A_1 & A_2 & A_3 \\ \hline A_1 & A_0 & -iA_3 & iA_2 \\ A_2 & iA_3 & A_0 & -iA_1 \\ A_3 & -iA_2 & iA_1 & A_0 \end{array}\right)$$

is a representation of the biquaternions.

Obviously it is isomorphic to $\mathbb{C}^4$ as a vector space. We will list a basis below which will reveal the relationship of the matrices and the biquaternions. Let x denote the matrix above in which $A_1 = 1$ and the other $A_i = 0$.

That is

$$x = \Phi(0, 1, 0, 0) = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{array}\right)$$

In the same way we define matrices

$y := \Phi(0, 0, 1, 0)$

$z := \Phi(0, 0, 0, 1)$

$I = \Phi(1, 0, 0, 0)$, the identity matrix of order 4.

Now $xy = iz$ and $x^2 = y^2 = z^2 = I$ and $xy = -yx$. Then the basis $\{ix,\ iy,\ iz,\ I\}$ obviously has the relations defining the biquaternions.

There is another representation of the biquaternions in which the traceless matrices are given by

$$F = \begin{pmatrix} 0 & \vec{A}^T \\ \vec{A} & \times(i\vec{A}) \end{pmatrix}$$

These matrices differ from the previous set in Example 4 by changing the $-i$ to $+i$. If we denote the set of matrices of Example 4 by S, let $\overline{S}$ denote the set of matrices of the form F above.

Now let $\{X,\ Y,\ Z,\ I\}$ be the complex conjugates of $\{x,\ y,\ z,\ I\}$ respectively. These new elements satisfy $XY = -iZ$ and $X^2 = Y^2 = Z^2 = I$ and $XY = -YX$. So the basis $\{-iX,\ \ -iY,\ \ -iZ,\ \ I\}$ obviously has the relations defining the biquaternions for $I \oplus \overline{S}$.

Now it happens that any $F \in S$ commutes with any $G \in \overline{S}$. That is $F\overline{G} = \overline{G}F$ for $F, G \in S$. This gives rise to a pairing $(I \oplus S) \otimes (I \oplus \overline{S}) \to M_4(\mathbb{C})$ given by $A \otimes B \mapsto AB$ where the product AB is in the space of 4×4 complex matrices. This pairing is an isomorphism of rings. This can be seen by observing that the following set of sixteen matrices forms a basis of $M_4(\mathbb{C})$:

Theorem 4.1. *The set of sixteen matrices*

a)
$$\begin{array}{rrrr} I, & xX, & yY, & zZ, \\ x, & X, & yZ, & zY, \\ y, & Y, & xZ, & zX, \\ z, & Z, & xY, & yX \end{array}$$

forms a basis for $M_4(\mathbb{C})$, the vector space of 4×4 complex matrices.

b) The square of each of the matrices in the basis is I.

c) Each matrix is Hermitian, so real linear combinations of the basis are the 4×4 Hermitian matrices.

d) Every matrix has zero trace except for I.

Proof. Theorem 3.3 of [Gottlieb(2001)].

It is easy to calculate any 4×4 matrix in terms of this basis using MATLAB. Below I produce a matrix whose first column is x written as a column vector of length 16. (this is done by x(:), which counts from 1 down the first column and then down the next column until you arrive at the 4×4 term which is the last number of the vector). The remaining columns are given in the order as shown below in the definition of Total.

(4.1)

Total = [x(:) X(:) y(:) Y(:) z(:) Z(:) xY(:) yX(:) yZ(:) zY(:) zX(:) xZ(:) xX(:) yY(:) zZ(:) I(:)];

Now any 4×4 matrix M can be converted into a vector M(:). The command Total \ M(:) gives the vector of coefficients which when multiplied with the basis in the order found in Total above will give the linear combination of M in terms of the basis.

Now since $M_4(\mathbb{C})$ is the complex Clifford algebra $\mathcal{C}\ell(4)$, there must be generators $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ so that $\alpha_i\alpha_j + \alpha_j\alpha_i = \delta_{ij}I$. One such set of α's is given by

$$\begin{aligned}\alpha_0 &= x\\ \alpha_1 &= y\\ \alpha_2 &= zX\\ \alpha_3 &= zY.\end{aligned}$$

Theorem 4.2. *Let $F, G \in S$ satisfy $Fu = \vec{A}$ and $Gu = \vec{B}$. Then*

a) $FG + GF = (\vec{A}\cdot\vec{B})I$

b) $F\overline{G} = \overline{G}F$

c) $[F, G]u = 2i\vec{A}\times\vec{B}$

d) $e^F = \cosh(\lambda_F)I + \frac{\sinh(\lambda_F)}{\lambda_F}F$ *where λ_F is an eigenvalue of F.*

Proof. Corollary 4.7, Theorem 4.8, Corollary 4.4, and Theorem 8.5 of [Gottlieb(1998)] respectively. $\square$

Now every nonsingular matrix $A \in M_4(\mathbb{C})$ gives rise to an inner automorphism of $M_4(\mathbb{C})$ given by $B \mapsto A^{-1}BA$. These maps transform the basis into a new basis with the same algebraic properies, but the form of the representative matrices can be quite different. We will end this section discussing what distinguishes our representation from the other representations.

The matrices of S (or $\overline{S}$) are skew symmetric with respect to the Minkowski metric $-+++$. That is equivalent to the property $F^T = -\eta F\eta$ where $F \in S$, F^T is the transpose of F and $\eta =$ the diagonal matrix with $-1, 1, 1, 1$ down the main diagonal. A popular set of matrices are the skew symmetric matrices with respect to the Euclidean metric. They satisfy $A = -A^T$. Now $\eta^{1/2}F\eta^{-1/2}$ is a skew symmetric matrix if $\eta^{1/2}$ and $\eta^{-1/2}$ equal the diagonal matrix $\pm i, 1, 1, 1$ respectively. Hence if

$$F = \left(\begin{array}{c|c} 0 & \vec{A}^T \\ \hline \vec{A} & \mp i(\times\vec{A}) \end{array}\right) \tag{4.2}$$

then

$$\eta^{1/2}F\eta^{-1/2} = -i\left(\begin{array}{c|c} 0 & -\vec{A}^T \\ \hline \vec{A} & \pm(\times\vec{A}) \end{array}\right) \tag{4.3}$$

Thus $M_4(\mathbb{C})$ is the tensor product $(I + \eta^{1/2}S\eta^{-1/2}) \otimes (I + \eta^{1/2}\overline{S}\eta^{-1/2})$. This means that the transformed S matrices still have squares equal to a multiple of the identity, and it satisfies the same exponential equation as in Theorem 4.2d. And the transformed S and $\overline{S}$ still commute, but they are no longer the complex conjugate of each other. It is this property which gives our representation its distinctive advantage, because the "modulus squared map" is a multiplicative homomorphism on S.

The matrices of the form

$$F = \left(\begin{array}{c|c} 0 & \vec{A}^T \\ \hline \vec{A} & \times\vec{C} \end{array} \right) \tag{4.4}$$

are the skew symmetric matrices with respect to the Minkowski inner product. So S and $\overline{S}$ are skew symmetric matrices with respect to the Minkowski inner product. The only skew symmetric matrices with respect to the Minkowski inner product whose squares are multiples of I are precisely the matrices of S and $\overline{S}$. [Gottlieb(1998)], see Theorem 4.5 .

Now note that if $F \in S$, then both the complex conjugate $\overline{F}$ and the transpose F^T are both in $\overline{S}$. Thus the pseudo automorphisms *conjugation* : $A \mapsto \overline{A}$, which is antilinear in that it changes the sign of i, and *transpose* : $A \mapsto A^T$, which reverses the order of multiplication, interchange S and $\overline{S}$. In terms of our basis, $ax + by + cz \mapsto \overline{a}X + \overline{b}Y + \overline{c}Z$ under conjugation and $ax + by + cz \mapsto aX + bY + cZ$ under transposition. The composition of conjugation and transposition yields the Hermitian conjugate $\dagger$: $ax + by + cz \mapsto \overline{a}x + \overline{b}y + \overline{c}z$ which is an antilinear isomorphism which preserves S and $\overline{S}$.

On the other hand, S and $\overline{S}$ are interchanged by the inner automorphism $A \mapsto \eta A \eta$. That follows since $\eta F \eta = -F^T$ when $F \in S$. In terms of our basis, $ax + by + cz \mapsto -aX - bY - cZ$.

5. The Modulus squared map

We define the modulus squared map and list several of its properties in this section.

Definition. *The modulus squared map is a multiplicative homomorphism* $\mathfrak{m} : (I + S) \to M_4(\mathbb{R})$ *given by* $A \mapsto \mathfrak{m}(A) = \overline{A}A$. *Its image* $\mathfrak{m}(I + S)$ *is denoted by* $\mathfrak{M}$.

To show that this definition is well–defined, we must show that its image is in the set of real matrices; and that it preserves matrix multiplication. The following lemma does that.

Lemma 5.1. *Suppose A and B square matrices. Then*

a) $\overline{A}A$ *is a real matrix if and only if A and $\overline{A}$ commute.*

b) $\mathfrak{m}(AB) = \mathfrak{m}(A)\mathfrak{m}(B)$

Proof.

a) A matrix is real if and only if it is equal to its own complex conjugate. Now $A\overline{A} = \overline{A}A = \overline{A\overline{A}}$ since A and $\overline{A}$ commute. Conversely, suppose $A\overline{A}$ is real. Now $A = C + iD$ where C and D are real. So $A\overline{A} = (C + iD)(C - iD) = C^2 - D^2 + i[D, C]$. Since $A\overline{A}$ is real, the commutator $[D, C] = 0$. This implies that $A\overline{A} = \overline{A}A$.

b) First of all , note that $I+S$ is closed under multiplication. See Lemma 6.2. Then $\mathfrak{m}(AB) = AB\overline{AB} = A\overline{A}B\overline{B} = \mathfrak{m}(A)\mathfrak{m}(B)$.

We will call $\mathfrak{m}$ the modulus squared map in analogy with the complex absolute value squared of a complex number.

Now $\mathfrak{m}$ has many striking properties. The following are the most interesting.

Theorem 5.2. *The set $\mathfrak{M}$ is homeomorphic to the cone over the projective space PC^3*

Proof. As a vector space $(I+S)$ is isomorphic to $\mathbb{C}^4$. The modulus map $\mathfrak{m}$ has fibres S^1 over all points of $\mathfrak{M}$ (except for 0) since $\mathfrak{m}(F) = \mathfrak{m}(\alpha F)$ when α is a complex number of unit modulus. Then $\mathfrak{m}$ can easily be seen to be an identification map, and the identification of $\mathbb{C}^4$ by identifying any vector to its multiple by a scalar with the same modulus is the cone over PC^3 with 0 as the vertex of the cone.

Corollary 5.3. *The image of $\mathfrak{m}$ restricted to the unit 7-sphere in $I+S$ is the complex projective space PC^3*

The *Lorentz group* is the set of linear transformations L on Minkowski space which preserves the Minkowski metric, that is $< Lu, Lv > = < u, v >$. It has four connected components. The component containing the identity is called the proper Lorentz group and is denoted by $SO^+(3,1)$.

The *complex Lorentz group* is the set of linear transformations on complexified Minkowski space $\mathbb{R}^{3,1} \otimes \mathbb{C}$ which preserve in Minkowski metric. The complex Lorentz group, $L(\mathbb{C})$, has two connected components. It plays a role in physics, [Wightman(2000)].

The identity component of the complex Lorentz group intersects $I+S$ in a subgroup, which I will call the *biquaternion Lorentz group.* Similarly, the identity component of the Lorentz group intersects $I+\overline{S}$ in a subgroup which is isomorphic to the other by compex conjugation. The other complex Lorentz group component is disjoint from both biquaternions.

Theorem 5.4. *The image of $\mathfrak{m}$ restricted to the biquaternion Lorentz group, which consists of the set $\{aI+F \mid a^2I-\lambda^2 = 1\}$, is the real proper Lorentz group $SO^+(3,1)$.*

Corollary 5.5. *The Lorentz Group $SO^+(3,1)$ is exponential, that is it has a surjective exponential map from $\mathfrak{so}^+(3,1)$.*

We will prove Theorem 5.4 and Corollary 5.5 in the next section. Corollary 5.5 was proved in [Nishikawa (1983)]. In fact Nishikawa shows that $SO(n,1)$ is exponential.

Theorem 5.6. $\mathfrak{m}(S)$ = *The set of electromagnetic energy-momentum tensors.*

proof. Suppose $F \in S$. Then $Fu = \mathbf{E} + i\mathbf{B}$, and if we imagined $\mathbf{E}$ and $\mathbf{B}$ as electric and magnetic vectors, then the corresponding electro-magnetic tensor $T = \frac{1}{2}F\overline{F}$.

See Proposition 5.1 with Definition 3.8 in [Gottlieb(1998)] . See [Parrott (1987)] for a mathematical account of electro-magnetic energy-momentum tensors.

Theorem 5.7. $\mathfrak{m}(S^3) = SO(3)$ *where* S^3 *is the unit 3-sphere, that is the real unit quaternions.*

proof. The real unit quaternions are represented by $\{aI + bix + ciy + diz)$ where x, y, z are the basis matrices of section 4, and a, b, c, d satisfy $a^2 + b^2 + c^2 + d^2 = 1$ and are real numbers. If we multiply $\{aI + bix + ciy + diz)$ by a unit modulus complex number, the element remains in the real quaternions if and only if the number is ± 1. Thus $\mathfrak{m}$ is a 2–1 covering map, so its image must be $SO(3)$. □

The real unit quaternions S^3 acts on the right of unit biquaternions $S^7 = \{aI + bx + cy + dz | a\overline{a} + b\overline{b} + c\overline{c} + d\overline{d} = 1\}$. The quotient map is the famous Hopf fibration $S^3 \to S^7 \to S^4$. Now $\mathfrak{m} : S^7 \to CP^3$ is a principal S^1-fibre bundle and is an equivariant map from the free S^3 action on S^7 to the induced $SO(3)$ action on CP^3. The action of $SO(3)$ on CP^3 is not free.

Consider the set of matrices in $1+S$ of the form $\{aI+F \mid a^2 = \lambda^2\}$, where λ is the eigenvalue of F. These matrices are those $aI + F$ such that $(aI + F)(aI - F) = 0$. In biquaternion jargon, these are called *nullquats* or singular quaternions. Since $F(\lambda I+F) = \lambda(\lambda I+F)$, we see that the image of $\lambda I+F$ consists of the eigenvectors of F corresponding to the eigenvalue λ. The fact that $(\lambda I+F)(\lambda I-F) = 0$ implies that the kernel of $F(\lambda I + F)$ consists of the eigenvalues of F corresponding to $-\lambda$. Thus $\lambda I+F$ has rank two. But it is not a spectral projection unless $\lambda = 1/2$. When $\lambda = 0$ we have the *null matrices* N such that $N^2 = 0$. Here the eigenvector space is both the image and the kernel of N. So N cannot be made into a projection by scalar multiplication. However, N does map $\mathbb{C}^4$ onto the subspace of eigenvectors of N.

Theorem 5.8. *The image of a nullquat under* $\mathfrak{m}$ *is a linear transformation from* $\mathbb{R}^4$ *to a real null 1-dimensional subspace of eigenvectors of the nullquat.*

proof. See Theorem 6.7c in [Gottlieb(1998)].

6. The Exponential Map

In this section we show that the exponential map for the proper Lorentz group is surjective using novel methods.

In order to discuss eigenvector spaces and exponential maps more fully, we will change our notation to emphasize the real matrices. We shall follow the notation of [Gottlieb(1998) and (2001)].

Let $F \in S$ now be denoted by cF where

$$cF := \begin{pmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & \times(-i\mathbf{A}) \end{pmatrix} \quad \text{where} \quad \mathbf{A} = \mathbf{E} + i\mathbf{B}$$

Then $cF := F - iF^*$ where F now denotes the real part of cF and $-F^*$ is the imaginary part. Thus

$$F = \begin{pmatrix} 0 & \mathbf{E}^T \\ \mathbf{E} & \times\mathbf{B} \end{pmatrix} \quad \text{and} \quad F^* = \begin{pmatrix} 0 & -\mathbf{B}^T \\ -\mathbf{B} & \times\mathbf{E} \end{pmatrix}.$$

Similarly we define $\overline{c}F := F + iF^*$.

Now F is a linear transformation on $\mathbb{R}^4$ which is skew symmetric with respect to the Minkowski metric, and cF will be called its *complexification* . We may regard F as a 1-1 tensor corresponding to a two-form $\hat{F}$. Then F^* corresponds to the Hodge dual $*\hat{F}$. If we apply the modulus squared map to cF, we get $\overline{c}FcF := 2T_F$ where T_F has the form of a multiple of the energy-momentum tensor of the electromagnetic field two-form $\hat{F}$ corresponding to F. On the other hand we may regard F as an element of the Lie algebra $\mathfrak{so}(3,1)$.

Theorem 6.1. *The exponential map Exp: $\mathfrak{so}(3,1) \to SO(3,1)^+$ given by $F \mapsto e^F$ is onto. That is, for every proper Lorentz transformation L, there exists an $F \in \mathfrak{so}(3,1)$ so that $L = e^F$.*

To prove the above theorem, we need to consider the complexification $\mathfrak{so}(3,1)\otimes\mathbb{C}$ operating on $\mathbb{R}^{3,1}\otimes\mathbb{C}$. This last is isomorphic to $\mathbb{C}^4$ and has an inner product which is of the type $-+++$ on $\mathbb{R}^{3,1}$ and extends to the complex vectors by $\langle i\vec{v}, \vec{w}\rangle = \langle \vec{v}, i\vec{w}\rangle = i\langle \vec{v}, \vec{w}\rangle$. See [Gottlieb(2001), Section 2] for more details.

Now let $c : \mathfrak{so}(3,1) \to \mathfrak{so}(3,1) \otimes \mathbb{C}$ given by $cF = F - iF^*$. The image of c, denoted S, is a three–dimensional complex vector space. The set of operators of the form $aI + bcF$ will be denoted by $I + S$. Note that $I + S$ is a vector space isomorphic to $\mathbb{R}^{3,1}\otimes\mathbb{C}$, and that $I+S$ is closed under multiplication, as the following lemma shows.

Lemma 6.2. *Let F and $G \in S$ denote cF and cG. Then $(aI + bF)(\alpha I + \beta G) = (a\alpha + b\beta\langle F, G\rangle)I + (b\alpha F + a\beta G + \frac{b\beta}{2}[F, G])$*

Now we say that $L \in I+S$ is a *biquaternion Lorentz transformation* if $\langle Lu, Lv\rangle = \langle u, v\rangle$. Any biquaternion Lorentz transformation L must have the form $L = aI+bF$, where $F \in S$, such that $a^2 - b^2\lambda_F^2 = 1$.

That is, $L^{-1} = aI - bF$.

Theorem 6.3. *Every complex Lorentz transformation L is an exponential, that is $L = e^F$ for some $F \in S$, except for $L = -I + N$ where $N \in S$ is null, that is $N^2 = 0$.*

Proof. Recall [Gottlieb(1998), Theorem 8.5] where $F \in S$ that

$$e^F = \cosh(\lambda_F)I + \frac{\sinh(\lambda_F)}{\lambda_F}F \qquad (**)$$

Now $L = aI + H$ where $H \in S$ and $a^2 - \lambda_H^2 = 1$. So the first obstruction to showing that L is an exponential is solving the equation $\cosh(\lambda) = a$. We shall show below that such a λ always exists. Next, if $\dfrac{\sinh(\lambda)}{\lambda} \neq 0$, then

$$L = aI + H = \cosh(\lambda)I + \frac{\sinh\lambda}{\lambda}\left(\frac{\lambda}{\sinh\lambda}H\right) =: \cosh(\lambda)I + \frac{\sinh\lambda}{\lambda}D = e^D$$

Hence L may not be an exponential if $\dfrac{\sinh(\lambda)}{\lambda} = 0$.

Now $\dfrac{\sinh\lambda}{\lambda} = 0$ exactly when $\lambda = \pi ni$ for n a non-zero integer. (Note that $\dfrac{\sinh(0)}{0} = 1$). Then

$$a = \cosh(\lambda) = \cosh(\pi ni) = \cos(\pi n) = (-1)^n.$$

If n is even, then $L = I + N = e^N$ where N must be null.

If n is odd, then $a = (-1)^n = -1$, so $L = -I + N$ where N must be null or zero. Now $e^B = -I$ where $B \in S$ has eigenvalue $(2k+1)\pi i$. But $-I + N = -e^{-N}$ cannot be an exponential, because it has a real eigenvector with negative eigenvalue. This proves Theorem 6.3 except for the following lemma.

Lemma 6.4.

a) $\cosh(\lambda) = a$ *always has a solution over the complex numbers.*

b) $\sinh(\lambda) = 0$ *if and only if* $\lambda = \pi ni$.

Proof. First we show b). Now $\sinh(\lambda) = \dfrac{e^\lambda - e^{-\lambda}}{2} = 0$.

Thus $e^{2\lambda} = 1$, hence $2\lambda = 2\pi ni$ so $\lambda = \pi ni$.

Next we show a). Now $\cosh(\lambda) = \dfrac{e^\lambda + e^{-\lambda}}{2} = a$. Hence $(e^\lambda)^2 - 2ae^\lambda + 1 = 0$

Hence $e^\lambda = \dfrac{2a \pm \sqrt{4a^2 - 4}}{2} = a \pm \sqrt{a^2 - 1}$.

Now $e^\lambda = b$ has a solution for all b except $b = 0$. But $a \pm \sqrt{a^2 - 1}$ cannot equal zero, hence we have shown there is a solution for each a.

Proof of Theorem 6.1. We show the exponential map is onto $SO(3,1)^+$ by showing the products of two exponentials is an exponential. That is $e^F e^G = e^D$ for $F, G, D \in \mathfrak{so}(3,1)$. Now $e^F = e^{\frac{1}{2}cF} e^{\frac{1}{2}\overline{c}F}$ where $\overline{c}F = F + iF^*$. This follows since cF and $\overline{c}F$ commute. Also for this reason, e^{cF} and $e^{\overline{c}G}$ commute. Thus $e^F e^G = e^{\frac{1}{2}cF} e^{\frac{1}{2}cG} e^{\frac{1}{2}\overline{c}F} e^{\frac{1}{2}\overline{c}G}$. Now $e^{\frac{1}{2}cF} e^{\frac{1}{2}cG}$ is a complex Lorentz transformation in $I + S$. So either it is an exponential e^{cD}, or it has the form $-I + cN = -e^{cN}$ by Theorem 6.3. Now Theorem 6.3 also holds for $I + \overline{S}$. Hence we have $e^F e^G = e^{2D}$ or $e^F e^G = (-e^{cN})(-e^{\overline{c}N}) = e^{2N}$. $\square$

Corollary 6.5. *The exponential map Exp* : $\mathfrak{so}(3,1) \otimes \mathbb{C} \to SO(\mathbb{R}^{3,1} \otimes \mathbb{C})$ *is not onto. If* $N \in \mathfrak{so}(3,1)$ *is null, then* $-e^N$ *is not an exponential even though* $-e^{cN}$ *is an exponential.*

Proof. As explained in [Gottlieb(2001)], we can extend duality F^* to skew symmetric matrices $\begin{pmatrix} 0 & \vec{E} \\ \vec{E} & \times\vec{B} \end{pmatrix}$ where $\vec{E}$ and $\vec{B}$ are complex vectors. Then $cF = F - iF^*$ and $\overline{c}F = F + iF^*$ satisfy the same properties as in the complexification of the real case. Now consider $e^F e^G$ where $F,\ G \in S$. Then $cF = \frac{1}{2}cF + \frac{1}{2}\overline{c}F$, so $e^F e^G = e^{\frac{1}{2}cF} e^{\frac{1}{2}\overline{c}F} e^{\frac{1}{2}cG} e^{\frac{1}{2}\overline{c}G}$. Now $cF = cA$ for some $A \in \mathfrak{so}(3,1)$, and $\overline{c}F = \overline{c}A'$ for $A' \in \mathfrak{so}(3,1)$, hence

$$e^F e^G = e^{cA} e^{\overline{c}A'} e^{cB} e^{\overline{c}B'} = (e^{cA} e^{cB})(e^{\overline{c}A'} e^{\overline{c}B'}),$$

and so $e^{cA}e^{cB}$ equals either e^{cD} or $-e^{cN}$. But $(-I)e^{cN} = e^{(2n+1)\pi i \overline{c}E} e^{cN} = e^{(2n+i)\pi i \overline{c}E + cN}$ where E has eigenvalue equal to 1. So in both cases $e^{cA}e^{cB}$ is an exponential.

Now $-e^{cN}$ is an exponential since $-e^{cN} = e^{\pi i \overline{c}E} e^{cN} = e^{\pi i \overline{c}E + cN}$ where E has eigenvalue $\lambda_{cE} = 1$. On the other hand $-e^N$, where N is the real part of a null cN, cannot be an exponential, since if $-e^N = e^F$, then s, the unique eigenvector for e^N, applied to this equation gives $-s = e^F s = e^{\lambda_F} s$, so $\lambda_F = (2n+1)\pi i$ for some n. Thus F has another linear independent null eigenvector, which contradicts $-e^N$ having only one. $\square$

7. Eigenvectors

In this section we give explicit formulas for the eigenvectors and eigenvalues of proper Lorentz transformations and their Lie algebra. We show the Doppler shift factor arises as a kind of Berry's phase.

Theorem 7.1. *Let $F \in \mathfrak{so}(3,1)$ and let λ_F be an eigenvalue of F and λ_T be an eigenvalue of T_F. The eigenvalue of cF is $\lambda_{cF} = \lambda_F - i\lambda_{F^*}$ and*

a) $\lambda_T = \sqrt{(\frac{E^2 - B^2}{2})^2 + (\mathbf{E}\cdot\mathbf{B})^2}$

b) $\lambda_F = \pm\sqrt{\lambda_T + \frac{(E^2 - B^2)}{2}}, \quad \lambda_{F^*} = \pm\sqrt{\lambda_T - \frac{(E^2-B^2)}{2}}.$

proof. This is Theorem 5.4 of [Gottlieb(1998)].

Now the image of $\lambda_{cF} I + cF$ is the 2-dimensional space of eigenvectors of cF with eigenvalue λ_{cF}. The image of $\lambda_{\overline{c}F} I + \overline{c}F$ is the 2-dimensional space of eigenvalues of $\overline{c}F$. Note that this is the complex conjugate of the eigenspace of $\lambda_{cF} I + cF$. Now let u be a vector of length -1 in the Minkowski metric, an observer in relativity theory. Then $s := (\lambda_{cF} I + cF)(\lambda_{\overline{c}F} I + \overline{c}F)u$ is in both eigenspaces, since the operators commute. And s is a real vector since u is. So s is not only an eigenvector for cF and $\overline{c}F$, but also for the real part F and the imaginary part F^*, and hence for the stress-energy tensor T_F and the Lorentz transformation e^F. See section 5, [Gottlieb(1998)].

Theorem 7.2. *The eigenvector $s := (\lambda_{cF} I + cF)(\lambda_{\overline{c}F} I + \overline{c}F)u$ for $F \in \mathfrak{so}(3,1)$ with $\mathbf{E} = Fu$ and $\mathbf{B} = -F^*u$ satisfies the following equation:*

$$s = 2\big((\lambda_T + \frac{E^2 + B^2}{2})u + \mathbf{E}\times\mathbf{B} + \lambda_F \mathbf{E} - \lambda_{F^*}\mathbf{B}\big). \tag{7.1}$$

proof. This is Corollary 6.8 of [Gottlieb(1998)].

Corollary 7.3. *For a null $N \in \mathfrak{so}(3,1)$, the eigenvector is*

$$s = 2\big((\frac{E^2+B^2}{2})u + \mathbf{E}\times\mathbf{B}\big) \tag{7.2}$$

proof. Now N null is the real part of the null cN. So $\lambda_{cN} = \lambda_N - i\lambda_{N^*} = 0$. Hence $\lambda_N = \lambda_{N^*} = \lambda_T = 0$. Then plug this into Theorem 7.2. □

Since there are at most two eigenvalues λ_{cF}, one the negative of the other, and since the null matrices have only one eigenvalue, 0, we see from the above results that there are two null real eigenvector spaces for the generic case and one null real eigenvector space for a null matrix.

Now we can use the above formulas to give us something like a connection on the eigenbundles of a field of $F \in \mathfrak{so}(3,1)$ on Minkowski space-time. And we can consider what occurs as we move around a closed time-like circuit in space-time, that is, two time-like paths starting with the same velocity at time 0 and ending at the same point at some positive time. Then the eigenvectors formulas will progress according to the formulas until they meet at a future time where they lie in the same 1-dimensional space, but they differ by a factor. We can calculate that factor. It only depends upon the tangent velocities u and u' at the point of intersection and the factor is real This differs from Berry's phase, in which the factor is complex and usually depends upon the history of the paths, yet it has the same feel to it.

We follow Scholium 8.2 of [Gottlieb(1998)]

Let s_u be an eigenvector of F corresponding to λ_F as seen by an observer u. Suppose

$$u' = \frac{1}{\sqrt{1-w^2}}(u+\mathbf{w}) \tag{7.3}$$

is another observer. Then u' sees a different eigenvector $s_{u'}$. But $s_{u'}$ must be a multiple of s_u since they are eigenvectors. So the question is, what is the multiple in terms of $\mathbf{E}, \mathbf{B}$ and $\mathbf{w}$? The answer is:

Theorem 7.4.

$$s_{u'} = \frac{1}{\sqrt{1-w^2}}\left[1 + \frac{-(\mathbf{E}\times\mathbf{B})\cdot\mathbf{w} + \lambda_F\mathbf{E}\cdot\mathbf{w} - \lambda_{F^*}\mathbf{B}\cdot\mathbf{w}}{\lambda_T + \dfrac{E^2+B^2}{2}}\right] s_u. \tag{7.4}$$

Proof. Define

$$\varphi(v) = \frac{\langle v, s_-\rangle}{\langle u, s_-\rangle} s_u \tag{7.5}$$

where s_- is an eigenvector corresponding to $-\lambda_F$. Then φ is a linear map whose image is the span of s_u and whose kernel is the space of vectors orthogonal to s_-. Now $\varphi(u) = s_u$.

Now $\Phi := (\lambda_{cF} I + cF) \circ (\overline{\lambda_{cF}} I + \overline{c}F)$ has the same properties and let $\Phi(u) := s_u$. Then $\Phi = \varphi$. Let $s_- = \Phi_-(u) = (-\lambda_{cF} I + cF) \circ (\overline{-\lambda_{cF}} I + \overline{c}F)u$.

Now

$$s_u = 2\left(\lambda_T u + \frac{E^2 + B^2}{2} u + \mathbf{E} \times \mathbf{B} + \lambda_F \mathbf{E} - \lambda_{F^*} \mathbf{B}\right) \tag{7.6}$$

from (7.2) and s_- is the same with the signs changed on λ_F and λ_{F^*}:

$$s_- = 2\left(\lambda_T u + \frac{E^2 + B^2}{2} u + \mathbf{E} \times \mathbf{B} - \lambda_F \mathbf{E} + \lambda_{F^*} \mathbf{B}\right) \tag{7.7}$$

Now $s_{u'} = \varphi(u') = \dfrac{\langle u', s_-\rangle}{\langle u, s_-\rangle} s_u$. Substituting (7.3) into this equation yields

$$s_{u'} = \frac{1}{\sqrt{1 - w^2}} \left(1 + \frac{\langle \mathbf{w}, s_-\rangle}{\langle u, s_-\rangle}\right) s_u. \tag{7.8}$$

Now

$$\langle u, s_-\rangle = -2\left(\lambda_T + \frac{E^2 + B^2}{2}\right) \tag{7.9}$$

using (7.7). Then using (7.7) to calculate $\langle \mathbf{w}, s_-\rangle$ and substituting this into (7.8) we obtain (7.4). $\square$

Now (7.4) holds for all $F \in \mathfrak{so}(3,1)$. If we restrict to null F we should see (7.4) reduce to a simpler form. In the null case $\lambda_F = \lambda_{F^*} = 0$ and $E = B$. So equation (7.4) reduces to

$$s_{u'} = \frac{1}{\sqrt{1 - w^2}} \left(1 - \mathbf{w} \cdot \frac{(\mathbf{E} \times \mathbf{B})}{E^2}\right) s_u. \tag{7.10}$$

Now $\mathbf{w} \cdot \dfrac{(\mathbf{E} \times \mathbf{B})}{E^2}$ is the component along the $\mathbf{E} \times \mathbf{B}$ direction. If we assume that $\mathbf{w} = \mathbf{w}_r$, that is $\mathbf{w}$ is pointing in the radial direction, then

$$s_{u'} = \sqrt{\frac{1 - w_r}{1 + w_r}} s_u. \tag{7.11}$$

Here $\sqrt{\dfrac{1 - w_r}{1 + w_r}}$ is the Doppler shift ratio. This suggests that null F propagate along null geodesics by parallel translation.

Now the fact that $I + S$ and $I + \overline{S}$ commute leads to a richer situation in analogy to Berry's phase considerations. If V is a 2–dimensional eigenspace for $F \in I + S$, then it is invariant under any $G \in I+\overline{S}$. In fact, any null 2–dimensional subspace of complexified Minkowski space is either an eigenspace of an $F \in S$ or an eigenspace of an $F \in \overline{S}$. The action of x, y, z on $\overline{V}$ is an irreducible action of the spin Lie algebra, and the action of X, Y, Z on V is also an irreducible action of the spin Lie algebra on V. The particular basis of the actions have a sign difference which [Ryder(1988)] calls left and right spin 1/2 actions.

Now, for example, the nullquat $(\lambda_{cF}I + cF)$ composed with $e^{\overline{c}G}$ and applied to a vector u must be an eigenvector of cF. So if these three quantities are varied, one gets a formula giving the progression of an eigenvector of cF.

8 Physical examples of eigenvectors and quantum probability

We will point out two examples of inner products of eigenvectors of F in $\mathfrak{M}$ which give probabilities underlying two important cases in [Sudbery (1986)]: Page 200, equation (5.84) which gives the probability of spin along an axis at angle θ from the spin direction of the particle. In this case the probability of spin $+1/2$ is equal to the Minkowski innerproduct

$$-\frac{1}{2}\langle u + \mathbf{v}, u + \mathbf{w}\rangle = \sin^2(\theta/2)$$

where u is an observer, i.e. $\langle u, u\rangle = -1$, and $\mathbf{v}$ and $\mathbf{w}$ are unit vector in the rest space of u pointing along the direction of spin of the particle and the direction of the measurement, usually the gradient of a pure $\mathbf{B}$ field. Note both $u+\mathbf{v}$ and $u+\mathbf{w}$ are both null vectors, and hence possible eigenvectors of some operators in $\mathfrak{M}$.

The other example is on P. 273, equation (6.121) of [Sudbery (1986)]. Here the distribution of electrons with specific velocity v is given by $1 - v\cos(\theta)$, where the electrons decay from a Cobalt 60 atom in a strong magnetic field $\mathbf{B}$. Here θ is the angle between the magnetic field B and the velocity of the electron v. If we let u represent the center of mass observer u and $u' = \frac{1}{\sqrt{(1-v^2)}}(u + \mathbf{v})$ represent the 4-velocity of the electron and $u + \frac{1}{B}\mathbf{B}$ be the normalised eigenvector of F representing the pure $\mathbf{B}$ field, then

$$-\langle\sqrt{(1-v^2)}u', u + \frac{1}{B}\mathbf{B}\rangle$$

equals this distribution.

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